The results of the Sixty-Sixth William Lowell Putnam Mathematical Competition, held December 3, 2005, follow. They have been determined in accordance with the regulations governing the Competition. The contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, an endowment established by Mrs. Putnam in memory of her husband. The annual Competition is held under the auspices of the Mathematical Association of America.

The first prize, $25,000, was awarded to the Department of Mathematics of Harvard University. The members of the winning team were Tiankai Liu, Alison B. Miller, and Tong Zhang; each was awarded a prize of $1,000.

The second prize, $20,000, was awarded to the Department of Mathematics of Princeton University. The members of the winning team were Ana Caraiani, Andrei Negut, and Aaron C. Pixton; each was awarded a prize of $800.

The third prize, $15,000, was awarded to the Department of Mathematics of Duke University. The members of the winning team were Nikifor C. Bliznashki, Jason Ferguson, and Lingren Zhang; each was awarded a prize of $600.

The fourth prize, $10,000, was awarded to the Department of Mathematics of the Massachusetts Institute of Technology. The members of the winning team were Timothy G. Abbott, Vladimir Barzov, and Daniel M. Kane; each was awarded a prize of $400.

The fifth prize, $5,000, was awarded to the Department of Mathematics of the University of Waterloo. The members of the winning team were Olena Bormashenko, Ralph Furmaniak, and Xiannan Li; each was awarded a prize of $200.

The six highest ranking individual contestants, the Putnam Fellows, in alphabetical order, were Oleg I. Golberg, Massachusetts Institute of Technology; Mathew M. Ince, Massachusetts Institute of Technology; Daniel M. Kane, Massachusetts Institute of Technology; Ricky I. Liu, Harvard University; Tiankai Liu, Harvard University; and Aaron C. Pixton, Princeton University. Each receives an award of $2,500.

The next ten highest ranking contestants, in alphabetical order, were: Robert M. Barrington Leigh, University of Toronto; Thomas D. Belulovich, Massachusetts Institute of Technology; Richard V. Biggs, Carnegie Mellon University; Steven J. Byrnes, Harvard University; Alexander R. Fink, University of Calgary; Po-Ru Loh, California Institute of Technology; Alison B. Miller, Harvard University; Thanasin Nampaisarn, Massachusetts Institute of Technology; Eric C. Price, Massachusetts Institute of Technology; and Kuat T. Yessenov, Massachusetts Institute of Technology. Each receives an award of $1,000.

The next eight highest ranking contestants, in alphabetical order, were: Ralph Furmaniak, University of Waterloo; Hyun Soo Kim, Massachusetts Institute of Technology; Zhiwei Calvin Lin, University of Chicago; Roger Mong, University of Toronto; Andrei Negut, Princeton University; Dimitar V. Ostrev, Yale University; Steven W. Sivek, Massachusetts Institute of Technology; and Lingren Zhang, Duke University. Each receives an award of $250.
The following teams, named in alphabetical order, received honorable mention: California Institute of Technology, with team members Justin Blanchard, Po-Ruh Loh, and Rumen Zarev; Carnegie Mellon University, with team members Richard V. Biggs, Paul Zagieboylo, and Chunhua Zhang; Stanford University, with team members John W. Hegeman, Serin Hong, and Robert D. Hough; University of Toronto, with team members Robert Barrington Leigh, David Tianyi Han, and Jacob Tsimerman; and Yale University, with team members Joshua D. Batson, Lazar Krstic, and Dimitar V. Ostrev.

Honorable mention was achieved by the following fifty-one individuals in alphabetical order: Timothy G. Abbott, Massachusetts Institute of Technology; Boris Alexeev, Massachusetts Institute of Technology; Jae M. Bae, Harvard University; Jongmin Baek, Massachusetts Institute of Technology; Joshua D. Batson, Yale University; Nath Bejraburin, Stanford University; Kshipra U. Bhowalkar, Duke University; Justin E. Blanchard, California Institute of Technology; Nikifor C. Bliznashki, Duke University; Olena Bormashenko, University of Waterloo; Ana Caraiani, Princeton University; Po-Ning Chen, Massachusetts Institute of Technology; Andrew J. Critch, Memorial University of Newfoundland; Fernando A. Delgado, University of Michigan; Ann Arbor; Anand R. Deopurkar, Massachusetts Institute of Technology; Gabriel E. Gauthier-Shalom, McGill University; Elyot J. L. Grant, University of Waterloo; Mathieu Guay-Paquet, McGill University; John W. Hegeman, Stanford University; Robert D. Hough, Stanford University; Nathaniel J. Ince, Massachusetts Institute of Technology; Lozan M. Ivanov, California Institute of Technology; Junehyuk Jung, University of Chicago; Nima Kamoosi, University of British Columbia; Dmytro Karabash, Columbia University; Pramod Khungurn, Massachusetts Institute of Technology; Sungyoon Kim, Massachusetts Institute of Technology; Aaron D. Kleinman, Princeton University; Roman L. Kogan, State University of New York, Stony Brook; Gabriel E. Kreindler, Princeton University; Joel B. Lewis, Harvard University; Joshua J. Lim, Massachusetts Institute of Technology; Yuncheng Lin, Massachusetts Institute of Technology; Aleksandar D. Lishkov, Princeton University; Po-Ling Loh, California Institute of Technology; Thomas J. Mildorf, Massachusetts Institute of Technology; Samuel A. Miner, Pomona College; Paul D. Nelson, Princeton University; Virgil C. Petrea, Massachusetts Institute of Technology; Natee Pitiwan, Williams College; Wei Quan Julius Poh, Cornell University; Vedran Sohinger, University of California, Berkeley; Kiat Chuan Tan, Stanford University; Matthew J. Thibault, Massachusetts Institute of Technology; David W. Vincent, Massachusetts Institute of Technology; Nathaniel G. Watson, Washington University, St. Louis; Ryan B. Williams, Stanford University; Yeol Yoon, California Institute of Technology; Chunhua Zhang, Carnegie Mellon University; Tong Zhang, Harvard University; and Yan Zhang, Harvard University.

The other individuals who achieved ranks among the top 99 students, in alphabetical order of their schools, were: California Institute of Technology, Yingkun Li, Oleg O. Rudenko; Carnegie Mellon University, Yinemeng N. Zhang; Duke University, Aaron J. Pollack; Harvard University, Luke A. Gustafson, Anatoly Preygel; Kansas State University, Jeffrey M. Amos; Massachusetts Institute of Technology, Nikolay V. Andreev, Vladimir V. Barzov, Daniel R. Gulotta, Anand R. Rajagopalan, Gary L. Sivek, Iliya T. Tsekov; Princeton University, Anton S. Malyshev; Queen’s University, Michael T. LeBlanc; Rice University, Max I. Glick; Simon Fraser University, Denis K. Dmitriev; University of British Columbia, Wei-Lung D. Tseng; University of California, Berkeley, Paul R. Monasterio; University of Mississippi, Sam S. Watson; University of Nebraska, Lincoln, Robert M. Brase; University of Toronto, Tianyi Han, Jacob Tsimerman; and University of Waterloo, Tor G. J. Myklebust.
The Elizabeth Lowell Putnam Prize, named for the wife of William Lowell Putnam and awarded "to a woman whose performance on the Competition has been deemed particularly meritorious," is awarded this year to Alison B. Miller, Harvard University. The winner is awarded a prize of $1,000.

There were 3545 individual contestants from 500 colleges and universities in Canada and the United States who participated in the competition of December 3, 2005. There were teams entered by 395 institutions. The Questions Committee for the sixty-sixth competition consisted of Hugh L. Montgomery (Chair), University of Michigan, Ann Arbor; Titu Andreescu, University of Texas, Dallas; and Steven Tschantz, Vanderbilt University; they composed the problems and were most prominent among those suggesting solutions. Alternative solutions have been published in Mathematics Magazine 79 (2006) 76–81.

PROBLEMS.

A-1. Show that every positive integer is a sum of one or more numbers of the form $2^r3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

A-2. Let $S = \{(a, b) : a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A rook tour of $S$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (i) $p_i \in S$, (ii) $p_i$ and $p_{i+1}$ are a unit distance apart for $1 \leq i < 3n$, (iii) for each $p \in S$ there is a unique $i$ such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$? (An example of such a rook tour for $n = 5$ is depicted.)

A-3. Let $p(z)$ be a polynomial of degree $n$ all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^n/2$. Show that all zeros of $g'(z)$ have absolute value 1.

A-4. Let $H$ be an $n \times n$ matrix all of whose entries are ±1 and whose rows are mutually orthogonal. Suppose $H$ has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

A-5. Evaluate $\int_0^1 \frac{\ln(x+1)}{x^2+1} \, dx$.

A-6. Let $n$ be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are $n$ randomly, independently and uniformly, chosen points on a circle. Consider the convex $n$-gon whose vertices are the $P_i$. What is the probability that at least one of the vertex angles of this polygon is acute?

B-1. Find a nonzero polynomial $P(x, y)$ such that $P([a], [2a]) = 0$ for all real numbers $a$. (Note: $[v]$ is the greatest integer less than or equal to $v$.)

B-2. Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$
B-3. Find all differentiable functions \( f : (0, \infty) \rightarrow (0, \infty) \) for which there is a positive real number \( a \) such that

\[
f'(\frac{a}{x}) = \frac{x}{f(x)}
\]

for all \( x > 0 \).

B-4. For positive integers \( m \) and \( n \), let \( f(m, n) \) denote the number of \( n \)-tuples \((x_1, x_2, \ldots, x_n)\) of integers such that \(|x_1| + |x_2| + \cdots + |x_n| \leq m\). Show that \( f(m, n) = f(n, m) \).

B-5. Let \( P(x_1, \ldots, x_n) \) denote a polynomial with real coefficients in the variables \( x_1, \ldots, x_n \), and suppose that

\[\begin{align*}
(a) \quad \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \ldots, x_n) &= 0 \quad \text{(identically)} \\
(b) \quad x_1^2 + \cdots + x_n^2 \text{ divides } P(x_1, \ldots, x_n).
\end{align*}\]

Show that \( P = 0 \) identically.

B-6. Let \( S_n \) denote the set of all permutations of the numbers \( 1, 2, \ldots, n \). For \( \pi \in S_n \), let \( \sigma(\pi) = 1 \) if \( \pi \) is an even permutation and \( \sigma(\pi) = -1 \) if \( \pi \) is an odd permutation. Also, let \( \nu(\pi) \) denote the number of fixed points of \( \pi \). Show that

\[
\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.
\]

SOLUTIONS. In the 12-tuples \((n_{10}, n_9, n_8, n_7, n_6, n_5, n_4, n_3, n_2, n_1, n_0, n_{-1})\) following each problem number, \( n_i \), for \( 10 \geq i \geq 0 \), is the number of students among the top 196 contestants achieving \( i \) points for the problem and \( n_{-1} \) is the number of those not submitting solutions.

A-1. (132, 17, 6, 0, 0, 0, 0, 0, 10, 4, 16, 11)

Solution. We argue by induction. Clearly \( 1 = 2^0 3^0 \), so the number 1 is represented. If \( n \) is even, then by the induction hypothesis \( n/2 \) has such a representation, and it suffices to multiply each summand by 2. Suppose \( n \) is odd, and let \( k \) be the largest integer such that \( 3^k \leq n \). If \( n = 3^k \), then we are done. If \( 3^k < n < 3^{k+1} \), then \( m = (n - 3^k)/2 \) is a positive integer. Since \( m < n \), it follows from the inductive hypothesis that we can write \( m = \sum_i 2^i 3^{u_i} \). Then \( n = 3^k + \sum_i 2^i 3^{u_i} \). It remains to show that no summand divides another. Using the inductive hypothesis, \( 2^i+1 3^{u_i} \) divides \( 2^j+1 3^{u_j} \) only when \( i = j \). Also, \( 2^i+1 3^{u_i} \) does not divide \( 3^k \) because 2 is a factor of \( 2^i+1 3^{u_i} \) but not a factor of \( 3^k \). Finally, \( m = (n - 3^k)/2 < (3^{k+1} - 3^k)/2 = 3^k/2 \). Thus, \( 2^i+1 3^{u_i} \) does not divide \( 3^k \).

A-2. (85, 44, 13, 0, 0, 0, 0, 24, 5, 12, 13)

Solution. Let \( T(n) \) be the number of rook tours of \( S_n = S \) that begin at \((1, 1)\) and end at \((n, 1)\), and let \( U(n) \) be the number of rook tours of \( S_n \) that begin at \((1, 1)\) and end at \((n, 3)\). Let \( j \) \((0 \leq j \leq n - 2)\) denote the number of moves to the right before the first vertical move is made. Once the first move up has been made, the only way to visit all squares in the first \( j + 1 \) columns without getting trapped is to move left \( j \) times, move up, and then move right \( j + 1 \) times. This leaves a player at \((j + 2, 3)\). It
remains to traverse a $3 \times (n - 1 - j)$ board, starting at $(j + 2, 3)$ and ending at $(n, 1)$. By definition, this can be done in $U(n - 1 - j)$ ways. Thus

$$T(n) = \sum_{j=0}^{n-2} U(n - 1 - j).$$

Similarly,

$$U(n) = \sum_{j=0}^{n-1} T(n - 1 - j),$$

with the convention that $T(0) = 1$. Clearly, $T(1) = 0$ and $U(1) = 1$. We claim that $T(n) = U(n) = 2^{n-2}$ when $n \geq 2$. This follows directly by induction from the preceding identities.

**A-3.** (14, 2, 5, 0, 0, 0, 0, 6, 4, 86, 79)

**Solution.** Write $p(z) = c(z - z_1)(z - z_2) \cdots (z - z_n)$. Thus

$$\frac{g'(z)}{g(z)} = \sum_{k=1}^{n} \left( \frac{1}{z - z_k} - \frac{1}{2z} \right).$$

Hence

$$2z \frac{g'(z)}{g(z)} = \sum_{k=1}^{n} \frac{z + z_k}{z - z_k}.$$  \hspace{1cm} (1)

Fix $k$, and consider the triangle with vertices $z$, $z_k$, and $-z_k$. Since the segment from $-z_k$ to $z_k$ is a diameter of the unit circle, the angle at $z$ is acute if $|z| > 1$. Thus in this case, $|\arg((z + z_k)/(z - z_k))| < \pi/2$, which is to say $\Re((z + z_k)/(z - z_k)) > 0$. Since this holds for each $k$, the right-hand side of (1) has positive real part when $|z| > 1$. Thus $g'(z) = 0$ has no solution when $|z| > 1$. Similarly, if $|z| < 1$, the angle at $z$ is obtuse, $\pi/2 < |\arg((z + z_k)/(z - z_k))| < \pi$, and $\Re((z + z_k)/(z - z_k)) < 0$ for all $k$. In this case the real part of the right-hand side of (1) is negative, so again $g'(z) = 0$ is impossible.

**A-4.** (20, 0, 7, 0, 0, 0, 0, 3, 0, 55, 111)

**Solution.** Write $H = [h_{ij}]$, and let $I$ and $J$ be intervals chosen so that the submatrix in question is obtained by restricting $i$ to $I$ and $j$ to $J$. Let the rows of $H$ be $r_1, \ldots, r_n$, and put $v = \sum_{i \in I} r_i$. On the one hand, since the rows are orthogonal, it follows that

$$\|v\|^2 = \sum_{i_1 \in I} \sum_{i_2 \in J} r_{i_1} \cdot r_{i_2} = \sum_{i \in I} r_i \cdot r_i = an.$$  

On the other hand, the $j$th coordinate of $v$ is $\sum_{i \in I} h_{ij}$, whence

$$\|v\|^2 = \sum_{j=1}^{n} \left( \sum_{i \in I} h_{ij} \right)^2.$$ 

Here all the summands are nonnegative, and for $j$ in $J$ the summand is $a^2$. Hence $\|v\|^2 \geq a^2 b$. It follows that $a^2 b \leq an$, which gives the stated result.

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A-5. \((10, 10, 0, 0, 0, 0, 0, 0, 2, 2, 40, 132)\)

Solution 1. Let \(I\) denote the integral in question. Because \(x = \tan(\arctan x)\) for all \(x\) in \([0, 1]\), with the substitution \(\arctan x = t\) we have

\[
I = \int_0^{\pi/4} \ln(1 + \tan t) \, dt.
\]

Let \(u = \pi/4 - t\). Then

\[
I = \int_{\pi/4}^0 \ln \left(1 + \tan \left(\frac{\pi}{4} - u\right)\right) (-du) = \int_0^{\pi/4} \ln \left(1 + \frac{1 - \tan u}{1 + \tan u}\right) \, du
\]

\[
= \int_0^{\pi/4} \ln \left(\frac{2}{1 + \tan u}\right) \, du = \int_0^{\pi/4} \ln 2 \, du - I.
\]

It follows that \(2I = (\pi/4) \ln 2\); that is, \(I = (\pi/8) \ln 2\).

Solution 2. Let \(I\) denote the integral in question. The substitution \(x = (1 - u)/(1 + u)\) leads directly to \(I = (\ln 2)\pi/4 - I\).

A-6. \((5, 0, 2, 0, 0, 0, 0, 0, 0, 3, 0, 65, 121)\)

Solution. Let \(A, B,\) and \(C\) denote three consecutive points on the circle. To say that \(\angle ABC\) is acute is equivalent to saying that the arc on the circle from \(A\) through \(B\) to \(C\) measures more than \(\pi\). Thus there is not enough room on the circle to have two acute angles if they are not adjacent. In the event under consideration, there is a unique point where the angle is obtuse followed (counterclockwise) by an acute angle at the next point. Suppose that \(0 \leq \theta \leq 1\) and that \(0 \leq \phi \leq 1/2\). The probability that exactly one of the \(n\) points lies on the arc \((2\pi\theta, 2\pi\theta + 2\pi\Delta \theta)\) and exactly one point lies on the arc \((2\pi(\theta + \phi), 2\pi(\theta + \phi) + 2\pi\Delta \phi)\) is approximately \(n(n-1)\Delta \theta \Delta \phi\). This estimate would be exact were it not for the possibility that two or more points might lie on one of these arcs or that the two arcs might overlap. These possibilities are negligible when \(\Delta \theta\) and \(\Delta \phi\) are small. Let \(A\) denote the point in the first arc and \(B\) denote the point in the second arc. In order for the angle at \(B\) to be acute all remaining points must lie in the semicircle from \(A\) to its antipode. The probability of this is \(1/2^{n-1}\). Let \(A'\) denote the antipode of \(A\), and \(B'\) the antipode of \(B\). We don’t want all of these remaining points to lie between \(A'\) and \(B'\). The probability of this latter event is at least \((\phi - \Delta \theta)^{n-2}\) but not more than \((\phi + \Delta \phi)^{n-2}\). Thus the probability of the event in question is

\[
n(n-1) \int_0^1 \int_0^{1/2} \left(\frac{1}{2^{n-2}} - \phi^{n-2}\right) \, d\phi \, d\theta
\]

\[
= n(n-1) \left(\frac{1}{2^{n-1}} - \frac{1}{2^{n-1}(n-1)}\right) = \frac{n(n-2)}{2^{n-1}}.
\]

B-1. \((155, 9, 22, 0, 0, 0, 0, 0, 7, 0, 3, 0)\)

Solution. One such polynomial is \(P(x, y) = (2x - y)(2x - y + 1)\). To see this, let \(a\) be a real number and \(n = \lfloor a \rfloor\). If \(a \in [n, n + 1/2)\) then \(\lfloor 2a \rfloor = 2\lfloor a \rfloor\), so \(P([a], [2a]) = 0\). If \(a \in [n + 1/2, n + 1)\) then \(\lfloor 2a \rfloor = 2\lfloor a \rfloor + 1\), so once again \(P([a], [2a]) = 0\).
B-2. \((148, 11, 4, 0, 0, 0, 0, 9, 1, 13, 10)\)

**Solution.** The possible solutions are:

\[
\begin{align*}
n & = 1, k_1 = 1; \\
n & = 3, \{k_1, k_2, k_3\} = \{2, 3, 6\}; \\
n & = 4, k_1 = k_2 = k_3 = k_4 = 4.
\end{align*}
\]

From the arithmetic mean-geometric mean (or Cauchy-Schwarz) inequality, we find that

\[
(k_1 + \cdots + k_n) \left( \frac{1}{k_1} + \cdots + \frac{1}{k_n} \right) \geq n^2,
\]

with equality if and only if \(k_1 = \cdots = k_n\). Hence \(5n - 4 \geq n^2\), which is equivalent to \(n(n - 4) \leq 0\). Thus \(n \in \{1, 2, 3, 4\}\). For \(n = 1\) or \(n = 4\), we have equality in (1), so \(k_1 = \cdots = k_n\). We obtain the solutions \(n = 1, k_1 = 1\), and \(n = 4, k_1 = k_2 = k_3 = k_4 = 4\). We are left with the cases \(n = 2\) and \(n = 3\). For \(n = 2\), the system of equations \(k_1 + k_2 = 6, 1/k_1 + 1/k_2 = 1\) is not solvable in positive integers. For \(n = 3\), we seek the triples \((k_1, k_2, k_3)\) such that \(k_1 + k_2 + k_3 = 11\) and \(1/k_1 + 1/k_2 + 1/k_3 = 1\). Let \(k_1k_2k_3 + k_3k_1k_2 = k_1k_2k_3 = q\). Then \(k_1, k_2,\) and \(k_3\) are positive integral solutions to the equation \(x^3 - 11x^2 + qx - q = 0\). It follows that a solution \(x\) is an integer different from 1 and 11, and that

\[
q = \frac{-x^3 + 11x^2}{x - 1} = -x^2 + 10x + 10 + \frac{10}{x - 1}.
\]

Because \(q\) is a positive integer, \(x - 1\) is a positive divisor of 10 and different from 10. Then \(x - 1 \in \{1, 2, 5\}\), so \(x \in \{2, 3, 6\}\). A simple case analysis shows that \(\{k_1, k_2, k_3\} = \{2, 3, 6\}\).

B-3. \((23, 34, 8, 0, 0, 0, 2, 8, 75, 46)\)

**Solution.** Define \(g(x) = f(x)f(a/x)\) for \(x\) in \((0, \infty)\). We claim that \(g\) is a constant function. Indeed, substituting \(a/x\) for \(x\) in the given condition yields \(f(a/x)f'(x) = a/x\) when \(x > 0\), so

\[
g'(x) = f'(x)f(a/x) + f(x)f'(a/x)(-a/x^2) = a/x - a/x = 0,
\]

ensuring that \(g\) is some positive constant \(b\).

From the original equation we can write

\[
b = g(x) = f(x)f\left(\frac{a}{x}\right) = f(x)\left(\frac{a}{x} \cdot \frac{1}{f'(x)}\right),
\]

which gives

\[
\frac{f'(x)}{f(x)} = \frac{a}{bx}.
\]
Integrating each side we obtain \( \ln f(x) = (a/b) \ln x + \ln c \), where \( c > 0 \). It follows that \( f(x) = cx^{a/b} \) for \( x > 0 \). Substituting back into the original equation yields

\[
c \cdot \frac{a}{b} \cdot \frac{a^{a/b-1}}{x^{a/b-1}} = \frac{x}{cx^{a/b}},
\]

which is equivalent to

\[
e^2 a^{a/b} = b.
\]

By eliminating \( c \) we find the family of solutions

\[
f_b(x) = \sqrt{b} \left( \frac{x}{\sqrt{a}} \right)^{a/b} \quad (b > 0).
\]

\textbf{B-4.} \quad (97, 15, 12, 0, 0, 0, 0, 0, 0, 13, 12, 27, 20)

\textit{Solution 1.} It suffices to show that

\[
f(m, n) = \sum_{k=0}^{\min(m,n)} 2^k \binom{m}{k} \binom{n}{k},
\]

since this is symmetric in \( n \) and \( m \). First we group the lattice points according to the number of nonzero coordinates. Call this number \( k \). There are \( \binom{\min(m,n)}{k} \) ways of choosing which coordinates are the nonzero ones. For each nonzero coordinate we must choose the sign of the coordinate. There are \( 2^k \) ways of choosing these signs. It remains to count the number of \( k \)-tuples \( (a_1, \ldots, a_k) \) of positive integers such that \( \sum a_i \leq m \). Put \( b_i = a_i - 1 \). Then \( (b_1, \ldots, b_k) \) is a \( k \)-tuple of nonnegative integers whose sum is at most \( m - k \). Let this sum be \( r \). It is well known that the number of \( k \)-tuples of nonnegative integers with sum \( r \) is exactly \( \binom{k+r-1}{k-1} \). Hence the number of \( k \)-tuples is

\[
\sum_{r=0}^{m-k} \binom{k+r-1}{k-1} = \binom{m}{k}.
\]

This leads to the stated formula for \( f(m, n) \).

\textit{Solution 2.} Extend the definition to include \( m = 0 \) or \( n = 0 \) but not both. For the case \( m = 0 \) the only \( n \)-tuple that works is \( (0, 0, \ldots, 0) \), so \( f(0, n) = 0 \) for \( n > 0 \). For the case \( n = 0 \) the only 0-tuple is \( \emptyset \), so \( f(m, 0) = 1 \). Thus \( f(0, n) = f(m, 0) = 1 \).

By considering the possible values of \( x_n \) among \( n \)-tuples we see that if \( n > 0 \), then

\[
f(m, n) = \sum_{i=-m}^{m} f(m - |i|, n)
\]

\[
= f(m, n - 1) + \sum_{i=1}^{m} 2 f(m - i, n - 1), \quad (1)
\]

for the last expression counts the \( n \)-tuples with \( x_n = i \) \( (-m \leq i \leq m) \).

Applying this to \( m - 1 \) for positive \( m \), we obtain

\[
f(m - 1, n) = f(m - 1, n - 1) + \sum_{i=1}^{m-1} 2 f((m - 1) - i, n - 1). \quad (2)
\]
Subtracting (2) from (1) leads to the recursion formula

\[ f(m, n) = f(m - 1, n) + f(m, n - 1) + f(m - 1, n - 1). \]  

(3)

The result now follows by strong induction on the sum \( m + n \) by using (3) twice, namely,

\[
\begin{align*}
    f(m, n + 1) &= f(m - 1, n + 1) + f(m, n) + f(m - 1, n) \\
    &= f(n + 1, m - 1) + f(n, m) + f(n - 1, m) \\
    &= f(n + 1, m).
\end{align*}
\]

\textbf{Solution 3.} Let \( S(m, n) \) denote the set of \( n \)-tuples counted by \( f(m, n) \), and let \( T(m, n) \) be the set of sequences

\[(a_1, b_1, e_1), (a_2, b_2, e_2), \ldots, (a_k, b_k, e_k)\]

such that the following conditions hold:

\[
k \geq 0,
\]

\[
0 < a_1 < a_2 < \ldots < a_k \leq m,
\]

\[
0 < b_1 < b_2 < \ldots < b_k \leq n
\]

\[
e_i = \pm 1 \text{ for } 1 \leq i \leq k.
\]

Clearly the correspondence \((a_i, b_i, e_i) \leftrightarrow (b_i, a_i, e_i)\) induces a bijection from \( T(m, n) \) to \( T(n, m) \). It therefore suffices to describe a bijection between \( T(m, n) \) and \( S(m, n) \).

Define \( \phi : T(m, n) \rightarrow S(m, n) \) as follows:

\[
\phi((a_1, b_1, e_1), (a_2, b_2, e_2), \ldots, (a_k, b_k, e_k)) = (x_1, x_2, \ldots, x_n),
\]

where

\[
x_i = \begin{cases} 
    e_j(b_j - b_{j-1}) & \text{if } i = a_j \text{ (Note: } b_0 \text{ is taken to be } 0), \\
    0 & \text{if } i \neq a_i \text{ for any } j \text{ with } 1 \leq j \leq k.
\end{cases}
\]

Define \( \psi : S(m, n) \rightarrow T(m, n) \) by

\[
\psi(x_1, x_2, \ldots, x_n) = ((a_1, b_1, e_1), (a_2, b_2, e_2), \ldots, (a_k, b_k, e_k)),
\]

where \( k = |\{x_i : x_i \neq 0\}| \) and

\[
a_i = \text{the index of the } i\text{th nonzero element in } (x_1, x_2, \ldots, x_n),
\]

\[
b_i = \sum_{1 \leq j \leq a_i} |x_j|,
\]

\[
e_i = \text{sign}(x_{a_i})
\]

for \( i = 1, 2, \ldots, k \). It is easy to show that \( \phi \circ \psi \) and \( \psi \circ \phi \) are identity functions, so we have a bijection.

\textbf{B-5.} (4, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 54, 137)
Solution. Write $P = \sum_k P_k$, where $P_k$ is homogeneous of degree $k$. If $P$ has the required properties, then each of the $P_k$ does. Thus we may assume that $P$ is homogeneous, say of degree $d$. Put $S = x_1^2 + \cdots + x_n^2$. The ring $\mathbb{R}[x_1, \ldots, x_n]$ is a unique factorization domain, and $S$ is irreducible (for otherwise $S$ would factor into linear factors, and then it would have nontrivial zeros in $\mathbb{R}^n$, which it doesn’t). Thus if $P \neq 0$, then there are a unique polynomial $Q$ in $\mathbb{R}[x_1, \ldots, x_n]$ and a unique $m$ such that $P = S^m Q$. By hypothesis (b) in the problem, $m > 0$. We now compute the second partial derivatives of $P$. First

$$\frac{\partial P}{\partial x_i} = mS^{m-1} \frac{\partial S}{\partial x_i} Q + S^m \frac{\partial Q}{\partial x_i}.$$ 

Taking the partial derivative with respect to $x_i$ again, we find that

$$\frac{\partial^2 P}{\partial x_i^2} = m(m - 1)S^{m-2} \left( \frac{\partial S}{\partial x_i} \right)^2 Q + mS^{m-1} \frac{\partial^2 S}{\partial x_i^2} Q + 2mS^{m-1} \frac{\partial S}{\partial x_i} \frac{\partial Q}{\partial x_i} + S^m \frac{\partial^2 Q}{\partial x_i^2}.$$ 

But $\partial S/\partial x_i = 2x_i$ and $\partial^2 S/\partial x_i^2 = 2$, so this reduces to

$$4m(m - 1)S^{m-2} x_i^2 Q + 2mS^{m-1} Q + 4mS^{m-1} x_i \frac{\partial Q}{\partial x_i} + S^m \frac{\partial^2 Q}{\partial x_i^2}. \quad (1)$$ 

Now $Q$ is homogeneous of degree $d - 2m$. From an identity of Euler it follows that

$$\sum x_i \frac{\partial Q}{\partial x_i} = (d - 2m)Q.$$ 

Thus when we sum the expression (1) over $i$ we discover that

$$0 = 4m(m - 1)S^{m-1} Q + 2mnS^{m-1} Q + 4m(d - 2m)S^{m-1} Q + S^m \sum \frac{\partial^2 Q}{\partial x_i^2}.$$ 

On cancelling $S^{m-1}$ from both sides, we deduce that

$$-2m(2m - 2 + n + 2d - 4m)Q = S \sum \frac{\partial^2 Q}{\partial x_i^2}.$$ 

Here the constant on the left side is positive, so it follows that $S$ divides $Q$, contrary to the assumption that $m$ is the highest power of $S$ in $P$.

B-6. (18, 0, 1, 0, 0, 0, 0, 0, 2, 1, 41, 133)

Solution. We sum over the number $v(\pi)$:

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{v(\pi)} + 1 = \sum_{i=0}^{n} \frac{1}{i + 1} \sum_{\pi \in S_n, v(\pi) = i} \sigma(\pi)$$

$$= \sum_{i=1}^{n} \frac{1}{i + 1} \sum_{\pi \in S_{n-i}, v(\pi) = 0} \binom{n}{i} \sigma(\pi)$$

$$= \sum_{i=1}^{n} \frac{1}{i + 1} \binom{n}{i} \sum_{\pi \in S_{n-i}, v(\pi) = 0} \sigma(\pi).$$
Now we claim that \( D_n = \sum_{\pi \in S_n, \nu(\pi) = 0} \sigma(\pi) \) is given recursively by
\[
D_n = -(n-1)(D_{n-1} + D_{n-2}).
\]

To see this, consider \( \pi \) in \( S_n \) with \( \nu(\pi) = 0 \). Either \( \pi^2(n) = n \) or \( \pi^2(n) \neq n \). If \( \pi^2(n) = n \), then \( (n, \pi(n)) \circ \pi \) leaves \( n \) and \( \pi(n) \) fixed, so can be identified with a derangement of \( S_{n-2} \). Otherwise, \( (n, \pi(n)) \circ \pi \) leaves \( n \) fixed, and can be identified with a derangement of \( S_{n-1} \). There are \( n-1 \) choices for \( \pi(n) \) and \( \sigma((n, \pi(n)) \circ \pi)) = -\sigma(\pi) \), which establishes the recurrence relation.

It is clear that \( D_2 = -1 \) and \( D_3 = 2 \), and we claim that \( D_n = (-1)^{n-1}(n-1) \). This is true for \( n = 2 \) and \( n = 3 \) and therefore by the recursion formula and the inductive hypothesis
\[
D_n = -(n-1) \left( (-1)^{n-2}(n-2) + (-1)^{n-3}(n-3) \right)
\]
\[
= (-1)^{n-1}(n-1)((n-2) - (n-3)) = (-1)^{n-1}(n-1).
\]

Then, noting that \( D_0 = 1 \) and \( D_1 = 0 \) (which also fit the formula), we obtain
\[
\sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi) + 1} = \sum_{i=1}^{n} \frac{1}{1+i} \binom{n}{i}(-1)^{n-i-1}(n-i-1)
\]
\[
= (-1)^n \left( \sum_{i=0}^{n} \frac{1}{1+i} \binom{n}{i}(-1)^{i+1}n - \sum_{i=0}^{n} \frac{1}{1+i} \binom{n}{i}(-1)^i(1+i) \right)
\]
\[
= (-1)^n \sum_{i=0}^{n} \frac{1}{1+i} \binom{n}{i}(-1)^{i+1}n
\]
\[
= (-1)^n \sum_{i=0}^{n} \frac{1}{n+1+i+1}(-1)^{i+1}
\]
\[
= (-1)^n \frac{n}{n+1} \left( \sum_{i=-1}^{n} \frac{n+1}{i+1}(-1)^{1+i} - \binom{n+1}{0} \right)
\]
\[
= (-1)^{n+1} \frac{n}{n+1}.
\]

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