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REALIZING DEGREE SEQUENCES WITH GRAPHS HAVING NOWHERE-ZERO 3-FLOWS*

RONG LUO†, RUI XU‡, WENAN ZANG§, AND CUN-QUAN ZHANG¶

Abstract. The following open problem was proposed by Archdeacon: Characterize all graphical sequences \( \pi \) such that some realization of \( \pi \) admits a nowhere-zero 3-flow. The purpose of this paper is to resolve this problem and present a complete characterization: A graphical sequence \( \pi = (d_1, d_2, \ldots, d_n) \) with minimum degree at least two has a realization that admits a nowhere-zero 3-flow if and only if \( \pi \neq (3^k, 2), (k, 3^k), (k^2, 3^{k-1}) \), where \( k \) is an odd integer.

Key words. degree sequence, graph, integer flow, characterization

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1. Introduction. Let \( G = (V, E) \) be a graph and let \( k \) be a positive integer. An ordered pair \( (D, \phi) \) is called a \( k \)-flow of \( G \) if \( D = (V, A) \) is an orientation of \( G \) and \( \phi : A \rightarrow Z_k \) is an assignment of flows such that, for every vertex \( v \),

\[
\sum_{e \in E^+(v)} \phi(e) \equiv \sum_{e \in E^-(v)} \phi(e) \pmod{k},
\]

where \( Z_k \) is the set \( Z/kZ \) of integers modulo \( k \), and \( E^+(v) \) (resp., \( E^-(v) \)) is the set of all arcs in \( A \) with tail \( v \) (resp., head \( v \)). We say that \( (D, \phi) \) is a nowhere-zero flow if \( \phi(e) \neq 0 \) for any \( e \in A \). This concept was introduced by Tutte [19], and the theory of nowhere-zero flows provides an interesting way to generalize theorems about region-coloring planar graphs to general graphs; major open problems in this area are Tutte’s celebrated 3-, 4-, and 5-flow conjectures. Interested readers are referred to Jaeger [8] and Seymour [17] for the main ideas of this subject and to Tutte [20] and Zhang [21] for in-depth accounts.

An integer-valued sequence \( \pi = (d_1, d_2, \ldots, d_n) \) is called graphical if there is a simple graph \( G \) so that the degree sequence of \( G \) is exactly the same as \( \pi \); such a graph \( G \) is called a realization of \( \pi \). For simplicity, we shall also write a graphical sequence in terms of multiplicities, for instance, \( (6, 4, 4, 3, 3, 3, 3) = (6, 4^2, 3^4) \). The problem of realizing degree sequences with graphs enjoying certain properties has been a subject of extensive research. Recently a surprising application of graph realization with 4-flows has been found in the design of critical partial Latin squares [5, 15], which leads to the proof [14] of the so-called simultaneous edge-coloring conjecture.

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by Keedwell [10, 11] and Cameron [2]. In this paper we study a closely related open problem proposed by Archdeacon.

**Problem 1.1** (see [1]). Characterize all graphical sequences \( \pi \) such that some realization of \( \pi \) admits a nowhere-zero 3-flow.

Our objective is to resolve this problem and to present a complete characterization.

**Theorem 1.2.** A graphical sequence \( \pi = (d_1, d_2, \ldots, d_n) \) with minimum degree at least two has a realization that admits a nowhere-zero 3-flow if and only if \( \pi \neq (3^k, 2), (k, 3^k), (k^2, 3^k) \), where \( k \) is an odd integer.

It is worthwhile pointing out that the most striking difference between [14] and the present paper is not the flow number but the proof technique. We say that a graph \( H \) is a \( k \)-flow contractible configuration if for every graph \( G \) containing \( H \) as a subgraph, \( G \) admits a nowhere-zero \( k \)-flow if and only if so does \( G/H \). Actually flow contractible configurations play important roles in both papers: For the 4-flow problem, every circuit of length at most four (cf. Seymour [16] and Catlin [3]) is contractible. The situation, however, becomes much more complicated for the 3-flow problem, and the digon is the only circuit that is 3-flow contractible [16]. So we have to appeal to other 3-flow contractible configurations. To be more precise, a graph \( G = (V, E) \) is \( k \)-connected [9] if for every \( b : V \to Z_k \) with \( \sum_{v \in V} b(v) \equiv 0 \) (mod \( k \)), there exist an orientation \( D = (V, A) \) of \( G \) and an assignment \( \phi : A \to \{1, 2, \ldots, k\} \) such that for every vertex \( v \),

\[
\sum_{e \in E^+(v)} \phi(e) - \sum_{e \in E^-(v)} \phi(e) \equiv b(v) \pmod{k}.
\]

As shown in [4, 13], \( Z_3 \)-connected graphs contain even wheels, triangularly connected graphs with a family of well-described exceptions, etc.; it is \( Z_3 \)-connected graphs that will serve as contractible configurations in our proof.

The remainder of this paper is organized as follows. In section 2, we exhibit some basic properties concerning graphical sequences and \( Z_3 \)-connectivity. In section 3, we describe some graphical sequences with \( Z_3 \)-connected realizations. In section 4, we characterize certain graphical sequences \( \pi \) such that some realization of \( \pi \) admits a nowhere-zero 3-flow and contains nontrivial \( Z_3 \)-connected subgraphs. In section 5, we present a proof of the main theorem (Theorem 1.2), which fully characterizes all graphical sequences that can be realized to admit nowhere-zero 3-flows.

We remark that a complete characterization of graphic sequences with \( Z_3 \)-connected realizations remains an interesting problem for further study.

**2. Preliminaries.** Let \( \pi = (d_1, d_2, \ldots, d_n) \) be a graphical sequence with \( d_1 \geq d_2 \geq \cdots \geq d_n \). Throughout we reserve the symbol \( \bar{\pi} \) for the sequence \((d_1 - 1, d_2 - 1, \ldots, d_{n-1} - 1, d_n, d_{n+1}, \ldots, d_{n-1})\), which is called the residual sequence obtained from \( \pi \) by laying off \( d_n \). We shall frequently use the following well-known results in our proof.

**Lemma 2.1.** Let \( \pi = (d_1, d_2, \ldots, d_n) \) be a sequence. Then

(a) \( \sum_{i=1}^{n} d_i \) is even if \( \pi \) is graphical;
(b) (Hakimi [6, 7]; Kleitman and Wang [12]) \( \pi \) is graphical if and only if so is \( \bar{\pi} \).

**Lemma 2.2** (Tutte [18]). A cubic graph admits a nowhere-zero 3-flow if and only if it is bipartite.

Lemma 2.2 can be further generalized in the following way.

**Lemma 2.3.** If a graph \( G \) admits a nowhere-zero 3-flow, then the subgraph of \( G \) induced by all degree-three vertices is bipartite.
If \( H \) is a connected subgraph of a graph \( G \), then \( G \) contracted by \( H \), denoted by \( G/H \), is the graph obtained from \( G \) by deleting all edges in \( H \) and then identifying \( V(H) \) into a single vertex. The following simple observations follow instantly from the definition of \( Z_k \)-connectivity.

**Lemma 2.4** (Jaeger [8]; Seymour [16]). Every circuit of length at most \( k - 1 \) is \( Z_k \)-connected.

**Lemma 2.5** (DeVos, Xu, and Yu [4]; Lai, Xu, and Zhang [13]). Let \( H \) be a \( Z_3 \)-connected subgraph of a graph \( G \).

(a) If \( G/H \) admits a nowhere-zero 3-flow, then so does \( G \).

(b) If \( G/H \) is \( Z_3 \)-connected, then so is \( G \).

In our proof these facts enable us to work on a reduced graph after a series of contractions of \( Z_3 \)-connected subgraphs. The reduced graphs/graphical sequences often enjoy much nicer properties than the original ones, and therefore are much easier to manipulate.

The following two lemmas are immediate corollaries of Lemmas 2.5 and 2.4.

**Lemma 2.6.** Let \( G \) be a \( Z_3 \)-connected graph, and let \( G' \) be obtained from \( G \) by adding a new vertex \( v \) and making it adjacent to at least two vertices of \( G \). Then \( G' \) is \( Z_3 \)-connected.

**Lemma 2.7.** Let \( G = (V, E) \) be a \( Z_3 \)-connected graph. Then for any \( u, v \in V \) with \( uv \notin E \), the graph obtained from \( G \) by adding an edge \( uv \) is \( Z_3 \)-connected.

In this paper we shall use \( G_{uv,uw} \) to stand for the graph \( G \cup \{uv\} \setminus \{uw, uv\} \).

**Lemma 2.8.** Let \( G = (V, E) \) be a graph, and let \( u, v, w \) be three vertices of \( G \) with \( uv, uw \in E \). If \( G_{uv,uw} \) is \( Z_3 \)-connected, then so is \( G \).

A graph \( G = (V, E) \) is triangularly connected if for every \( e, f \in E \) there exists a sequence of circuits \( C_1, C_2, \ldots, C_k \) such that \( e \in E(C_1) \), \( f \in E(C_k) \), and \( |E(C_i)| \leq 3 \) for \( 1 \leq i \leq k \), and \( E(C_j) \cap E(C_{j+1}) \neq \emptyset \) for \( 1 \leq j \leq k - 1 \). A wheel \( W_k \) is the graph obtained from a \( k \)-circuit \( C \) by adding a vertex \( v \) and then making it adjacent to all vertices on \( C \). By convention, we call \( v \) the hub and \( C \) the rim of \( W_k \). We also call \( W_k \) odd if \( k \) is odd and even otherwise.

The following lemma gives sufficient conditions for a graph to be \( Z_3 \)-connected.

**Lemma 2.9** (DeVos, Xu, and Yu [4]). Let \( G \) be a triangularly connected graph. Then \( G \) is \( Z_3 \)-connected, provided that one of the following conditions is satisfied:

(a) \( G \) contains a nontrivial \( Z_3 \)-connected subgraph;

(b) the minimum degree of \( G \) is at least four;

(c) \( G \) is an even wheel \( W_k \), with \( k \geq 4 \).

The following lemma establishes the “only if” part of our main theorem (Theorem 1.2).

**Lemma 2.10.** Let \( k \) be an odd integer. Then no realization of the graphical sequences \( (3^4, 2) \), \( (k, 3^k) \), \( (k^2, 3^{k-1}) \) admits a nowhere-zero 3-flow.

**Proof.** Observe the following:

- the only realization \( G_1 \) of \( (3^4, 2) \) is the graph obtained from \( K_4 \) (the complete graph with four vertices) by subdividing an edge precisely once;

- a realization \( G_2 \) of \( (k, 3^k) \) is either an odd wheel or several wheels sharing the hub, and at least one of these wheels is odd;

- the only realization \( G_3 \) of \( (k^2, 3^{k-1}) \) is \( \frac{k-1}{2} \) copies of \( K_4 \) sharing a common edge. (To justify this, let \( u \) and \( v \) be the two vertices of maximum degree. Then the subgraph obtained from \( G_3 \) by deleting \( u \) and \( v \) is 1-regular.)

By Lemma 2.3, neither \( G_1 \) nor \( G_2 \) admits a nowhere-zero 3-flow. To prove the statement concerning sequence \( (k^2, 3^{k-1}) \), let \( Q_1 \) be a copy of \( K_4 \) with vertex
set \( \{w_i, x_i, y_i, z_i\} \) for \( i = 1, 2, \ldots, \frac{k+1}{2} \). As described above, \( G_3 \) is obtained from \( Q_1, Q_2, \ldots, Q_{k-1} \) by first identifying all \( y_i \) as a single vertex \( y \) and all \( z_i \) as a single vertex \( z \), and then replacing all edges \( y, z \) with a single edge \( yz \). Assume to the contrary that \( G_3 \) admits a nowhere-zero 3-flow \((D, f)\). Then there must exist a 3-flow \((D, f_i)\) of each \( Q_i \) such that

- \( f(e) = f_i(e) \) for every edge \( e \in Q_i \) \( \{yz\} \) (that is, \( \text{supp}(f_i) \supseteq E(Q_i) \setminus \{yz\} \)), and
- \( f = \sum_{i=1}^{\frac{k+1}{2}} f_i \), which implies the existence of a subscript \( i \) such that \( f_i \) is nowhere-zero on \( Q_i \) \( (= K_4) \), contradicting Lemma 2.2. \( \square \)

### 3. \( Z_3 \)-connected realizations

The purpose of this section is to establish the following two theorems, which give some sufficient conditions for \( Z_3 \)-connected realizations.

**Theorem 3.1.** Let \( \pi = (d_1, d_2, \ldots, d_n) \) be a graphical sequence with \( d_1 \geq d_2 \geq \cdots \geq d_n \). If \( d_n \geq 3 \) and \( d_{n-3} \geq 4 \), then \( \pi \) has a \( Z_3 \)-connected realization.

**Theorem 3.2.** Let \( \pi = (d_1, d_2, \ldots, d_n) \) be a graphical sequence with \( n-1 = d_1 \geq d_2 \geq \cdots \geq d_n \geq 3 \). Then \( \pi \) has a \( Z_3 \)-connected realization if and only if \( \pi \neq (k, 3^k) \), \((k^2, 3^{k-1})\), where \( k \) is odd.

Let us establish a weaker version of Theorem 3.1 before presenting a proof.

**Lemma 3.3.** Let \( \pi = (d_1, d_2, \ldots, d_n) \) be a graphical sequence with \( d_1 \geq d_2 \geq \cdots \geq d_n \). If \( d_n \geq 3 \) and \( d_{n-2} \geq 4 \), then \( \pi \) has a \( Z_3 \)-connected realization.

**Proof.** Suppose the contrary: \( \pi = (d_1, d_2, \ldots, d_n) \) is a counterexample with smallest \( n \). According to the configuration of \( \pi \), we propose to consider four cases and construct a \( Z_3 \)-connected realization of \( \pi \) in each case, thereby reaching a contradiction. Notice that \( n \geq 5 \) as \( n-1 \geq d_1 \geq d_{n-2} \geq 4 \).

The lemma is to be proved step by step with the following observations and claims.

1. \( d_1 \geq 4 \). Assume the contrary: \( d_1 = 4 \). Let us consider the following two cases.
   - **Case 1.** \( d_1 = 4 \) and \( d_n = 4 \). In this case \( \pi = (4^n) \), the construction goes as follows: Let \( C \) be a circuit with \( n \) vertices \( v_1, v_2, \ldots, v_n \), and let \( G^1 = C \cup \bigcup_{i=1}^{n} \{v_i v_{i+2}\} \), where \( v_{n+1} = v_1 \) and \( v_{n+2} = v_2 \). Clearly, \( G^1 \) is 4-regular and is trianularly connected. By Lemma 2.9(b), \( G^1 \) is \( Z_3 \)-connected.
   - **Case 2.** \( d_1 = 4 \) and \( d_n = 3 \). Since \( d_1 = d_{n-2} = 4 \) and \( d_n = 3 \), by Lemma 2.1(a) we have \( \pi = (4^{n-2}, 3^2) \). Let \( G^1 \) be the graph constructed in Case 1, and let \( G^2 = G^1 \setminus \{v_2 v_n\} \). Clearly \( G^2 \) is a realization of \( \pi \). It remains to show that \( G^2 \) is \( Z_3 \)-connected.
     
     From the construction of \( G^1 \) and \( G^2 \), we see that \( G^2_{[v_1 v_2, v_1 v_3]} \) is trianularly connected and contains a 2-circuit \( v_2 v_3 v_2 \). Since any 2-circuit is \( Z_3 \)-connected (by Lemma 2.4), Lemma 2.9(a) implies that \( G^2_{[v_0 v_1, v_0 v_2]} \) is \( Z_3 \)-connected, and hence so is \( G^2 \) by Lemma 2.8. Therefore (1) holds.

2. \( d_2 \geq 4 \). Suppose to the contrary that \( d_2 < 5 \). By (1), we have \( d_1 \geq 5 \) and \( d_2 \geq 5 \).

Since \( n \geq 5 \) and \( d_{n-2} \geq 4 \), we get \( d_3 \geq 4 \). Hence it can be seen that the residual sequence \( \tilde{\pi} = (d_1, d_2, \ldots, d_{n-1}) \), with \( d_1 \geq d_2 \geq \cdots \geq d_{n-1} \) satisfies \( d_{n-1} \geq 3 \) and \( d_{n-3} \geq 4 \). Thus Lemma 2.1(b) and the assumption on \( \pi \) guarantee the existence of a \( Z_3 \)-connected realization \( \tilde{G} \) of \( \tilde{\pi} \). We can then get a realization \( G^3 \) of \( \pi \) from \( \tilde{G} \) by adding a new vertex \( v \) and \( d_n \) edges joining \( v \) to the corresponding vertices in \( \tilde{G} \). By Lemma 2.6, \( G^3 \) is \( Z_3 \)-connected. This proves (2).

We claim that (3) \( d_n = 3 \). Otherwise, \( d_n = 4 \). By (1) and (2), we have \( d_1 \geq 5 \) and \( d_2 = 4 \). So \( \pi = (d_1, 4, \ldots, 4) \). By Lemma 2.1(a), \( d_1 \) is even. Thus \( d_1 \geq 6 \).
Set \( k = \frac{d_1 - 4}{2} \). Let \( G^{i} \) be the 4-regular triangularly connected graph exhibited in Case 1 of (1). For each \( 1 \leq i \leq k \), we subdivide the edge \( v_{2i+2}v_{2i+3} \) once by a degree-two vertex \( u_i \). Since \( n - 1 \geq d_1 = 2k + 4 \), we have \( 2k + 3 \leq n - 2 \). Now let us identify all \( u_i \) with \( v_1 \). Then the resulting graph \( G^4 \) is simple and is clearly a realization of \( \pi \). To show that \( G^{4} \) is \( Z_3 \)-connected, we replace the path \( v_{2i+2}v_1v_{2i+3} \) with an edge \( v_{2i+2}v_{2i+3} \) for all \( 1 \leq i \leq k \); the resulting graph is precisely \( G^1 \). Since \( G^1 \) is \( Z_3 \)-connected, repeated applications of Lemma 2.8 imply that so is \( G^4 \). Thus (3) follows.

By (1), (2), and (3), we have \( d_1 \geq 5, \ d_2 = 4, \) and \( d_n = 3 \). So \( \pi \) is either \((d_1, 4, 4, 4, 3, 3)\) or \((d_1, 4, 4, 3, 3)\).

If \( d_{n-1} = 4 \), then the residual sequence \( \tilde{\pi} \) satisfies the conditions of the theorem, so it admits a \( Z_3 \)-connected realization \( \tilde{G} \), and hence so does \( G \). It remains to consider the case when \( d_{n-1} = 3 \). Since \( d_2 = d_3 = \cdots = d_{n-2} = 4 \) and \( d_{n-1} = d_n = 3 \), we see that \( d_1 \) is even. So \( d_1 \geq 6 \). Set \( k = \frac{d_1 - 4}{2} \). Let \( G^2 \) be the triangularly connected graph constructed in Case 2 of (1). For each \( 1 \leq i \leq k \), we subdivide the edge \( v_{2i+2}v_{2i+3} \) once by a degree-two vertex \( u_i \). Since \( n - 1 \geq d_n = 2k + 4 \), we have \( 2k + 3 \leq n - 2 \). Now let us identify \( u_i \) and \( v_1 \) for each \( 1 \leq i \leq k \). Then the resulting graph \( G^5 \) is simple and is clearly a realization of \( \pi \). To show that \( G^5 \) is \( Z_3 \)-connected, we replace the path \( v_{2i+2}v_1v_{2i+3} \) with an edge \( v_{2i+2}v_{2i+3} \) for all \( 1 \leq i \leq k \); then the resulting graph is precisely \( G^2 \). Since \( G^2 \) is \( Z_3 \)-connected, by Lemma 2.8 so is \( G^5 \), completing the proof of the lemma.

**Proof of Theorem 3.1.** Suppose the contrary: \( \pi = (d_1, d_2, \ldots, d_n) \) is a counterexample with smallest \( n \). Notice that \( n \geq 5 \) as \( n - 1 \geq d_1 \geq d_{n-3} \geq 4 \). By Lemma 3.3, we have

1. \( d_{n-2} = 3 \).
2. \( d_2 = 4 \). Otherwise, \( d_2 \geq 5 \) for \( d_2 \geq d_{n-3} \geq 4 \). So the residual sequence \( \tilde{\pi} \) satisfies the conditions of the theorem, and hence the assumption on \( \pi \) guarantees the existence of a \( Z_3 \)-connected realization \( \tilde{G} \) of \( \tilde{\pi} \). By Lemma 2.6, we can get a \( Z_3 \)-connected realization \( G \) of \( \pi \) from \( \tilde{G} \) by adding a new vertex \( v \) and \( d_n \) edges joining \( v \) to the corresponding vertices in \( \tilde{G} \). This contradiction implies (2).

Combining (1), (2), and the hypothesis of the theorem, we get

3. \( \pi = (d_1, 4^{n-4}, 3^3) \). So \( d_1 \) is odd by Lemma 2.1(a) and hence at least 5.

4. \( n \geq 8 \). Suppose to the contrary that \( n \leq 7 \). Since \( d_1 \geq 5 \) and is odd by (3), we have \( d_1 = 5 \) and \( n \geq 6 \). So \( \pi = (5, 4^{n-4}, 3^3) \).

For \( n = 6 \), let \( G \) be the graph obtained from the complete bipartite graph \( K_{2,3} \) by adding a new vertex and then making it adjacent to each vertex in \( K_{2,3} \). Clearly each edge of \( G \) is contained in a wheel \( W_4 \). Since \( W_4 \) is \( Z_3 \)-connected by Lemma 2.9(c), so is \( G \) by Lemma 2.5(b).

For \( n = 7 \), let \( G \) be the graph obtained from \( W_4 \) by adding two adjacent vertices \( v_1 \) and \( v_2 \) and then making \( v_1 \) adjacent to the hub of \( W_4 \) and a rim vertex and making \( v_2 \) adjacent to two other rim vertices. Since both \( W_4 \) and the graph obtained from \( G \) by contracting \( W_4 \) (which results in a triangle with parallel edges) are \( Z_3 \)-connected, so is \( G \) by Lemma 2.5(b); this contradiction establishes (4).

Let us distinguish between two cases according to the value of \( d_1 \).

**Case 1.** \( d_1 \geq n - 3 \). Set \( k = \frac{d_1 - 5}{2} \geq 0 \) and \( m = \left\lfloor \frac{n-4}{2} \right\rfloor \). Take a wheel \( W_{n-4} \) with hub \( w \) and rim \( u_1u_2u_3 \ldots u_{n-4}u_1 \). Let \( H \) be the graph obtained from this wheel by adding \( k \) edges \( u_iu_{i+m} \) for \( i = 1, 2, \ldots, k \). Then the degree sequence of \( H \) is \((n-4, 4^{k-5}, 3^{n-4-2k}) = (n-4, 4^{d_1-5}, 3^{n-d_1+1}) \). To get a graph \( G \) with \( n \) vertices and degree sequence \( \pi = (d_1, 4^{n-4}, 3^3) \), we need to add three vertices and...
\([d_1 + 4(n-4) + 9 - (n-4) - 4(d_1 - 5) - 3(n - d_1 + 1)]/2 = 7\) edges to \(H\). The construction of \(G\) goes as follows: We first add a path \(P = v_1v_2v_3\) to \(H\), then we connect \(w\) and \(d_1 - (n-4)\) vertices on \(P\), and finally add precisely one edge between each of \(n - d_1 + 1\) degree-three vertices on \(H\) and \(P\), so that there are precisely two edges between each of \(v_1\) and \(v_3\) and \(H\), and there is precisely one edge between \(v_2\) and \(H\).

By (4), \(n \geq 8\). Note that if \(n = 8\), then \(H = W_4\); if \(n \geq 9\), then at least one edge is added to \(W_{n-4}\), which implies that \(H\) contains an even wheel. So, by Lemma 2.9(c), \(H\) contains a \(Z_3\)-connected subgraph (an even wheel) in either case. Clearly, \(G/H\) is triangularly connected and contains a 2-circuit. Since a 2-circuit is \(Z_3\)-connected, so is \(G/H\) by Lemma 2.9(a). It follows from Lemma 2.5(b) that \(G\) is \(Z_3\)-connected, a contradiction.

Case 2. \(d_1 \leq n - 4\). By (3), \(d_1 \geq 5\). So in this case \(n \geq d_1 + 4 \geq 9\). Let us consider the sequence \(\sigma = (d_1 - 1, 4^{n-7}, 3^2)\). From the construction of \(G^2\) and \(G^5\) of the proof of Lemma 3.3, we deduce that \(\sigma\) has a \(Z_3\)-connected realization \(H\); let \(v_1, v_2, v_3\) denote the vertices of \(H\) with degree three and degree \(d_1 - 1\), respectively. Let \(H^2\) be the graph obtained from \(H\) by first adding a complete graph with four vertices \(v_1, v_2, v_3, v_4\), then deleting edge \(v_1v_3\), and finally adding a matching of size three between \(\{v_1, v_2, u_3\}\) and \(\{v_1, v_2, v_3\}\). Clearly, \(G\) is a realization of \(\pi\). Note that \(G/H\) is a wheel \(W_5\) with hub \(v_2\), so by Lemma 2.9(c) it is \(Z_3\)-connected, and hence so is \(G\) by Lemma 2.5(b). This contradiction completes the proof of the theorem.

The proof of Theorem 3.2 is based on the following lemma.

**Lemma 3.4.** Let \(\pi = (d_1, d_2, \ldots, d_n)\) be a graphical sequence with \(d_1 \geq d_2 \geq \cdots \geq d_n \geq 2\). Then \(\pi\) has a connected realization \(G\) that contains an even circuit if and only if \(\pi \neq (2^n), (n-1, 2^{n-1})\), where \(n\) is odd.

**Proof.** It is easy to see that when \(n\) is odd

- the unique connected realization of \(\pi = (2^n)\) is an odd circuit;
- the unique connected realization of \(\pi = (n-1, 2^{n-1})\) is \(\frac{n-1}{2}\) triangles sharing a common vertex.

Clearly neither of these two graphs contains an even circuit, so the “only if” part is established.

Let us proceed to the “if” part. Assume that \(\pi \neq (2^n), (n-1, 2^{n-1})\), where \(n\) is odd, but no connected realization of \(\pi\) contains an even circuit. We further assume that \(\pi\) is chosen with minimum \(n\). Let us make some simple observations.

(1) \(n \geq 5\). By the assumption on \(\pi\), we have \(n \neq 3\). Hence \(n \geq 4\). If \(n = 4\), then \(\pi = (2^4)\) or \((3^2, 2^2)\) or \((3^4)\). In each case \(\pi\) has a connected realization that contains an even circuit. This contradiction yields (1).

(2) \(d_n = 2\). Otherwise, \(d_n \geq 3\). By (1), the residual sequence \(\pi\) satisfies the condition of the lemma. Hence it admits a connected realization \(H\) that contains an even circuit by the assumption on \(\pi\). We can then get a desired realization \(G\) of \(\pi\) from \(H\) by adding a new vertex and making it adjacent to corresponding vertices in \(H\), a contradiction. So (2) holds.

(3) \(d_2 \geq 3\). Otherwise, by (2) we have \(d_2 = 2\), and so \(\pi = (d_1, 2^{n-1})\), where \(d_1 = \geq 2k\) for some integer \(k \geq 1\). Let \(H\) be the graph obtained from \(k\) disjoint triangles and then gluing them at a common vertex \(v\). Note that the number of vertices in \(H\) is \(2k + 1\). According to the assumption on \(\pi\), we have \(n \neq 2k + 1\) and \(k \geq 2\). So \(2k + 1 \leq n - 1\). Let \(G\) be the graph obtained from \(H\) by inserting a degree-two vertex into the edge not containing \(v\) in the first triangle, and inserting the remaining degree-two vertices (if any) into the edge not containing \(v\) in the second triangle. Then \(G\) is a connected realization of \(\pi\) that contains an even circuit (of length four), a contradiction. So (3) is proved.
Consider the residual sequence $\bar{\pi}$ of $\pi$. By (3), we have $d_1 - 1 \geq d_2 - 1 \geq 2$. If $\bar{\pi}$ satisfies the conditions of the lemma, then the assumption on $\pi$ guarantees a connected realization $H$ of $\bar{\pi}$ that contains an even circuit. From $H$ we can obviously get a desired realization of $\pi$. This contradiction implies that $\bar{\pi} = (2^{n-1})$ or $(n-2, 2^{n-2})$, where $n-1$ is odd and thus $n$ is even.

If $\bar{\pi} = (2^{n-1})$, then $\pi = (3^2, 2^{n-2})$ by (2). We can get a desired realization of $\pi$ from an $n$-circuit by adding a chord.

If $\bar{\pi} = (n-2, 2^{n-2})$, then the unique connected realization $H$ of $\bar{\pi}$ is $\frac{n-2}{2}$ triangles sharing a common vertex $v$. Note that $\pi = (n-1, 3, 2^{n-2})$ by (2). Let $G$ be the graph obtained from $H$ by adding a new vertex and making it adjacent to $v$ and one other vertex. Clearly, $G$ is a connected realization of $\pi$ and contains an even circuit. This contradiction completes the proof.

**Proof of Theorem 3.2.** The “only if” part follows instantly from Lemma 2.10. It remains to show the “if” part.

Consider the sequence $\sigma = (d_2 - 1, d_3 - 1, \ldots, d_n - 1)$. Note that $\sigma \neq (2^{n-1}), (n-2, 2^{n-2})$, where $n-1$ is odd, for otherwise $\pi = (n-1, 3^{n-1})$ or $(n-1)^2, 3^{n-2})$, contradicting the hypothesis on $\pi$. By Lemma 3.4, $\sigma$ has a connected realization $H$ that contains an even circuit. Let $G$ be the graph obtained from $H$ by adding a new vertex and making it adjacent to each vertex of $H$. Clearly $G$ is a realization of $\pi$. Since $G$ is triangularly connected and contains an even wheel, from Lemma 2.9 we deduce that $G$ is $Z_3$-connected.

4. **Partially $Z_3$-connected realizations.** We propose to establish the following two theorems in this section.

**Theorem 4.1.** Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 3$ and $d_3 \geq 5$. Then $\pi$ has a realization $G$ such that

(a) $G$ admits a nowhere-zero 3-flow; and

(b) $G$ has a $Z_3$-connected subgraph $H$ that contains all vertices of $G$ with degree at least four.

**Theorem 4.2.** Let $\pi = (d_1, d_2, 4^{n-k-2}, 3^k)$ be a graphical sequence with $n - 2 \geq d_1 \geq d_2 \geq 4, d_1 + d_2 \geq 11, n - 3 \geq k \geq 4$, and $n \geq 9$. Then $\pi$ has a realization $G$ such that

(a) $G$ admits a nowhere-zero 3-flow; and

(b) $G$ has a $Z_3$-connected subgraph $H$ that contains all vertices of $G$ with degree at least four.

Let us introduce three operations before proving these theorems, which will be used frequently in our proofs.

Let $H_1$ and $H_2$ be two disjoint graphs. A graph $G$ obtained by adding $H_2$ onto $H_1$ via Operation A, B, or C is defined below.

Operation A. Let $u_i v_i$ for $i = 1, 2, \ldots, k$ be $k$ edges of $H_2$. The graph $G$ is obtained from the union of $H_1$ and $H_2$ by first cutting each $u_i v_i$ into two edges $u_i x_i$ and $y_i v_i$ and then identifying each of $x_i$ and $y_i$ with a vertex of $H_1$.

Operation B. Let $u_i v_i$ for $i = 1, 2, \ldots, k$ be $k$ edges of $H_2$. The graph $G$ is obtained from the union of $H_1$ and $H_2$ by inserting a degree-two vertex $x_i$ into each $u_i v_i$ and then identifying each $x_i$ with a vertex in $H_1$.

Operation C. Let $u$ be a vertex in $H_2$ with $d(u) = k \geq 2$, and let $u_1, u_2, \ldots, u_t$ be $t$ neighbors of $u$. The graph $G$ is obtained from the union of $H_1$ and $H_2$ by splitting $u$ into $t + 1$ vertices $u_1', u_2', \ldots, u_t', u'$ such that $u_i$ is the only neighbor of $u_i'$ and that $d(u') = d(u) - t$, and then identifying each of those $t + 1$ vertices with a vertex in $H_1$. 


Lemma 4.3. Let $H_1$ be a $Z_3$-connected graph, and let $H_2$ be a cubic bipartite graph with at least four vertices. ($H_2$ is simple if it has at least six vertices or contains precisely two 2-circuits otherwise.) Then the graph $G$ obtained by adding $H_2$ onto $H_1$ via Operation A, B, and/or C admits a nowhere-zero 3-flow.

Remark. Obviously, the new graph $G$ obtained via Operation C is simple as long as $H_1$ and $H_2$ are simple. Let $H_3$ be the graph obtained from $H_2$ by cutting all $u_iv_i$. Then the new graph $G$ obtained via Operation A is simple if, first, $H_3$ and $H_1$ are simple; second, the edges in $H_3$ joining the same vertex of $H_1$ form a matching in $H_3$. If $G$ is obtained via Operation B, then $G$ is simple if both $H_1$ and the graph obtained from $H_2$ by inserting a new degree-two vertex into each edge $u_iv_i$ are simple and the edges $u_iv_i$ form a matching in $H_2$.

Proof. Note that $H_1$ remains intact in the new graph $G$, so it is still $Z_3$-connected (as a subgraph of $G$). By Lemma 2.5(a), $G$ admits a nowhere-zero 3-flow if and only if $G/H_1$ admits a nowhere-zero 3-flow.

Since $H_2$ is a cubic bipartite graph, by Lemma 2.2, it admits a nowhere-zero 3-flow. Furthermore, $G/H_1 = H_2$ if only Operation C is applied, and if Operation A or B is applied, then $G/H_1$ can be obtained from $H_2$ by subdividing some edges once and then identifying the new degree-two vertices as one vertex. Therefore, $G/H_1$ also admits a nowhere-zero 3-flow.

Lemma 4.4. Let $\pi = (d_1, d_2, 4^{n-4}, 3^2)$ be a sequence with $n - 1 \geq d_1 \geq d_2 \geq 4$ and $n \geq 5$. Then $\pi$ is graphical, provided that $d_1 + d_2$ is even.

Proof. Assume the contrary: $\pi$ is a counterexample with minimum $n$. Then $n \geq 6$, for otherwise $n = 5$, so $\pi = (4, 3^2)$, and thus the graph obtained from $K_5$ by deleting one edge is a realization of $\pi$, a contradiction.

If $d_2 \geq 5$, then the residual sequence $\bar{\pi} = (d_1 - 1, d_2 - 1, 4^{n-5}, 3^2)$. From the assumption on $\pi$, we see that $\bar{\pi}$ is graphical and hence so is $\pi$, by Lemma 2.1(b); this contradiction yields $d_2 = 4$. Therefore $\pi = (d_1, 4^{n-3}, 3^2)$.

Since $d_1 + d_2 = d_1 + 4$ is even, so is $d_1$. It follows that the graph $G^2$ (resp., $G^5$) in the proof of Lemma 3.3 is a realization of $\pi$ if $d_1 = 4$ (resp., $d_1 \geq 6$).

Lemma 4.5. Let $\pi = (d_1, d_2, 5, 4^{n-3-k}, 3^k)$ be a graphical sequence with $n - 2 \geq d_1 \geq d_2 \geq 5$, $n - 3 \geq k \geq 5$, and $n \geq 9$. Then $\pi$ has a realization $G$ such that

(a) $G$ admits a nowhere-zero 3-flow;
(b) $G$ has a $Z_3$-connected subgraph $H$ that contains all vertices of $G$ with degree at least four.

Proof. Let us distinguish between two cases according to the parity of $k$.

Case 1. $k$ is even. Since $k \geq 5$ and since the degree sum of $\pi$ is even, in this case we have

(1) $k \geq 6$, and $d_1 + d_2$ is odd.

We propose to construct a realization $G$ of $\pi$ with properties (a) and (b) using Lemma 4.3, such that the $Z_3$-connected graph $H_1$ (recall Lemma 4.3) has degree sequence $\pi^* = (d_1^*, d_2^*, 4^{n-2-k-1}, 3^k)$, where $d_1^*$ and $d_2^*$ are to be determined, and the cubic bipartite graph $H_2$ has $k - 2$ vertices.

We are to determine $d_1^*$ and $d_2^*$ by letting $A_i = \max\{4, d_i - (k - 2)\}$ and $B_i = \min\{n - k + 1, d_i - 1\}$ for $i = 1, 2$. Then

(2) $A_i \leq B_i$, and equality holds if and only if $A_i = B_i = 4$. To verify this, note that $4 \leq n - k + 1$ (because $k \leq n - 3$), $d_i - (k - 2) < d_i - 1$ (because $6 \leq k$), $4 \leq d_i - 1$ (because $5 \leq d_i$), and $d_i - (k - 2) < n - k + 1$ (because $d_i \leq n - 2$). Combining these inequalities yields $A_i \leq B_i$ and $d_i - (k - 2) < B_i$. It follows that $A_i = B_i$ if and only if both of them are four. So (2) is true.

Clearly (2) guarantees the existence of $d_1^*$ and $d_2^*$ such that
\begin{itemize}
\item $A_i \leq d_i^* \leq B_i$ for $i = 1, 2$;
\item $d_1^* + d_2^*$ is even.
\end{itemize}

By Lemma 4.4, the sequence $\pi^\ast = (d_1^*, d_2^*, 4^{n-2-k}, 3^2)$ is graphical, and hence by Theorem 3.1 it admits a $Z_3$-connected realization $H_1$. By the definitions of $A_i$, $B_i$, and $d_i^*$, we have $d_i - (k - 2) \leq d_i^* \leq d_i - 1$ for $i = 1, 2$, so
\begin{equation}
1 \leq d_i - d_i^* \leq k - 2.
\end{equation}

It follows that $2 \leq (d_1 + d_2) - (d_1^* + d_2^*) \leq 2(k - 2)$. Since $d_1 + d_2$ is odd (by (1)) while $d_1^* + d_2^*$ is even, the above inequalities can be strengthened as $3 \leq (d_1 + d_2) - (d_1^* + d_2^*) \leq 2(k - 2) - 1$, or equivalently
\begin{equation}
2 \leq \frac{(d_1 + d_2) - (d_1^* + d_2^*)+1}{2} \leq k - 2.
\end{equation}

Using Lemma 4.3, a realization of $\pi$ can be constructed as follows. Let $H_2$ be a cubic bipartite graph with $k - 2$ ($\geq 4$) vertices, where $H_2$ is simple if $k \geq 8$ and contains precisely two disjoint $2$-circuits, $C_1$ and $C_2$, if $k = 6$. Then there exist two disjoint perfect matchings $M_1, M_2$ in $H_2$ such that $M_i \cap C_i \neq \emptyset$ for $i = 1, 2$ if $k = 6$. Without loss of generality, we assume that $d_1 - d_1^*$ is even. So $d_2 - d_2^*$ is odd. By (3), (4) and the selection of $M_i$, we can find a subset $F_i$ of $M_i$ such that $|F_1| = \frac{d_1 - d_1^*}{2}$, $|F_2| = \frac{d_2 - d_2^*+1}{2}$, and $(F_1 \cup F_2) \cap C_1 \neq \emptyset$ for $i = 1, 2$ if $k = 6$. Let $v_i$ be the vertex with degree $d_i^*$ in $H_1$ for $i = 1, 2$, and let $ab$ be a special edge in $F_2$. For $i = 1, 2$, let us subdivide each edge in $F_i$ once by a degree-two vertex (let $c$ denote this vertex on the special edge $ab$), then identifying all these degree-two vertices with $v_i$. At this stage, the resulting graph has the degree sequence $(d_1, d_2 + 1, 4^{n-2-k}, 3^2)$. Finally, switch the edge $ca$ away from vertex $v_2$ to a degree-four vertex in $H_1 \setminus \{v_1, v_2\}$. Clearly, the resulting graph $G$ is simple and is a desired realization of the sequence $\pi$.

Case 2. $k$ is odd. In this case, we have
\begin{enumerate}
\item[(5)] $d_1 + d_2$ is even. According to the value of $k$, we consider two possibilities.
\item[(Subcase 2.1)] $k \leq n - 4$. The proof of this subcase goes along the same line as that of Case 1: We propose to construct a realization $G$ of $\pi$ with properties (a) and (b), using Lemma 4.3, such that the $Z_3$-connected graph $H_1$ (recall Lemma 4.3) has degree sequence $\pi^\ast = (d_1^*, d_2^*, 4^{n-3-k}, 3^2)$, where $d_1^*$ and $d_2^*$ are to be determined, and the cubic bipartite graph $H_2$ has $k - 1$ vertices.
\end{enumerate}

We are to determine $d_1^*$ and $d_2^*$ by letting $A_i = \max\{4, d_i - (k - 1)\}$ and $B_i = \min\{n - k, d_i - 1\}$ for $i = 1, 2$. It is a routine matter to check that
\begin{equation}
A_i \leq B_i, \text{ and equality holds if and only if } A_i = B_i = 4.
\end{equation}

Thus (6) guarantees the existence of $d_1^*$ and $d_2^*$ such that
\begin{itemize}
\item $A_i \leq d_i^* \leq B_i$ for $i = 1, 2$;
\item $d_1^* + d_2^*$ is even.
\end{itemize}

By Lemma 4.4, the sequence $\pi^\ast = (d_1^*, d_2^*, 4^{n-3-k}, 3^2)$ is graphical, and hence by Theorem 3.1 it admits a $Z_3$-connected realization $H_1$. By the definitions of $A_i$, $B_i$, and $d_i^*$, we have $d_i - (k - 1) \leq d_i^* \leq d_i - 1$ for $i = 1, 2$, so
\begin{equation}
1 \leq d_i - d_i^* \leq k - 1.
\end{equation}

Let $H_2$ be a cubic bipartite graph with $k - 1$ ($\geq 4$) vertices, where $H_2$ is simple if $k \geq 7$, or contains precisely two disjoint $2$-circuits, $C_1$ and $C_2$, if $k = 5$. Renaming the subscripts if necessary, we assume that $d_1 - d_1^* \geq d_2 - d_2^*$. Set $t_1 = (d_1 - d_1^*)/2$ and $t_2 = (d_2 - d_2^*)/2$ if $d_1 - d_1^*$ is even, or, set $t_1 = (d_1 - d_1^* + 1)/2$ and $t_2 = (d_2 - d_2^* - 1)/2$ otherwise. Since $d_1 + d_2$ and $d_1^* + d_2^*$ have the same parity, $t_1$ and $t_2$ are both integers. Moreover, by (7) we have
\begin{enumerate}
\item[(8)] $t_2 \leq t_1 \leq (k - 1)/2$, and $t_2 < t_1$ if $d_1 - d_1^*$ is odd.
\end{enumerate}
Let \( M_1, M_2, M_3 \) be three disjoint perfect matchings in \( H_2 \). In view of (8), we can find a subset \( F_i \) of \( M_i \) such that \( |F_i| = t_i \) for \( i = 1, 2 \), \( |F_3| = 1 \), and \( (F_1 \cup F_2 \cup F_3) \cap C_i \neq \emptyset \) for \( i = 1, 2 \) if \( k = 5 \). Let \( v_i \) be the vertex with degree \( d_i^* \) in \( H_1 \) for \( i = 1, 2 \), let \( ab \) be a special edge in \( F_1 \) such that \( a \) is covered by no edge in \( F_2 \) if \( d_1 - d_1^* \) is odd (such edge is available as \( t_2 < t_1 \)), and let \( cd \) be the edge in \( F_3 \). For \( i = 1, 2 \), let us subdivide each edge in \( F_i \) once by a degree-two vertex (let \( x \) denote this vertex on the special edge \( ab \)) and identify all these degree-two vertices with \( v_i \). At this stage, the resulting graph has the degree sequence \( (d_1, d_2, 4^{n-3-k}, 3^{k+1}) \) if \( d_1 - d_1^* \) is even, or \( (d_1 + 1, d_2 - 1, 4^{n-3-k}, 3^{k+1}) \) if \( d_1 - d_1^* \) is odd. Then switch edge \( xa \) away from vertex \( v_1 \) to \( v_2 \) if \( d_1 - d_1^* \) is odd, and finally cut \( cd \) (\( \in F_3 \)) into two edges \( cx \) and \( yd \) and identify \( x \) (resp., \( y \)) with a degree-four (resp., degree-three) vertex in \( H_1 \setminus \{v_1, v_2\} \).

Clearly, the resulting graph \( G \) is simple and is a desired realization of the sequence \( \pi \).

Subcase 2.2. \( k = n - 3 \). In this subcase, by the hypothesis of the theorem we have

\[(9) \quad \pi = (d_1, d_2, 5, 3^k), \quad \text{where} \quad n - 2 \geq d_1 \geq d_2 \geq 5 \quad \text{and} \quad n \geq 9.
\]

According to the hypothesis of Case 2, \( k \) is odd. From \( k = n - 3 \) and \( n \geq 9 \) we deduce that

\[(10) \quad k \geq 7.
\]

By Lemma 4.4, the sequence \( (4^3, 3^2) \) is graphical and hence, by Theorem 3.1, it admits a \( Z_3 \)-connected realization \( H_1 \): let \( x_1, x_2, x_3 \) be the three vertices of degree four in \( H_1 \). Since \( k \geq 7 \) by (10), we can find a simple cubic bipartite graph \( H_2 \) with \( k - 1 \) vertices. Let \( u \) be a vertex of \( H_2 \), let \( v_1, v_2, v_3 \) be the neighbors of \( u \) in \( H_2 \), and let \( M_1, M_2, M_3 \) be three disjoint perfect matchings of \( H_2 \). Renaming the subscripts if necessary, we assume \( u v_i \in M_i \) for \( i = 1, 2, 3 \). Let \( G^* \) be the graph obtained from the union of \( H_1 \) and \( H_2 \) by first splitting \( u \) into three vertices \( \{u_1, u_2, u_3\} \) (so the three edges incident with \( u \) in \( H_2 \) become \( u_1 v_1, u_2 v_2, u_3 v_3 \)) and then identifying \( u_i \) with \( x_i \) in \( H_1 \) for \( i = 1, 2, 3 \). At this stage, the resulting graph has the degree sequence \( (5^3, 3^k) \).

Set \( t_1 = (d_1 - 5)/2 \) and \( t_2 = (d_2 - 5)/2 \) if \( d_1 \) is odd, and set \( t_1 = (d_1 - 4)/2 \) and \( t_2 = (d_2 - 6)/2 \) otherwise. Since \( d_1 + d_2 \) is even by (5), \( t_1 \) and \( t_2 \) are both integers. Moreover, since \( k = n - 3 \) and \( n - 2 \geq d_1 \geq d_2 \geq 5 \), we have

\[(11) \quad t_2 \leq t_1 \leq (k - 3)/2, \quad \text{and} \quad t_2 < t_1 \quad \text{if} \quad d_1 \quad \text{is even}.
\]

Since \( M_i - \{u v_i\} \) contains \( (k - 3)/2 \) edges, we can find a subset \( F_i \) of \( M_i - \{u v_i\} \) such that \( |F_i| = t_i \) for \( i = 1, 2 \). Let \( ab \) be a special edge in \( F_1 \) such that \( a \) is covered by no edge in \( F_2 \) if \( d_1 \) is even (such an edge is available as \( t_2 < t_1 \)). We construct a graph from \( G^* \) as follows: For \( i = 1, 2 \), subdivide each edge in \( F_i \) once by a degree-two vertex (let \( c \) denote this vertex on the special edge \( ab \)), then identify all these degree-two vertices with \( x_i \), and finally switch edge \( ca \) away from vertex \( x_1 \) to \( x_2 \) if \( d_1 \) is even. Clearly, the resulting graph \( G \) is simple and is a desired realization of the sequence \( \pi \).

Now we are ready to establish the main results of this section.

**Proof of Theorem 4.1.** Assume the contrary: \( \pi \) is a counterexample with minimum \( n \). By Theorem 3.1, we have the following:

\[(1) \quad d_n = 3.
\]

\[(2) \quad \text{The residual sequence} \bar{\pi} \text{ does not satisfy the hypothesis of the theorem.}
\]

Otherwise, from the assumption on \( n \) we deduce that \( \bar{\pi} \) has a realization \( \bar{G} \) such that

- \( \bar{G} \) admits a nowhere-zero 3-flow;
- \( \bar{G} \) has a \( Z_3 \)-connected subgraph \( \bar{H} \) that contains all vertices of \( \bar{G} \) with degree at least four.
Let $G$ be the realization of $\pi$ obtained from $\bar{G}$ by adding a new vertex $v$ and three edges between $v$ and corresponding vertices in $\bar{G}$ (recall (1)). Let $H$ be the subgraph induced by $V(H) \cup \{v\}$ in $G$. Since $\bar{H}$ contains all vertices of $\bar{G}$ with degree at least four, it also contains all vertices of $G$ with degree at least four as $d_3 \geq 5$ and $d_n = 3$. So the degree of $v$ in $H$ is also three. Hence $H$ is $Z_3$-connected by Lemma 2.6. Note that the existence of a nowhere-zero 3-flow is preserved under edge contractions, so $\bar{G}/H$ and hence $G/H$ (as $\bar{G}/H = G/H$) admits a nowhere-zero 3-flow. It follows from Lemma 2.5(a) that so does $G$. This contradiction implies (2).

(3) $d_1 \leq n - 2$. Otherwise, $d_1 = n - 1$. Since $d_3 \geq 5$, $\pi$ has a $Z_3$-connected realization by Theorem 3.2, a contradiction. So we get (3).

(4) $d_3 = 5$. Otherwise, $d_3 \geq 6$; combining this with (1), we see that $\bar{\pi}$ satisfies the hypothesis of the theorem, contradicting (2). So (4) holds.

Throughout the proof, let $m_k$ denote the multiplicity of $k$ in $\pi$. Then

(5) $m_3 \geq 5$. Otherwise, by Theorem 3.1, we have $m_3 = 4$. Thus precisely three entries of $\bar{\pi}$ are three by (1) and (4). It follows from Theorem 3.1 that $\bar{\pi}$ has a $Z_3$-connected realization. Hence so does $\pi$ by Lemma 2.6. This contradiction yields (5).

(6) $n \geq 9$. Otherwise, $n \leq 8$. From (4) and (5) we deduce that $n = 8$ and $m_3 = 5$. In view of (3), $d_1 \leq 6$. So $\pi = (5^4, 3^5)$ or $(6^2, 5, 3^5)$.

For $\pi = (5^3, 3^5)$, let $G$ be the graph obtained from the disjoint union of a $W_4$, with hub $v_0$ and rim $v_1v_2v_3v_4v_1$, and a path $v_5v_6v_7$ by adding edges $v_5v_1, v_5v_3, v_5v_1, v_5v_3, v_5v_0$. Then $G/W_4$ is triangularly connected and contains two 2-circuits. Since 2-circuits are $Z_3$-connected, so is $G/W_4$ by Lemma 2.9(a). It follows from Lemmas 2.9(c) and 2.5(b) that $G$ is $Z_3$-connected.

For $\pi = (6^2, 5, 3^5)$, we have $\bar{\pi} = (5^2, 4^3)$. Let $G$ be the graph obtained from the disjoint union of a $W_4$ and a $W_3$ by identifying one rim edge of $W_4$ with a rim edge of $W_3$. Then $G$ is a realization of $\bar{\pi}$. Using the same proof employed in the preceding paragraph, we can justify that $G$ is $Z_3$-connected. Now let $G$ be the graph obtained from $G$ by adding a new vertex $v$ and three edges between $v$ and vertices of degree at least four in $G$. Clearly, $G$ is a realization of $\pi$ and is $Z_3$-connected by Lemma 2.6. This contradiction proves (6).

From (3), (4), (5), (6), and Lemma 4.5, we deduce the following:

(7) $d_4 = 5$.

(8) $d_2 = 5$. Otherwise, $d_2 \geq 6$. In view of (7), $\bar{\pi}$ satisfies the conditions of the theorem, contradicting (2).

(9) $d_5 \leq 4$. Suppose to the contrary that $d_5 \geq 5$. By (7), we have $d_5 = 5$. It follows that $d_1 = 5$ and $d_6 \leq 4$, for otherwise $\bar{\pi}$ satisfies the conditions of the theorem, contradicting (2). Thus $\pi = (5^5, 4^{m_4}, 3^{m_3})$. Consider the sequence $\pi^* = (4^{m_4+4}, 3^2)$. By Lemma 4.4, $\pi^*$ is graphical. Thus it has a $Z_3$-connected realization $H_1$ by Theorem 3.1. Note that $m_3$ is odd since $\pi = (5^5, 4^{m_4}, 3^{m_3})$ (by Lemma 2.1). Let $H_2$ be a bipartite cubic graph on $m_3 - 1 \geq 4$ vertices, where $H_2$ is simple if $m_3 \geq 7$, or contains precisely two 2-circuits, $C_1$ and $C_2$, if $m_3 = 5$. We take three edges $e_1, e_2, e_3$ in $H_2$ such that $e_1$ and $e_2$ are independent and that $e_i \in C_i$ for $i = 1, 2, 3$ if $m_3 = 5$. Let us construct a graph from the disjoint union of $H_1$ and $H_2$ as follows: Cut each of $e_1$ and $e_2$ into two edges (let $u_1, u_2, u_3, u_4$ denote the new vertices), then subdivide $e_3$ once by a degree-two vertex $v$, and finally identify $u_1, u_2, u_3, u_4$ with four degree-four vertices in $H_1$, respectively, and $v$ with a degree-three vertex in $H_1$. Clearly, $G$ is a realization of $\pi$. It follows from Lemma 4.3 that $G$ admits a nowhere-zero 3-flow. This contradiction implies (9).

From the above observations, we conclude
Lemma 4.3, such that the degree $H$ graph $\pi$ is a desired realization of the sequence $\pi$. By (13), there exist two disjoint matchings $M$ such that the degree $H$ graph $\pi$ is a desired realization of the sequence $\pi$. Thus (15) guarantees the existence of $H$ that $|\pi| = 2$, and ($\pi$ is even. So $|\pi| = 3$, we have $d_i - (m_3 - 1) \leq d_i \leq 3$, so
\[0 \leq d_i - d_i^* \leq m_3 - 1, \quad \text{and hence } 0 \leq \frac{d_i - d_i^*}{2} \leq \frac{m_3 - 1}{2} \quad (\text{recall Case 1}).\]
Let $H_2$ be a cubic bipartite graph with $m_3 - 1$ (≥ 4) vertices, where $H_2$ is simple if $m_3 \geq 7$ and contains precisely two disjoint 2-circuits, $C_1$ and $C_2$, if $m_3 = 5$ (see (11)). By (13), there exist two disjoint matchings $M_1, M_2$ in $H_2$ such that $|M_1| = \frac{d_i - d_i^*}{2}$, $|M_2| = 2$, and $(M_1 \cup M_2) \cap C_i \neq \emptyset$ for $i = 1, 2$ if $m_3 = 5$. Let us first subdivide each edge in $M_1$ once by a new vertex and identify all these vertices with the vertex of degree $d_i^*$ in $H_1$, and then cut each edge in $M_2$ into two edges (let $v_1, v_2, v_3, v_4$ be the new vertices) and identify $v_1, v_2, v_3, v_4$ with three degree-four vertices in $H_1$, respectively, and $v_5$ with a degree-three vertex in $H_2$. Clearly, the resulting graph $G$ is simple and is a desired realization of the sequence $\pi$. By Lemma 4.3, $G$ admits a nowhere-zero 3-flow.

Case 2. $d_1$ is odd. From (10) we see that $m_3 \geq 6$ by (5).

We propose to construct a realization $G$ of $\pi$ with properties (a) and (b), using Lemma 4.3, such that the $Z_3$-connected graph $H_1$ (recall Theorem 3.1) has degree sequence $\pi^* = (d_1^*, 4^{n-m_3-2}, 3^2)$, where $d_1^*$ is to be determined, and the cubic bipartite graph $H_2$ has $m_3 - 2$ vertices.

To this end, set $A = \max\{4, d_1 - (m_3 - 2)\}$ and $B = \min\{n - m_3 + 1, d_1\}$. By virtue of (3), (5), (10), and (14), we get
\[0 \leq d_1 - d_1^* \leq m_3 - 2, \quad \text{and hence } 1 \leq \frac{d_1 - d_1^* + 1}{2} \leq \frac{m_3 - 2}{2} \quad (\text{as } d_1 \text{ and } d_1^* \text{ have different parities}).\]
Let $H_2$ be a cubic bipartite graph with $m_3 - 2$ (≥ 4) vertices, where $H_2$ is simple if $m_3 \geq 8$, or contains precisely two disjoint 2-circuits, $C_1$ and $C_2$, if $m_3 = 6$ (see (14)). By (16), there exist a pair of edge-disjoint matchings $M_1, M_2$ in $H_2$ such that $|M_1| = \frac{d_1 - d_1^* + 1}{2}$, $|M_2| = 1$, and $(M_1 \cup M_2) \cap C_i \neq \emptyset$ for $i = 1, 2$ if $m_3 = 6$. Let $ab$ be a special edge in $M_1$. We construct a graph from the disjoint union of $H_1$ and $H_2$ as follows: First subdivide each edge in $M_1$ once by a new vertex (let $x$ be the new vertex on the special $ab$) and identify all these new vertices with the vertex $v_1$ of degree $d_1^*$ in $H_1$, then cut the edge in $M_2$ into two edges (let $y, z$ be
the new vertices) and identify $y,z$ with two degree-four vertices in $H_1$, respectively. At this stage, the resulting graph has the degree sequence $(d_1, 1, 5^2, 4^{n-3}, 3^3)$. Finally, switch the edge $ax$ away from $v_1$ to some degree-four vertex. Clearly, the resulting graph $G$ is simple and is a desired realization of the sequence $\pi$ (see (10)). By Lemma 4.3, $G$ admits a nowhere-zero 3-flow. This contradiction completes the proof of the theorem.  

Proof of Theorem 4.2. The proof goes along the same line as that of Lemma 4.5, so we give only a sketch here. Let us consider four cases according to the values of $k$ and $d_4$: In each case we propose to construct a realization $G$ of $\pi$ with properties (a) and (b) using Lemma 4.3; the degree sequence $\pi^*$ of the $Z_3$-connected graph $H_1$ (recall Lemma 4.3) and the number of vertices in the cubic bipartite graph $H_2$ are given below.

Case 1. $k$ is even and at least six. The number of vertices in $H_2$ is $k - 2$, and the degree sequence $\pi^*$ of $H_1$ is $(d_1^*, d_2^*, 4^{n-2-k}, 3^2)$. In order to determine $d_1^*$ and $d_2^*$, we

- set $A_i = \max\{d_i, (k - 2)\}$ for $i = 1,2$,
- set $B_i = \min\{n - k + 1, d_i - p\}$, where $p = 3$ if $d_2 = 4$ and 2 if $d_2 \geq 5$, and
- set $B_i = \min\{n - k + 1, d_2 - q\}$, where $q = 0$ if $d_2 = 4$ and 1 if $d_2 \geq 5$.

Case 2. $k = 4$. The number of vertices in $H_2$ is $k$, and the degree sequence $\pi^*$ of $H_1$ is $(d_1^*, d_2^*, 4^{n-4-k}, 3^2)$. In order to determine $d_1^*$ and $d_2^*$, we

- set $A_i = \max\{d_i, (k - 2)\}$ for $i = 1,2$,
- set $B_i = \min\{n - k + 1, d_i - p\}$, where $p = 3$ if $d_2 = 4$ and 2 if $d_2 \geq 5$, and
- set $B_i = \min\{n - k + 1, d_2 - q\}$, where $q = 0$ if $d_2 = 4$ and 1 if $d_2 \geq 5$.

Case 3. $k$ is odd and $d_4 = 4$. The number of vertices in $H_2$ is $k - 1$, and the degree sequence $\pi^*$ of $H_1$ is $(d_1^*, d_2^*, 4^{n-3-k}, 3^2)$. From $d_4 = 4$ it can be seen that $n - 3 - k \geq 1$. In order to determine $d_1^*$ and $d_2^*$, we

- set $A_i = \max\{d_i, (k - 2)\}$ for $i = 1,2$,
- set $B_i = \min\{n - k + 1, d_i - p\}$, where $p = 3$ if $d_2 = 4$ and 2 if $d_2 \geq 5$, and
- set $B_i = \min\{n - k + 1, d_2 - q\}$, where $q = 0$ if $d_2 = 4$ and 1 if $d_2 \geq 5$.

In each of the above three cases, it is a routine matter to check that $A_i \leq B_i$ for $i = 1,2$, and equality holds if and only if $A_i = B_i = 4$. Thus there exist $d_1^*$ and $d_2^*$ such that

- $A_i \leq d_i^* \leq B_i$ for $i = 1,2$;
- $d_1^* + d_2^*$ is even.

By Lemma 4.4, the sequence $\pi^*$ is graphical, and hence by Theorem 3.1 it admits a $Z_3$-connected realization $H_1$. Clearly, we can choose $H_2$ so that it is simple if it has at least six vertices and contains precisely two disjoint 2-circuits otherwise. Note that $H_2$ contains three disjoint perfect matchings. By subdividing or cutting a certain number (at least two) of edges in these matchings, we can get a realization $G$ of $\pi$, as desired.

Case 4. $k$ is odd and $d_4 = 3$. In this case $n - k - 2 = 1$, so $k = n - 3$. Hence $\pi = (d_1, d_2, 4^{3k})$, $k \geq 7$, and $n$ is even (recall the hypothesis of the theorem). By Lemma 2.1(a), $d_1 + d_2$ is odd. From $d_1 + d_2 \geq 11$ we further deduce that $n - 3 \geq d_1 \geq 7$ if $d_1$ is odd.

Let $H_1$ be a $Z_3$-connected realization of $(4^3, 3^2)$ in which vertices $x_1, x_2, x_3$ are of degree four, and let $H_2$ be a cubic bipartite graph with $k - 1$ vertices. Let $u$ be a vertex of $H_2$, let $v_1, v_2, v_3$ be the neighbors of $u$ in $H_2$, and let $M_1, M_2, M_3$ be three disjoint perfect matchings of $H_2$. Renaming the subscripts if necessary, we assume $u v_i \in M_i$ for $i = 1, 2, 3$. For odd $d_1$, let $G^*$ be the graph obtained from the union of $H_1$ and $H_2$ by identifying $x_1$ and $u$. For even $d_1$, let $G^*$ be the graph obtained from the union of $H_1$ and $H_2$ by first splitting $u$ into three vertices $\{u_1, u_2, u_3\}$ (so
the three edges incident with $u$ in $H_2$ become $u_1v_1, u_2v_2, u_3v_3$ and then identifying $u_1, v_2$ with $x_1$ and $u_3, v_1$ with $x_2$.

Set $t_1 = (d_1 - 7)/2$ and $t_2 = (d_2 - 4)/2$ if $d_1$ is odd, and set $t_1 = (d_1 - 6)/2$ and $t_2 = (d_2 - 5)/2$ otherwise. Since $d_1 + d_2$ is odd, $t_1$ and $t_2$ are both integers. Moreover, since $k = n - 3, n - 2 \geq d_1 \geq d_2 \geq 4$, and $d_1 \leq n - 3$ if $d_1$ is odd, we have

- $t_1 \leq (k - 1)/2 - 3$ if $d_1$ is odd and $t_1 \leq (k - 1)/2 - 2$ otherwise, and
- $t_2 \leq (k - 1)/2 - 1$ if $d_1$ is odd and $t_2 \leq (k - 1)/2 - 1$ otherwise.

For odd $d_1$, let $F_1$ be a subset of $M_1$ such that $F_1$ covers none of $v_1, v_2, v_3$ and $|F_1| = t_1$, and let $F_2$ be a subset of $M_2$ such that $uv_2 \not\in F_2$ and $|F_2| = t_2$. For even $d_1$, let $F_1$ be a subset of $M_1$ such that $F_1$ covers neither $v_1$ nor $v_2$ and $|F_1| = t_1$, and let $F_2$ be a subset of $M_2$ such that $uv_3 \not\in F_2$ and $|F_2| = t_2$. Finally, we construct a graph from $G^*$ as follows: For $i = 1, 2$, subdivide each edge in $F_i$ once by a degree-two vertex, and then identify all these degree-two vertices with $x_i$. Clearly, the resulting graph $G$ is simple and is a realization of $\pi$ that admits a nowhere-zero 3-flow, a contradiction. 

5. Realizations with 3-flows. To establish the main theorem of this paper, we shall break the proof into two parts and turn to proving the following two theorems.

**Theorem 5.1.** Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 3$. Then $\pi$ has a realization $G$ that admits a nowhere-zero 3-flow if and only if $\pi = (k, 3^k), (k, 3^k-1)$, where $k$ is even.

**Theorem 5.2.** Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphical sequence with $d_1 \geq d_2 \geq \cdots \geq d_n = 2$. Then $\pi$ has a realization $G$ that admits a nowhere-zero 3-flow if and only if $\pi = (3^i, 2)$.

The following lemma will be used repeatedly in our proof.

**Lemma 5.3.** Let $k \geq 4$ be an even integer, and let $\pi = (a_1, a_2, a_3, 3^k)$ be a sequence with $0 \leq a_i \leq k$. Then $\pi$ has a realization that admits a nowhere-zero 3-flow if one of the following holds:

- (a) $a_i$ is even for $i = 1, 2, 3$, and $a_1 + a_2 + a_3 \geq 4$ if $k = 4$;
- (b) one of $a_1, a_2, a_3$ is even, and the remaining two are odd and at least three.

**Proof.** (a) Let $H$ be a cubic bipartite graph with $k$ vertices, where $H$ is simple if $k \geq 6$, or contains precisely two disjoint 2-circuits, $C_1$ and $C_2$, if $k = 4$. Then the edge set of $H$ can be decomposed into three perfect matchings $M_1, M_2, M_3$. Since $a_i \leq k$ and since $a_1 + a_2 + a_3 \geq 4$ if $k = 4$, we can find a subset $F_i$ of $M_i$ such that $|F_i| = a_i^+$ for $i = 1, 2, 3$ and that $(F_1 \cup F_2 \cup F_3) \cap C_i \neq \emptyset$ for $i = 1, 2$ if $k = 4$. For $i = 1, 2, 3$, let us subdivide each edge in $F_i$ once by a degree-two vertex and then identify all these degree-two vertices as a single vertex $v_i$. Then the resulting graph $G$ is a realization of the sequence $\pi$. By Lemma 2.2, $H$ admits a nowhere-zero 3-flow, and so does $G$.

(b) Renaming the subscripts if necessary, we may assume that $a_3$ is even. By Lemma 2.1, $k$ is even and so $a_i \leq k - 1$ for $i = 1, 2$. Let $H$ be a simple cubic bipartite graph with $k + 2$ vertices, and let $M_1, M_2, M_3$ be three disjoint perfect matchings of $H$. Take an arbitrary edge $u_1v_1$ in $M_1$, and take a subset $F_1$ of $M_i$ such that $|F_1| = a_i - 3$ for $i = 1, 2$, $|F_3| = a_3^+$, and $F_1$ (resp., $F_2$) covers no vertices of $N(u_1)$, the neighborhood of $u_1$ (resp., $N(v_1)$). Let us now subdivide each edge of $F_1$ (resp., $F_2$) once by a degree-two vertex and then identify all these degree-two vertices with $u_1$ (resp., $v_1$), and subdivide each edge of $F_3$ once by a degree-two vertex and then identify all these degree-two vertices. Then the resulting graph $G$ is a realization of the sequence $\pi$. By Lemma 2.2, $H$ admits a nowhere-zero 3-flow, and so does $G$. 

**Proof of Theorem 5.1.** Since the “only if” part is already established by Lemma 2.10, let us proceed to the “if” part.
Assume the contrary: $\pi$ is a counterexample with minimum $n$. By Theorems 3.1, we have

(1) $m_3 \geq 4$.

From Theorem 4.1 it can be seen that

(2) $d_3 \leq 4$.

Let us further make some simple observations.

(3) $4 \leq d_1 \leq n - 2$. The upper bound follows instantly from Theorem 3.2. To justify lower bound, we assume to the contrary that $d_1 = 3$. So $\pi = (3^n)$. Using the upper bound, we have $n \geq 5$. From Lemma 2.1(a) we deduce that $n$ is even and so $n \geq 6$. Thus $\pi$ can be realized by a bipartite cubic graph, which, by Lemma 2.2, admits a nowhere-zero 3-flow, a contradiction. So (3) holds.

(4) $d_2 \geq 4$. Otherwise, $d_2 = 3$. Thus $\pi = (d_1, 3^{n-1})$. If $n - 1$ is even, then so is $d_1$. Set $a_1 = d_1$, $a_2 = a_3 = 0$, and $k = n - 1$. By (3) and Lemma 5.3(a), $\pi$ has a realization that admits a nowhere-zero 3-flow, a contradiction. This proves (4).

(5) $d_3 = 4$. By (2), $d_3 \leq 4$. We prove by contradiction and assume $d_3 = 3$. Thus $\pi = (d_1, d_2, 3^{n-2})$. Observe that $d_1 + d_2$ is odd, for otherwise (1), (3), and Lemma 5.3 (with $a_i = d_i$ for $i = 1, 2$ and $a_3 = 0$) would guarantee the existence of a realization of $\pi$ that admits a nowhere-zero 3-flow, a contradiction. It follows that $n$ and exactly one of $d_1$ and $d_2$ are odd.

If $d_1 \leq n - 3$, then, by Lemma 5.3(b) (with $a_i = d_i$ for $i = 1, 2$, $a_3 = 3$, and $n - 3 \geq 4$ because $n$ is odd), $\pi$ has a realization that admits a nowhere-zero 3-flow; this contradiction implies that $d_1 = n - 2$. Hence $d_1$ is odd and $d_2$ is even. Now let us take odd wheel $W_{n-2}$ with hub $v_0$ and rim $v_1v_2\ldots v_{n-2}v_1$ and take $M = \{v_2v_{2i+1} : 0 \leq i \leq d_2/2 - 1\}$. Clearly, $M$ is a matching of size $d_2/2$. Let us subdivide each edge in $M$ once and identify all the new vertices as a single vertex. Then the resulting graph is a realization of $\pi$ and admits a nowhere-zero 3-flow (to find it, direct each of edges $v_0u$, $v_0v_2$, $v_0v_3$ from $v_0$ to the other end, where $u$ is the vertex subdividing $v_0v_1$; then directions of the remaining edges can be determined accordingly). This contradiction implies (5).

(6) $n \geq 9$. Otherwise, $n \leq 8$. By (5) and (1), we have $d_3 = 4$ and $m_3 \geq 4$. So $n \geq 7$.

If $n = 7$ then, by (3), (5), and Lemma 2.1(a), we have $\pi = (4^3, 3^4)$ or $(5^2, 4, 3^4)$. For $\pi = (4^3, 3^4)$, clearly $K_3, 4$ is a realization of $\pi$ that admits a nowhere-zero 3-flow. For $\pi = (5^2, 4, 3^4)$, let $G$ be the graph obtained from the union of $W_4$ and $W_3$ by identifying a rim edge of $W_4$ with a rim edge of $W_3$. Then $G$ is a realization of $\pi$ that, by Lemma 2.9(c) and (a), admits a nowhere-zero 3-flow. So we have $n = 8$.

Since $d_3 = 4$ by (5), we have $m_3 \leq 5$: combining this with (1), we further have $m_3 = 4$ or 5. Since $d_1 \leq 6$ by (3), one of the following cases must occur:

- $\pi = (6^2, 4^2, 3^4)$,
- $\pi = (6, 4^4, 3^4)$,
- $\pi = (4^2, 3^4)$,
- $\pi = (5^2, 4^2, 3^4)$,
- $\pi = (5, 4^2, 3^5)$, or
- $\pi = (6, 5, 4, 3^5)$.

For each $\pi$, we shall exhibit a realization $G$ that admits a nowhere-zero 3-flow, thereby reaching a contradiction.
For $\pi = (6^2, 4^2, 3^4)$, let $\bar{G}$ be the graph obtained from the union of $W_4$ and $W_3$ by identifying a rim edge of $W_4$ with a rim edge of $W_3$. Then $\bar{G}$ is the realization of the residual sequence $\bar{\pi}$ and $\bar{G}$ is $Z_3$-connected. So it is easy to obtain a realization $G$ of $\bar{\pi}$ from $\bar{G}$ such that $G$ admits a nowhere-zero 3-flow, a contradiction.

For $\pi = (6, 4^4, 3^4)$, let $G$ be the graph obtained from $W_6$ by adding an edge between two nonadjacent vertices, then subdividing two independent edges once each, and finally identifying these new vertices as one vertex. (Since $W_6$ is $Z_3$-connected, so is the graph obtained from $W_6$ by adding an edge by Lemma 2.7.)

For $\pi = (4^4, 3^4)$, let $G$ be the graph obtained from the cubic bipartite graph with four vertices by subdividing each of the four multiple edges once and then connecting these four degree-two vertices with a 4-circuit. (Note that $G$ can be decomposed into a subdivision of a cubic bipartite graph and a 4-circuit.)

For $\pi = (5^2, 4^2, 3^4)$, let $G$ be the graph obtained from the union of two $W_4$'s by identifying a rim edge of one $W_4$ with a rim edge of the other $W_4$. (In fact $G$ is $Z_3$-connected.)

For $\pi = (5, 4^2, 3^3)$, let $G$ be the graph obtained from a $K_{4,3}$ with color classes $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ by subdividing $u_i v_i$ once with a degree-two vertex $w_i$ for $i = 2, 3$, and then adding a triangle $u_1 w_2 w_3 u_1$.

For $\pi = (6, 5, 4, 3^3)$, let $G$ be the graph obtained from $W_4$ (in which $u_1 u_2$ is a rim edge) by adding a path $v_1 v_2 v_3$ and then adding edges $v_1 u_1, v_1 u_2, v_2 u_1, v_3 u_1$. (Note that $G$ is triangularly connected and contains $W_4$. So it is $Z_3$-connected, by Lemma 2.9(a).)

It is a routine matter to check that $G$ is a realization of $\pi$ and admits a nowhere-zero 3-flow in each case. This contradiction implies (6).

From (3), (5), (6), and Theorem 4.2, we deduce that

$$d_1 + d_2 \leq 10.$$  

Since $d_1 \geq d_2 \geq 4$, we see that $(d_1, d_2)$ is $(6, 4)$, or $(5, 5)$, or $(5, 4)$, or $(4, 4)$. So the following is the complete list of all possible configurations of $\pi$:

- $\pi = (6, 4^m, 3^m)$,
- $\pi = (5^2, 4^m, 3^m)$,
- $\pi = (5, 4^m, 3^m)$,
- $\pi = (4^m, 3^m)$,

where $m_k$ is the multiplicity of $k$ in $\pi$. Let us process these cases one by one: For each $\pi$, we shall construct a realization $G$ that admits a nowhere-zero 3-flow, thereby reaching a contradiction.

Case 1. $\pi = (4^m, 3^m)$. Note that $m_3$ is even. Depending on the value of $m_3$, we consider two subcases.

Subcase 1.1. $m_3 = 4$. Our proof relies on the following statement.

(8) The sequence $(4^k, 2^4)$, with $k \geq 1$, can be realized by a simple connected graph $H$ that admits a nowhere-zero 3-flow. To justify this, we apply induction on $k$. For $k = 1$, the graph $H_1$ obtained from two triangles by gluing them at a common vertex is as desired. Suppose that $H_k$ is the desired realization of $(4^k, 2^4)$. Let $e, f$ be two independent edges in $H_k$, and let $H_{k+1}$ be the graph obtained from $H_k$ by first subdividing each of $e, f$ once with a degree-two vertex and then identifying these degree-two vertices. Clearly, $H_{k+1}$ is a realization of $(4^{k+1}, 2^4)$ and admits a nowhere-zero 2-flow since it is Eulerian. So (8) holds.

By (6), we have $m_4 \geq 5$. In view of (8), we can find a connected realization $H$ of the sequence $(4^{m_4 - 4}, 2^4)$. Let $G$ be the graph obtained from $H$ by adding a bipartite cubic graph $F$ with four vertices, then subdividing each of the four multiple edges in
$F$ with a degree-two vertex, and finally identifying these four degree-two vertices with the degree-two vertices of $H$, respectively. Clearly, $G$ is a realization of $\pi$ and admits a nowhere-zero 3-flow.

**Subcase 1.2.** $m_3 \geq 6$. For $m_4 \geq 5$, let $H_1$ be a bipartite cubic simple graph with $m_3$ vertices, and let $H_2$ be a $Z_3$-connected realization of $(4^{m_4-2}, 3^2)$ (recall Lemmas 4.4 and 3.3). Then we can get a desired realization $G$ of $\pi$ by cutting one edge of $H_1$ and then connecting these two half edges to the degree-three vertices of $H_2$, respectively. For $3 \leq m_4 \leq 4$, let $G$ be the graph obtained from the union of $H_1$ and an $m_4$-circuit $H_3$ by subdividing $m_4$ edges of $H_1$ once with degree-two vertices and then identifying these new vertices with $m_4$ vertices of $H_3$, respectively. Clearly, $G$ is a realization of $\pi$ and admits a nowhere-zero 3-flow.

**Case 2.** $\pi = (5, 4^{m_4}, 3^{m_3})$. Note that $m_3$ is odd, so $m_3 \geq 5$ by (1). We distinguish two subcases according to the value of $m_3$.

**Subcase 2.1.** $m_3 = 5$. Recall that $n \geq 9$ by (6). For $n = 9$, let $H_1$ be the graph obtained from $W_3$ by adding a new vertex and joining it to two vertices of the $W_3$. Clearly, $H_1$ admits a nowhere-zero 3-flow and has degree sequence $(4^2, 3^2, 2)$. For $n \geq 10$, we have $n - m_3 - 2 \geq 3$. By Lemma 4.4, the sequence $(4^{n-3-2}, 3^2)$ is graphical and hence, by Lemma 3.3, admits a $Z_3$-connected realization $F$. Let $H_1$ be the graph obtained from $F$ by subdividing one edge once. Clearly, $H_1$ admits a nowhere-zero 3-flow and has degree sequence $(4^{n-3-2}, 3^2, 2)$. Let $H_2$ be the cubic bipartite graph on four vertices in which both $u_1v_1$ and $u_2v_2$ are of multiplicity two, and let $G$ be the graph obtained from the union of $H_1$ and $H_2$ by subdividing $u_iv_i$ once for $i = 1, 2$, and then identifying one new vertex with the degree-two vertex of $H_1$ and the other new vertex with a degree-three vertex of $H_1$. Clearly, $G$ is a realization of $\pi$ and admits a nowhere-zero 3-flow.

**Subcase 2.2.** $m_3 \geq 7$. Since $d_3 = 4$ by (5), we have $m_4 \geq 2$. Let $H_1$ be a realization of $(4^{n-m_3-2}, 3^2, 2)$ as exhibited in the preceding paragraph, and let $H_2$ be a cubic bipartite simple graph with $m_3 - 1$ vertices. Using $H_1$ and $H_2$ and following the same argument as the preceding paragraph, we can obviously get a realization $G$ of $\pi$ that admits a nowhere-zero 3-flow.

**Case 3.** $\pi = (6, 4^{m_4}, 3^{m_3})$. Let $H$ be an arbitrary realization of $\pi$, and let $u$ be the vertex of degree six in $H$. Then the configuration of $\pi$ implies the existence of two nonadjacent neighbors $v, w$ of $u$ in $H$. Let $H'$ be the graph obtained from $H$ by replacing path $vw$ with edge $vw$. Then the degree sequence of $H'$ is $\pi' = (4^{m_4+1}, 3^{m_3})$. By Case 1, $\pi'$ has a realization $G'$ that admits a nowhere-zero 3-flow. Moreover, if $\pi' = (4^5, 3^4)$, by Subcase 1.1, $G'$ can be chosen such that there is a degree-four vertex $x$ and an edge $e$ such that $x$ is not incident with $e$ and is not adjacent to the end-vertices of $e$. Let $x$ be a degree-four vertex in $G'$. Let $X_1 = N(x) \cup \{x\}$ and $X_2 = V(G') \setminus X_1$. Then there must be an edge $e$ not incident with $x$ such that $x$ is not adjacent to the end-vertices of $e$ if $\pi' \neq (4^5, 3^4)$. Otherwise, $G'$ is connected and $X_2$ is an independent set. Let $[X_1, X_2]$ denote the set of edges with one end in $X_1$ and the other in $X_2$. Then $(n - 5) \times 3 \leq \sum_{u \in X_2} d(u) = |X_1, X_2| \leq \sum_{v \in X_1 \setminus \{x\}} (d(v) - 1) \leq 4 \times 3 = 12$. Hence, $n \leq 9$ with equality if and only if $X_1$ consists of all degree-four vertices and $X_2$ consists of all degree-three vertices. By (6), we have $n = 9$. Therefore, the degree sequence of $G'$ is $(4^5, 3^4)$, a contradiction to the assumption that $\pi' \neq (4^5, 3^4)$. Therefore, in any case, we can find a degree-four vertex $a$ and an edge $e$ such that $a$ is not incident with $e$ and is not adjacent to the end-vertices of $e$. Let $G$ be the graph obtained from $G'$ by subdividing $e$ once and then identifying the new vertex with $a$. Clearly, $G$ is a realization of $\pi$ and admits a nowhere-zero 3-flow.

**Case 4.** $\pi = (5^2, 4^{m_4}, 3^{m_3})$. Since $\pi$ is graphical, using the same argument
employed in the preceding paragraph we deduce that the sequence \((5,4^m,3^{m+1})\) is graphical and hence, by Case 2, has a realization \(H\) that admits a nowhere-zero 3-flow. Let \(x\) be a degree-three vertex. Let \(X_1 = N(x) \cup \{x\}\) and \(X_2 = V(H) \setminus X_1\). We first show that there exists an edge \(e\) such that \(x\) is not incident with \(e\) and is not adjacent to the end-vertices of \(e\). Otherwise, \(H\) is connected and \(X_2\) is independent. Let \([X_1, X_2]\) denote the set of edges with one end in \(X_1\) and the other in \(X_2\). Since the degree of each vertex in \(X_2\) is at least three and the degree of each vertex in \(X_1\) is at most five, we have 
\[
(n-4) \times 3 \leq \sum_{u \in X_2} d(u) = ||X_1, X_2|| \leq \sum_{v \in X_1 \setminus \{x\}} (d(v) - 1) \leq 4 \times 3 = 12.
\] Therefore, \(n \leq 4\), contradicting (6), i.e., that \(n \geq 9\). Let \(e\) be an edge and \(x\) be a degree vertex not incident with \(e\) such that \(x\) is not adjacent to any end-vertices of \(e\). Let \(G\) be the graph obtained from \(H\) by subdividing an edge \(e\) once and then identifying the new vertex with a degree-three vertex not incident to \(e\). Clearly, \(G\) is a realization of \(\pi\) and admits a nowhere-zero 3-flow. This completes the proof of Theorem 5.1. □

Let us make some preparation before presenting the proof of Theorem 5.2.

Lemma 5.4. Let \(k\) be an integer with \(k = 2\) or \(k \geq 4\), and let \(\pi = (k, 3^k, 2)\) or \((k^2, 3^{k-1}, 2)\). If \(\pi\) is graphical, then it has a realization that admits a nowhere-zero 3-flow.

Proof. Note that if \(k = 2\), then \(\pi = (3^2, 2^2)\). Let \(G\) be the graph obtained from \(W_3\) by deleting one edge. Clearly, \(G\) is a realization of \(\pi\) and admits a nowhere-zero 3-flow. So we assume

(1) \(k \geq 4\). According to the configurations of \(\pi\), we consider two cases.

Case 1. \(\pi = (k, 3^k, 2)\). If \(k\) is even, then, by Lemma 5.3(a), \(\pi\) has a realization that admits a nowhere-zero 3-flow. It remains to consider the subcase when \(k\) is odd. Thus \(k \geq 5\) by (1). Let \(H\) be a bipartite cubic simple graph with \(k+1\) vertices, let \(u\) be a vertex of \(H\), and let \(\{v_1, v_2, v_3\}\) be the neighbors of \(u\). Then \(H \setminus \{u, v_1, v_2\}\) contains a matching \(M\) of size \((k-3)/2\). Let \(G\) be the graph obtained from \(H\) by subdividing each edge in \(M\) once, then identifying all the degree-two vertices with \(u\), and finally subdividing one edge \(uv_3\) once. Clearly, \(G\) is a realization of \(\pi\) and admits a nowhere-zero 3-flow.

Case 2. \(\pi = (k^2, 3^{k-1}, 2)\). In this case \(k\) is odd, so \(k \geq 5\) by (1). Write \(k = 2t + 1\). Let \(H\) be the graph obtained from the disjoint union of a 4-circuit and \(t-1\) triangles by gluing them at a common vertex \(x\). Then \(H\) has \(2t+2\) vertices and degree sequence \((2t, 2^{t+1})\). Let \(G\) be the graph obtained from \(H\) by adding a new vertex \(y\) and making it adjacent to all vertices of \(H\) except precisely one degree-two vertex in a triangle. Then the degree sequence of \(G\) is \((2t+1, 3^t, 2)\), which is exactly \(\pi\). Since \(G\) is triangularly connected and contains \(W_4\), it is \(Z_3\)-connected by Lemma 2.9 and hence admits a nowhere-zero 3-flow. □

Proof of Theorem 5.2. The “only if" part is already established by Lemma 2.10, so we proceed to the “if" part.

Assume the contrary: \(\pi\) is a counterexample with minimum \(n\). Observe that

(1) \(d_2 \geq 3\). Otherwise, \(d_2 = d_3 = \cdots = d_n = 2\). So \(d_1\) is even. Thus each realization of \(\pi\) admits a nowhere-zero 2-flow; this contradiction leads to (1).

(2) The sequence \(\sigma = (d_1, d_2, \ldots, d_{n-1})\) is not graphical. Assume to the contrary that \(\sigma\) is graphical. Then \(\sigma \neq (3^4, 2^2), (k, 3^k), (k^2, 3^{k-1}), \) where \(k\) is an odd integer, for otherwise \(\pi = (3^4, 2^2), (k, 3^k), (k^2, 3^{k-1})\), so \(\pi\) has a realization that admits a nowhere-zero 3-flow by Lemma 5.3(a) or Lemma 5.4, a contradiction. By Theorem 5.1 and the assumption on \(\pi\), the sequence \(\sigma\) has a realization \(H\) that admits a nowhere-zero 3-flow. Let \(G\) be a graph obtained from \(H\) by subdividing an edge once. Clearly,
$G$ is a realization of $\pi$ and admits a nowhere-zero 3-flow; this contradiction implies (2).

(3) Let $\bar{G}$ be an arbitrary realization of the residual sequence $\bar{\pi} = (d_1 - 1, d_2 - 1, d_3, \ldots, d_n - 1)$ and let $v_i$ be a vertex of $\bar{G}$ with degree $d_i - 1$ for $i = 1, 2$. Then $v_1v_2$ is an edge of $\bar{G}$.

Otherwise, $v_1$ and $v_2$ are nonadjacent in $\bar{G}$. Thus the graph obtained from $\bar{G}$ by adding edge $v_1v_2$ is a realization of the sequence $\sigma = (d_1, d_2, \ldots, d_n - 1)$, contradicting (1). So (3) holds.

(4) The residual sequence $\bar{\pi}$ is $(3^d, 2)$, $(k, 3^k)$, or $(k^2, 3^{k-1})$, where $k$ is an odd integer.

Otherwise, $\bar{\pi}$ has a realization $\bar{G}$ that admits a nowhere-zero 3-flow. Let $v_i$ be a vertex of $\bar{G}$ with degree $d_i - 1$ for $i = 1, 2$. By (4), $v_1v_2$ is an edge of $\bar{G}$. Let $G$ be the graph obtained from $\bar{G}$ by adding a new vertex $w$ and making it adjacent to both $v_1$ and $v_2$. Since $G$ contains the triangle $wv_1v_2w$ and since $G$ admits a nowhere-zero 3-flow, it is easy to see that so does $G$. Hence (4) is justified.

From (4) we deduce that one of the following four cases must occur:

- $\pi = (4, 3^4, 2)$,
- $\pi = (4^2, 3^2, 2^2)$,
- $\pi = (k + 1, 4, 3^{k-1}, 2)$ or
- $\pi = ((k + 1)^2, 3^{k-1}, 2)$,

where $k$ is an odd integer. In each case we shall construct a realization of $\pi$ that admits a nowhere-zero 3-flow, thereby reaching a contradiction.

For $\pi = (4, 3^4, 2)$, $\pi$ has a realization $G$ obtained by subdividing one edge once of a $W_4$. Since $W_4$ admits a nowhere-zero 3-flow, so does $G$.

For $\pi = (4^2, 3^2, 2^2)$, let $G$ be the graph obtained from a $W_3$ by adding a new vertex, making it adjacent to two vertices of the $W_3$, and then subdividing an edge. Clearly $G$ is a realization of $\pi$. To see that $G$ admits a nowhere-zero 3-flow, let $H$ be the graph obtained from $W_3$ by duplicating an edge. Then $H$ is triangularly connected and contains a 2-circuit. By Lemma 2.9, $H$ is $Z_3$-connected. So $G$ admits a nowhere-zero 3-flow as it is a subdivision of $H$.

For $\pi = (k + 1, 4, 3^{k-1}, 2)$, let $G$ be a graph obtained from $W_k$ by adding a new vertex and making it adjacent to the hub and a rim vertex. It is easy to see that $G$ is a realization of $\pi$ and admits a nowhere-zero 3-flow.

For $\pi = ((k + 1)^2, 3^{k-1}, 2)$, let $G$ be the graph obtained from the disjoint union of $\frac{k-1}{2}$ copies of $W_3$ by gluing all of them along an edge $uv$, and then adding a new vertex and making it adjacent to both $u$ and $v$. Clearly, $G$ is a realization of $\pi$ and admits a nowhere-zero 3-flow.

This completes the proof of Theorem 5.2 and hence of Theorem 1.2. 

REFERENCES

REALIZING DEGREE SEQUENCES WITH GRAPHS HAVING 3-FLOWS

519


