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Abstract—In this paper, we discuss the initial value problem and its stability for fractional autonomous order systems in the usual sense. Our result in the linear case is equivalent to the one known in literature; this establishes the mathematic technique in order to solve the problem with initial trajectory that will be presented in future studies. The conditions that are shown are simpler to verify than the ones that are commonly known and have a close relationship with the calculations for the integer case.

I. INTRODUCTION

The use of fractional operators allow to study systems from a different perspective. In [1] the authors claimed that many systems exhibit the fractional phenomena, such as motion in complex media/environments, random walk of bacteria in a fractal substance, and the chemotaxi behaviour and food seeking of microbes [2], these phenomena are always related to the complexity and hierarchy of systems due to the fractional properties of systems components, such as fractional viscoelastic material, the fractional circuit element and the fractal structure [3].

The stability problem for fractional linear autonomous dynamical systems has been studied and used by a large number of authors. In particular when the system is commensurate order, there is a very useful tool, namely Matignon’s stability theorem [4]. This theorem gives conditions to ensure stability of the systems through the location in the complex plane of the eigenvalues. In [5] the stability of n-dimensional linear fractional differential systems with commensurate order $1 < \alpha < 2$ and the corresponding perturbed systems is investigated. In [6] the authors deal with Linear Matrix Inequality (LMI) stability conditions for fractional order systems. In [7] extends some basic results from the area of finite time and practical stability to nonlinear, perturbed, fractional order time-delay systems where a robust stability test procedure is proposed. On the stability of fractional order nonlinear systems, in [8] the conditions for stability are on the order of convergence of the dynamics of the autonomous system. In [1], [9], [10], [11] the authors proposed the Fractional Lyapunov direct method, where we need to know the fractional derivative of the Lyapunov function candidate, in general this is not an easy work, but this result is important in the development of our results. The stability problem for fractional nonlinear systems has been studied in earlier works but it remains an open problem under the consideration of some authors [12].

The rest of this paper is organized as follows. In Section II is given a brief note about fractional derivatives, Mittag-Leffler type function, fractional nonlinear systems and stability results in the usual manner. In Section III we discuss the stability of the solution $x_a = 0$ in the sense of Lyapunov for fractional nonlinear systems. In Section IV we present examples. Finally in the Section V we give some conclusions.

II. FRACTIONAL CALCULUS

In this Section we give the definitions of the Riemann-Liouville fractional Integral and fractional derivative-

Definition 1 ([13]): The Riemann-Liouville fractional integral of order $\alpha \in (0,1)$ of $g(t)$ is defined as

$$aI^\alpha \ g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t g(\tau)(t-\tau)^{\alpha-1} \, d\tau,$$

(1)

where $\Gamma$ is the Gamma function. □

If $g$ is integrable in the usual sense then the Riemann-Liouville fractional integral exits and is defined almost surely.

Definition 2 ([13]): The Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ of $g(t)$ is defined as

$$aD^\alpha \ g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t g(s)(t-s)^{-\alpha} \, ds,$$

(2)

where $\Gamma$ is the Gamma function. □

Definition 3 ([13]): The Caputo fractional derivative of order $\alpha \in (0,1)$ of $g(t)$ is defined as

$$a_{\ capacities}(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t g(s)(t-s)^{-\alpha} \, ds,$$

(3)

where $\Gamma$ is the Gamma function. □

When $g \in AC[0,h]^1$, the Riemann-Liouville and Caputo fractional derivatives of $g(t)$ exist.

Lemma 1 ([13]): If $1 \leq p \leq \infty$ and $g = aI^\alpha \psi$, $\psi \in L_p(a,b)$, then

$$aI^\alpha \ aD^\alpha \ g(t) = g(t).$$

$1 AC(G) = \{ f: G \subset \mathbb{R} \rightarrow \mathbb{R} : f$ is an absolutely continuous function$\}$
Theorem 1 ([14]): In order that a function \( g(t) \) be representable as \( g = \int_a^t \psi \, ds \), \( \psi \in L_p(a,b) \), \( -\infty < a < b < \infty \), where \( 0 < \alpha < 1 \), \( 1 < p < \infty \), it is necessary and sufficient that \( g \in L_p(a,b) \) and there exist \( \lim_{t \to 0^+} \psi(t) \) in \( L_p(a,b) \) where

\[
\psi(t) = \int_{a}^{t-c} \frac{g(t) - g(s)}{(t-s)^{\alpha+1}} ds, \quad c > 0.
\]

A. Mittag-Leffler type function

The Mittag-Leffler function is used to solve fractional differential equations in a similar way like the exponential function does in integer order systems [13].

Definition 4 ([13]): The Mittag-Leffler function with two parameters is defined as

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z, \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0. \quad (4)
\]

In the particular case when \( \alpha = \beta = 1 \), we have that \( E_{1,1}(z) = e^z \).

B. Fractional nonlinear systems: existence and uniqueness

We quote a remark on existence and uniqueness for the local and global solution of fractional order nonlinear systems. For the following initial value problem, with \( \beta \in (0,1) \),

\[
D_{0}^{\beta} x(t) = f(t, x(t)), \quad x(0) = x_0, \quad (5)
\]

where, \( f : G \to \mathbb{R} \) is a Lipschitz continuous function on \( A \subset \mathbb{R} \) and \( G = [0, h^{*}) \times A \). We use the following results.

Remark 1: With above conditions on (5), it is guaranteed existence and uniqueness of the solution \( x \in C^1[0, h] \), where \( h \) depends on \( G \) and \( f \), and an integral representation for the system (5):

\[
x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_{0}^{t} f(s, x(s))(t-s)^{\beta-1} ds. \quad (6)
\]

If \( f \in C^1(G) \), then \( x \in C^1(0, h] \cap C[0, h] \) and \( x(t) = O(t^{\beta-1}) \) as \( t \to 0 \) (in particular if \( f \in C^1(G) \) then \( f \) is a locally Lipschitz continuous function).

If \( h^{*} \to \infty \) and \( A = \mathbb{R} \) then \( h \to \infty \). For details of these facts see [15].

Now we give two usual definitions and a result on stability that we use in our problem statement.

Definition 5 ([11]): The constant \( x_e \) is an equilibrium point of the system (5), if and only if,

\[
f(t, x_e) = D_{0}^{\beta} x_e = 0.
\]

We take the case when the equilibrium point is \( x_e = 0 \). There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables, see [1] for details.

Definition 6 ([16]): A continuous function \( \alpha : [0, t] \to [0, \infty) \) is said to belong to class-\( K \) if it is strictly increasing and \( \alpha(0) = 0 \).

Theorem 2 ([11]): Let \( x = 0 \) be an equilibrium point for the nonautonomous fractional order system (5). Assume that there exists a Lyapunov function \( V(t, x(t)) \) and class-\( K \) functions \( \alpha_i \, (i = 1, 2, 3) \) satisfying:

\[
\alpha_1(||x(t)||) \leq V(t, x(t)) \leq \alpha_2(||x(t)||), \quad (7)
\]

\[
\int_{0}^{t} V(t, x(t)) \leq -\alpha_3(||x(t)||), \quad (8)
\]

where \( \alpha \in (0, 1) \). Then the equilibrium point of system (5) is asymptotically stable.

This theorem gives sufficient conditions for asymptotic stability of fractional order systems, but in most cases is very complicated to calculate fractional derivatives explicitly, we will give simple conditions that imply asymptotic stability.

III. AN APPROACH TO THE INITIAL VALUE PROBLEM

In this section we motivate our problem to be solved according to the following observations. Consider the following fractional order system with the enough conditions such that Remark 1 is satisfied.

\[
D_{0}^{\beta} x(t) = f(t, x(t)), \quad x(0) = x_0, \quad t \geq 0. \quad (9)
\]

According to Remark 1 the problem (9) is equivalent to

\[
x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_{0}^{t} f(s, x(s))(t-s)^{\beta-1} ds, \quad t \geq 0. \quad (10)
\]

If we take the time \( t_0 \) fixed and we would like to study the solution for \( t \geq t_0 \) we can rewrite (10) as follows

\[
x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_{0}^{t_0} f(s, x(s))(t-s)^{\beta-1} ds
\]

\[
+ \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} f(s, x(s))(t-s)^{\beta-1} ds,
\]

\[
= \mu(t) + \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} f(s, x(s))(t-s)^{\beta-1} ds, \quad t \geq t_0. \quad (11)
\]

Now, If we take the integer case\(^2\), that is:

\[
Dy(t) = g(t, y(t)), \quad y(0) = y_0, \quad t \geq 0. \quad (12)
\]

\(^2D = \frac{d}{dt}\) in the usual sense
If \( g \) is a Lipschitz continuous function, the problem (12) is equivalent to

\[
y(t) = y_0 + \int_0^t g(s, y(s)) \, ds, \quad t \geq 0.
\]

(13)

If we take the time \( t_0 \) fixed and we would like to study the solution for \( t \geq t_0 \) we can rewrite (13) as follows:

\[
y(t) = y_0 + \int_0^{t_0} g(s, y(s)) \, ds + \int_{t_0}^t g(s, y(s)) \, ds, \quad t \geq t_0.
\]

(14)

Note that in (11) \( \mu \) is a function that depends the time \( t \), while in equation (14) \( \mu_1(t) \) is a constant which is usually written as \( \mu_1(t) = y(t_0) \).

The structure of the equation (14) has been related to physical interpretation and for some authors [17], [18], [19], [20]. This structure would have to be met in fractional systems, so that the initial value problem in the usual sense has to be rethought, due to that when we make translations of time in the system the initial condition is not sufficient. To solve the system is necessary to know the initial trajectory (\( \mu(t) \) in our structure). A proposal we have is precisely to consider systems with the structure of the equation (11), where \( f \) and \( \mu \) are known, and apart from this, we analyze the stability of the solution in the sense of Lyapunov. From our perspective the usual initial value problem also helps to model systems where is not possible to know the initial trajectory of the system and is a good enough tool for modelling.

As a first step to solve the stability problem (11), in this paper we analyze the case of the initial value problem in the usual sense \( \mu \) is a constant. The general case will appear elsewhere.

We consider the initial value problem for a fractional nonlinear autonomous system with \( \beta \in (0, 1) \),

\[
\frac{\partial}{\partial t} D_{t}^{\beta} x(t) = f(x(t)), \quad x(0) = x_0,
\]

(15)

where, \( x \in \mathbb{R}, f : \mathbb{R} \to \mathbb{R} \). For the system (15) we assume the following conditions:

**H1**: \( f \in C^1(\mathbb{R}) \),

**H2**: \( f(0) = 0 \),

**H3**: \( f : \mathbb{R}^+ \to \mathbb{R}^- \) and \( f : \mathbb{R}^- \to \mathbb{R}^+ \).

The assumptions above, help us to characterize different properties of the system. With the hypothesis H1 we guarantee existence of a unique differentiable solution, see Remark 1. The hypothesis H2 indicates that the equilibrium point is \( x_e = 0 \). The hypothesis H3 is very important because characterizes the dynamic properties, we discuss this later.

**Lemma 2**: If \( x_0 \in \mathbb{R}^+ (x_0 \in \mathbb{R}^-) \) and H1-H3 are met, then the solution of the system (15) satisfies \( x(t) \in \mathbb{R}^+ (x(t) \in \mathbb{R}^-) \).

**Proof.** We assume that there is a time \( \tau \) where the solution \( x(\tau) = 0 \) and iterating leads to a contradiction.

We have that \( \dot{x}(t) \in C^1[0, \tau] \), due to H1 see Remark 1, therefore if we use the equation (6) we can obtain the following expression

\[
\dot{x}(t) = \frac{d}{dt} \left( x_0 + \frac{1}{\Gamma(\beta)} \int_0^t f(x(t))(t-s)^{\beta-1} \, ds \right),
\]

(16)

\[
\dot{x}(t) = 0 D_{t}^{1-\beta} f(x(t)).
\]

(17)

The following Lemma will be used in a result of \( f \), so that, we can rewrite the right side of the equation (17).

**Lemma 3**: If H1 holds, then \( |\dot{x}(t)| \leq M t^{\beta-1}, \quad t \leq T, \) for some constant \( M > 0 \).

**Proof.** Let us define the following function for \( t \leq T \):

\[
I_{[b,T]}(s) = \begin{cases} 1 & \text{if } s \in [b,T], \\ 0 & \text{otherwise}. \end{cases}
\]

Now using hypothesis H1, see Remark 1, we have that \( \exists M_0 > 0, K > 0, \delta > 0 \) such that:

\[
|\dot{x}(t)| \leq M_0 \frac{1}{t^{1-\beta}} I_{[0,\delta]}(t) + |\dot{x}(t)| I_{[\delta,T]}(t),
\]

(18)

\[
\leq M_0 \frac{1}{t^{1-\beta}} + K \frac{1}{t^{1-\beta}} = M' \frac{1}{t^{1-\beta}},
\]

(19)

thus the proof is complete.

**Lemma 4**: Let \( f \in C^1(G) \) then \( f(x) \in AC(G) \).

**Proof.** By hypothesis \( f \in C^1(G) \), so for a \( t, \) and using the fundamental theorem of calculus we obtain:

\[
f(x(t)) = f(x(a)) + \int_a^t f'(x(s)) \dot{x}(s) \, ds,
\]

applying the limit as \( a \to 0 \).

\[
f(x(t)) = f(x(0)) + \lim_{a \to 0} \int_a^t f'(x(s)) \dot{x}(s) \, ds.
\]

Now we can apply the Lemma 3 and the dominated convergence theorem:

\[
f(x(t)) = f(x(0)) + \int_0^t f'(x(s)) \dot{x}(s) \, ds,
\]

this proves that \( f \in AC(G) \).

Therefore by Lemma 3 the equation (17) can be written as [13]:

\[
\dot{x}(t) = D_{t}^{1-\beta} f(x(t)).
\]
Now we give some results about Lemma 6:

Where the last expression is obtained taking the variable $t$ from 20 and Lemma 3 we have

\[ \Gamma(\beta-x(t)) = 1 \]

is a constant. Applying the change of variable $u = s/t$ we have:

\[ \Gamma(\beta-x(t)) \leq 1 \]

Hence for continuity of the right side of (21) there exist to > 0 such that $\dot{x}(t) \in \mathbb{R}^-$ in (0, t0).

Lemma 6: If $x_0 \in \mathbb{R}^+$ ($x_0 \in \mathbb{R}^-$) and H1-H3 hold, then $\dot{x}(t) \in \mathbb{R}^-$ ($\dot{x}(t) \in \mathbb{R}^+$) for all $t \in (0, t_0)$.

Proof. Lemma 5 is used to build a demonstration by contradiction.

Lemma 7: If H1 holds, then for $f(x(t))$ we have that

\[ I_{x_0}^{1-\beta} D_{x_0}^{1-\beta} f(x(t)) = f(x(t)). \]

Proof. Using the Lemma 4 when $s, t \in (0, T], s < t$,

\[ f(x(t)) = f(x(0) + \int_0^t f'(x(r)) \dot{x}(r) dr, \]

\[ f(x(s)) = f(x(0) + \int_0^s f'(x(r)) \dot{x}(r) dr, \]

we have

\[ \int_{x_0}^{x(t)} - f(x(s)) = \left| \int_{x_0}^{s} f(x(t)) \dot{x}(r) dr \right| \leq M \int_{x_0}^{s} \dot{x}(r) dr, \]

\[ \leq M M' \frac{1}{s^{1-\beta}} \int_{x_0}^{s} t - dr = MM' \frac{t - s}{s^{1-\beta}}. \]

Where the inequality (27) holds due to Lemma 3, applying the hypothesis H1; in such a way that:

\[ \int_{x_0}^{s} f(x(t)) - f(x(s)) \frac{t - s}{(t - s)^{2-\beta}} ds \leq C \int_{0}^{t} \frac{ds}{s^{1-\beta}(t - s)^{1-\beta}} \leq C \frac{1}{(1-\beta)} \int_{x_0}^{t} \dot{x}(s) (t - s)^{-\beta} ds, \]

Where the last expression is obtained taking the variable $u = s/t$ as in the Lemma 5. Therefore applying Theorem 1, Dominated convergence theorem and the Lemma 1 to the equation (28) we conclude the proof.

Now to talk about stability we need to ensure the existence and uniqueness of the solution $x(t)$ for the system (15) when $t \to \infty$, which is guaranteed by the following result.

Lemma 8: If we have a system as (15) and we suppose that H1-H3 are met, then the solution $x(t)$ of the system (15) exists and is unique for all $t \geq 0$.

Proof. Using the behavior of the solution and its derivative are satisfied the hypotheses of Remark 1.

We conclude the results with the following theorem for global asymptotic stability of a family of fractional order nonlinear autonomous systems, which is the main result in this paper.

Theorem 3: Let $x_0 = 0$ be an equilibrium point of the system (15) where $f$ is such that H1-H3 hold. Also assume that there exists a Lyapunov function $V(x(t))$ with nondecreasing derivative and class-K functions $\alpha_i$ ($i=1,2,3$) satisfying

\[ \alpha_1 \|x(t)\| \leq V(x(t)) \leq \alpha_2 \|x(t)\|, \]

\[ \frac{dV(x(t))}{dx} f(x(t)) \leq -\alpha_3 \|x(t)\|, \]

then the equilibrium point of system (15) is globally asymptotically stable.

Proof. We calculate the following fractional derivative

\[ \phi D_{x_0}^{\beta} V(x(t)) = \frac{1}{1-\beta} \int_0^t \frac{f(x(t))}{dx} \dot{x}(s) (t - s)^{-\beta} ds. \]

Case a). If $x_0 = 0$, as we have asked for the origin to be an equilibrium point of the system (15) we have $x(t) = x_0 = x_e$ and hence $x_e = 0$ is globally asymptotically stable.

Case b). If $x_0 \in \mathbb{R}^+$ then $x(t) > 0$ by Lemma 2, and $x(t) < 0$, $\forall t$ due to Lemma 6, therefore $\lim_{t \to \infty} x(t) = \epsilon$ exists. Let us suppose that $\epsilon > 0$ thus $\epsilon < x(t) < 0$. Using that $V(x(t))$ has a nondecreasing derivative then $\frac{dV}{dx} \leq 0$ and the equation (31) implies:

\[ \phi D_{x_0}^{\beta} V(x(t)) \leq \frac{1}{1-\beta} \int_0^t \frac{f(x(t))}{dx} \dot{x}(s) (t - s)^{-\beta} ds. \]

Using the relation (17) and the Lemma 7:

\[ \phi D_{x_0}^{\beta} V(x(t)) \leq \frac{1}{1-\beta} \int_0^t \frac{dV}{dx} \frac{f(x(t))}{dx} (t - s)^{-\beta} ds, \]

\[ = \frac{dV}{dx} \epsilon f(x(t)) + \frac{dV}{dx} (\phi D_{x_0}^{\beta} f)((x(t))) \leq -\alpha_3 \|x(t)\|, \]

Where (32) is followed by (30), note that $\alpha_3 \|x(t)\|$ remains a class-K function, but by Theorem 2 we have that
lim_{t \to \infty} x(t) = 0 which contradicts our supposition that \( \epsilon > 0 \) then \( \epsilon = 0 \) and therefore the equilibrium point \( x_e = 0 \) is globally asymptotically stable for \( x_0 > 0 \).

\textbf{Case c).} If \( x_0 \in \mathbb{R}^- \), and using the Lemmas for this case, the proof is similar to \textbf{Case b)}. And thus we have concluded the proof of our main result.

IV. EXAMPLES AND SIMULATIONS

In this section we present the application of the previous results, where we use numerical solutions of the kind used in [21].

\textbf{Example 1:} Let us take the following system with \( \beta \in (0,1), n \in \mathbb{N} \) and \( a > 0 \).

\[
\begin{align*}
\frac{\partial}{\partial t} x(t) &= -ax(t)^{2n+1} \quad x(0) = x_0 \\
(33)
\end{align*}
\]

In this example \( f(x(t)) = -ax(t)^{2n+1} \), so we have that \( H1-H3 \) hold, let \( V(x(t)) = x(t)^2 \), so \( \frac{dV(x(t))}{dx} = 2x(t) \) is nondecreasing, then.

\[
\begin{align*}
\frac{1}{2} \|x(t)\|^2 &\leq V(x(t)) \leq 2\|x(t)\|^2, \\
(34)
\end{align*}
\]

\[
\begin{align*}
\frac{dV(x(t))}{dx} f(x(t)) &= 2x(t) (-ax(t)^{2n+1}) = -2ax(t)^{2n+2} \\
&\leq -a\|x(t)\|^{2n+2} \\
(35)
\end{align*}
\]

hence the equilibrium point \( x_e = 0 \) is globally asymptotically stable, see Fig. 1.

Now when \( n = 0 \) we have that:

\[
\begin{align*}
\frac{\partial}{\partial t} x(t) &= -ax(t) \quad x(0) = x_0, \\
(36)
\end{align*}
\]

we know that the system in globally asymptotically stable if and only if \(-a < 0 \) [4]. This is true because we take \( a > 0 \). In this case \( f(x) = -ax \) and hypotheses \( H1-H3 \) hold immediately, note the importance of our hypothesis \( H3 \). Even more, our main result and the classical result [4] are equivalent.

On the other hand the solution of (36) is given in [13].

\[
\begin{align*}
x(t) &= x_0 e_{\alpha,1} (-a t^\beta). \\
(37)
\end{align*}
\]

If we take the time \( t_0 \) fixed and we would like to study the solution for \( t \geq t_0 \) and we thing in the similar manner like the integer case, we have:

\[
\begin{align*}
\int_{t_0}^{t} x(t) = -ax(t), \quad x(t_0) = x_{t_0}, \\
(38)
\end{align*}
\]

the solution of equation (38) is given in [13].

\[
\begin{align*}
x(t) &= x_{t_0} e_{\alpha,1} (-a(t-t_0)^\beta). \\
(39)
\end{align*}
\]

For the integer case we have:

\[
\begin{align*}
 Dy(t) &= -ay, \quad y(0) = y_0, \\
(40)
\end{align*}
\]

\[
\begin{align*}
 Dy(t) &= -ay, \quad y(t_0) = y_{t_0}, \\
(41)
\end{align*}
\]

with the following respective solutions

\[
\begin{align*}
y(t) &= y_0 e^{-at} \\
(42)
\end{align*}
\]

\[
\begin{align*}
y(t) &= y_{t_0} e^{-a(t-t_0)} \\
(43)
\end{align*}
\]

Note that the equations (42) and (43) are equivalent because \( y_{t_0} e^{at_0} = y_0 \) but the equations (37) and (39) are not equivalent because \( E_{\alpha,1} \) does not satisfies the semi-group property. This show the importance of \( \mu(t) \) in the propose structure equation (11). Despite this the equilibrium point for the systems (36) and (38) is globally asymptotically stable.

\textbf{Example 2:} Let us take the following system, with \( \beta \in (0,1) \).

\[
\begin{align*}
\frac{\partial}{\partial t} x(t) &= -|x(t)|, \quad x(0) = x_0, \\
(44)
\end{align*}
\]

in this example \( f(x(t)) = -|x(t)| \).

\textbf{Case a.)} If \( x_0 = 0 \), then \( x \equiv 0 \). Thus equilibrium point \( x_e = 0 \) is asymptotically stable.
Case b) Now we assume $x_0 > 0$, then $f(x(t)) = -x(t)$, and we have that H1-H3 hold, let $V(x(t)) = x(t)^2$, then
\[
\frac{dV(x(t))}{dx} = 2x(t) \text{ is nondecreasing then }
\]
\[
\frac{1}{2} \|x(t)\|^2 \leq V(x(t)) \leq 2\|x(t)\|^2.
\] (45)

From Theorem 3 the equilibrium point $x_e = 0$ is asymptotically stable.

Case c) Finally, when $x_0 < 0$ we have $f(x(t)) = x(t)$ and H1-H2 hold, but H3 is not true, because $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ so we can not conclude in an analytical form that the equilibrium point $x_e = 0$ is asymptotically stable for $x_0 < 0$. In particular, trough the numerical solution we can observe that in this case the equilibrium point $x_e = 0$ is unstable, see Figure 2b.

![Figure 2: Here $\beta = 0.8$, and the simulation time is $t = 10$ sec.](image)

V. CONCLUSION

This paper present sufficient conditions for global asymptotic stability of a family of autonomous nonlinear fractional dynamical systems, these conditions are similar to those needed for integer order systems, so the verification of the stability of such systems is straightforward. As possible future works one can study the problems of global stability for non autonomous systems and the case of vector systems.

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