Deterministic discrete-event representations of linear continuous-variable systems

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Conditions are derived, under which the nondeterministic discrete-event behaviour of a quantised continuous-variable system becomes deterministic.

Abstract

The paper concerns linear continuous-variable systems whose state can be measured only through a quantiser. As for an observer the system together with the quantiser behaves like a discrete-event system, the problem of representing this system by some deterministic finite state machine is considered. The main difficulty of this representation problem occurs due to the nondeterminism introduced by the state quantisation. It is investigated under what conditions on the linear continuous-variable system and on the state quantisation the discrete-event behaviour is deterministic. The main results are sufficient conditions for this property.

Keywords: Quantisation; Discrete-event representation; Hybrid systems

1. Introduction

Until recently, the theories of continuous-variable and of discrete-event systems have been developed quite independently. They have resulted in different modelling formalisms, analysis and simulation methods and design procedures. Now, hybrid systems have become the headline of a new research direction which tries to bring both lines of systems theory together.

One among several motivations for this kind of research results from the fact that control tasks of, for example, chemical or electrical processes have to be solved on different levels of abstraction. For the primary control loops, which have to stabilise given operation points, the systems under consideration act continuously or at least quasi-continuously, where the attribute “continuous” concerns both the signal values and the time.

On the other hand, for supervisory control tasks, which concern the selection of the operation point, the supervision of the main signals, diagnosis of possible faults, etc. the systems are considered from a more abstract point of view. Only rough assessments of the signal values are relevant for these tasks. Consequently, the process is considered as a discrete-event system whose global state switches from one value to another. The system is discrete, where the attribute “discrete” concerns both the signal values and the time. The models used to solve supervisory control tasks have typically the form of Petri nets or automata.

For both kinds of control tasks, the system under consideration is the same, but the representation differs. Therefore, the question occurs how a discrete-event model can be set up for a continuous-variable system. This abstraction process, which combines both viewpoints of systems theory, is the subject of this paper.

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occurrence of nondeterminism concerning the future behaviour of the system has to be considered if systems with virtually continuous-variable behaviour shall be described from a discrete-event point of view.

The problem to be investigated is illustrated in Fig. 1. The core of the system under consideration is a continuous-variable system, which can be described, for example, by means of a state-space model. The system is controlled through an injector, which transforms the discrete input signal \([u(t_k)]\) into a real-valued signal \(u(t)\) that is continuous in time \(t\). On the other hand, the system output \(y(t)\) can only be measured by means of a quantiser. Consequently, the measurement information \([y(t_k)]\) has discrete values and occurs only at discrete-time instances \(t_k\). The overall system depicted in Fig. 1 is called the quantised system.

From the viewpoint of the observer who knows only the discrete input and output values, the system behaves as a discrete-event system. The question to be answered is:

How can a deterministic discrete-event model of the quantised system be set up?

As an answer to this representation problem, relations between linear continuous-variable systems together with the quantiser and deterministic automata as representation of the quantised system will be given in this paper.

1.1. Outline of the paper

When abstracting a discrete-event representation from a continuous-variable system description the nondeterminism of the discrete-event behaviour occurs as an important phenomenon. How large the nondeterminism is depends on the properties of the continuous-variable system and on the quantiser and injector used. Uncertainties concerning the system behaviour result from three sources. First, the initial state is only qualitatively known. This uncertainty is propagated from one time step to the next. Second, the qualitative information about the input signal brings about additional uncertainties in each time step. The third source of uncertainty results from the qualitative nature of the output measurement. The main focus of this paper is under which conditions the nondeterminism that is brought about by these uncertainties vanishes.

For the determination of the temporal behaviour of the quantised system the easiest problem occurs if the system has no input and the state rather than the output can be qualitatively observed. We restrict the main part of this paper to this situation and show that even in this “easiest” case nearly all systems have a nondeterministic behaviour. Consequently, in the more general situation with qualitative inputs and the restriction of the measurements to outputs rather than the system state, nondeterminism will occur for almost all systems.

The paper is organised as follows. In Section 2, the quantised system is described as a composition of a continuous-variable linear system and a quantiser that brings about a partitioning of the state space. In Section 3 the problem of nondeterministic behaviour of the quantised system occurring under arbitrary partitioning of the signal spaces is discussed.

Section 4 deals with the question for which partitionings of the state space the discrete event behaviour is deterministic and, thus, makes an exact representation by means of a finite state machine possible. The main results show that natural partitionings and granulated natural partitionings introduced in Algorithms 1–4 yield deterministic behaviour of the quantised system. As the discussion of these results and the example presented in Section 7 shows, such partitionings have a very complex structure. Hence, the discrete-event behaviour of practically relevant systems is, in general, nondeterministic.

1.2. Relevant literature

This paper is concerned with links between continuous-variable and discrete-event systems theory by investigating how discrete-event models of continuous-variable systems can be set up. Similar investigations are reported in the literature on hybrid systems and on qualitative modelling.

In the literature on hybrid systems those papers are relevant that concern the discrete-event modelling of continuous-variable (sub)systems or of the overall hybrid system. Lunze (1992) proposed to use Petri nets as discrete abstractions of the continuous system. Lemmon et al. (1996) used a discrete-event description of a hybrid system to design controllers by applying the supervisory control theory proposed by Wonham. Raisch (1997) showed that by identifying sequences of output symbols with the automaton states, discrete-event descriptions of different model accuracy can be derived for the quantised system. Franke (1994) considered deterministic automata as discrete-event descriptions of continuous-variable systems and derived analysis and design methods in analogy to the well-known control systems theory.

The second relevant line of research concerns qualitative modelling with surveys given by Weld and de Kleer (1990) and Lunze (1995, 1998b). Lunze (1994) and Lichtenberg (1998) adopted the discrete-time viewpoint and
2. Quantised continuous-variable systems

2.1. Quantitative system description and state-space quantisation

In this paper, linear autonomous continuous-variable systems are considered which can be described by means of the state-space model

\[ \dot{x}(t) = A_s x(t), \quad x(0) = x_0, \]

(1)

where \( x(t) \in \mathbb{R}^n \) denotes the state vector. For any given initial state \( x_0 \) the system (1) generates the trajectory

\[ x(t) = e^{At} x_0. \]

(2)

In the following, we will consider the system (1) in a quantised state space (Fig. 2). The quantiser maps the state space onto a finite set \( \mathcal{Q}_x = \{0, 1, 2, \ldots, N\} \) of qualitative values. That is, the quantiser introduces a partitioning of the state-space \( \mathbb{R}^n \) into a finite number of disjoint sets \( \mathcal{Q}_x(i) \):

\[ \mathcal{Q}_x(i) \cap \mathcal{Q}_x(j) = \emptyset \quad \forall i \neq j, \ i, j \in \mathcal{Q}_x. \]

(3)

\[ \bigcup_{i=0}^{N} \mathcal{Q}_x(i) = \mathbb{R}^n. \]

(4)

\( \mathcal{Q}_x(i) \) denotes the set of states \( x \in \mathbb{R}^n \) with the same qualitative value \( i \). The mapping invoked by the quantiser is symbolised by \( \lfloor . \rfloor \). That is,

\[ \lfloor x \rfloor = i \iff x \in \mathcal{Q}_x(i). \]

(5)

The sets \( \mathcal{Q}_x(i) \ (i = 1, \ldots, N) \) are assumed to be bounded while \( \mathcal{Q}_x(0) \) is the unbounded “remaining” subset of \( \mathbb{R}^n \). For the bounded sets, \( \delta \mathcal{Q}_x(i) \) denotes the hull of \( \mathcal{Q}_x(i) \).

Fig. 2 illustrates the state quantisation. The shaded region represents the set of all states with qualitative value 4.

It is assumed in the sequel that only the qualitative state information is available. We call \( \lfloor x \rfloor \) the qualitative state and the trajectory \( [x(t)] \) the qualitative state trajectory.

2.2. Temporal quantisation

The system trajectories are considered at discrete-time instances \( t_k \). That is, instead of the continuous-time trajectory \( [x(t)] \) the sequence

\[ [X(0..T)] = ([x(0)], [x(t_1)], \ldots, [x(T)]) \]

(6)

is used to represent the system behaviour, where \( T \) denotes the time horizon. \([X(0..T)]\) describes the behaviour of the quantised system. It is also said to be the qualitative behaviour of the system (1).

In the following two temporal quantisations of the system are considered. First, the system is considered as a sampled-data system and second as a discrete-event system where the interesting time instances \( t_k \) are generated by the system itself.

2.2.1. Discrete-time quantised systems

The most common temporal quantisation results from sampling of signals at equidistant time instances, where the interesting time instances \( t_k \) are given by \( t_k = kT_s \) with \( T_s \) being the sampling time. Then the trajectory (6) becomes

\[ [X(0..T)] = ([x(0)], [x(T)], \ldots, [x(TT_s)]) \]

and can be written as

\[ [X(0..T)] = ([x(0)], [x(1)], \ldots, [x(T)]) \]

(7)

with a simpler enumeration of the time instances. In the further investigation the quantised system will be called the discrete-time quantised system if the temporal abstraction is based on sampling.

Trajectory (7) results from a quantisation of the discrete-time trajectory of the continuous-variable system,
which is the solution of the discrete-time state-space model

\[ x(k + 1) = A_d x(k), \quad x(0) = x_0 \]  

of system (1) with the regular matrix \( A_d = \exp(A_d T) \). For given initial state \( x_0 \) the quantised trajectory is unique.

Fig. 3a shows the quantitative and the qualitative trajectories of an oscillating second-order system. The sequence \( x_1(k) \) of the first state variable is shown over the sampling time \( k \). The state quantisation is depicted by intervals that show which values the first state variable \( x_1 \) can assume if system (1) is in the corresponding qualitative states 1..6. For the given initial state, the qualitative trajectory \( [X(0..27)] \) of the quantised system is illustrated by the grey rectangles. The rectangle shown for time \( k \) signifies to which interval \( x_1(k) \) belongs. The qualitative dynamics of the quantised system is represented by the rectangles. Due to the quantiser, the quantitative information provided by the circles is not available for an observer outside the quantised system.

2.2.2. Discrete-event quantised systems

In the discrete-event context, the time instances \( t_k \) are determined in accordance with the system behaviour. An event is a change of the qualitative state.

**Definition 1.** An event \( e_{ij} \) is said to occur in system (1) at time \( t_k \) if the relations

\[
[x(t_k)] = i \quad \text{and} \quad [x(t_k - \delta t)] = j
\]  

hold for some \( i, j \in \mathcal{N}_x \) for \( \delta t \to 0 \) and \( i \neq j \).

In the discrete-event setting, the quantiser does not only determine which event occurs but also the occurrence time \( t_k \) of the event. The events are denoted by \( e(t) \). In Fig. 2 for example \( e(t) = e_{15} \) signifies that the continuous-variable system changes its qualitative state from 5 to 1 at time \( t \).

The event sequence is denoted by

\[ E(1..T) = (e(t_1), e(t_2), \ldots, e(t_T)) \]

In discrete-event modelling the quantitative information about time is neglected and only the order of the events is kept. Hence, the sequence of events is written as

\[ E(1..T) = (e(1), e(2), \ldots, e(T)). \]

For any given initial state \( x_0 \) the quantised system generates a unique event sequence \( E(1..T) \). If the discrete-event point of view is adopted, the system under consideration is called a discrete-event quantised system.

Fig. 4a shows a continuous trajectory for quantitatively given initial state (thick line) together with the event sequence \( E(1..T) = (e_{21}, e_{32}, e_{43}, \ldots) \) generated by the system.

From any sequence of events the sequence of qualitative states

\[ [X(0..T)] = ([x(0)], [x(1)], [x(2)], \ldots, [x(T)]) \]  

can be derived and vice versa. Here \([x(k)]\) denotes the qualitative state after the \( k \)th event has occurred.

2.3. Comparison of both representations

Formally, representations (7) and (10) are identical. The system trajectory is described at discrete time instances, which are enumerated by \( k \), by the current qualitative state \([x(k)]\). This representation enables us to investigate the discrete-time and the discrete-event quantised system in close analogy.

As a consequence of the different time-bases inherent in the different representations, the discrete-time system may produce identical qualitative states at successive
time instances, whereas the qualitative state of the discrete-event system changes between successive time instances. The discrete-event quantised system may eventually stop its event sequence. This phenomenon occurs if the continuous-variable system remains within a certain partition $Q_{x}(i)$ of the state space. That is why for a given time horizon $T$ the state trajectory (10) may not exist.

3. Nondeterminism of the qualitative dynamics

The main difficulties in finding a qualitative representation of the quantised system result from the nondeterminism of the qualitative behaviour. In the preceding sections, the qualitative state sequence $[X(0..T)]$ has been considered for known initial state $x_0$ and it has been stated that this sequence is unique. However, if the state of the system can only be qualitatively measured, then the same holds true for the initial state $x_0$. Hence, the qualitative trajectories $[X(0..T)]$ have to be considered for qualitatively given initial state $[x_0]$.

The inherent difficulty is, that for qualitatively given initial state the trajectory of the quantised system is not defined uniquely. The quantised system behaves nondeterministically, because it may produce any trajectory $[X(0..T)]$ of the set of possible qualitative state trajectories denoted by $[\tilde{X}(x_0)]$:

$$[\tilde{X}(x_0)] = \{[X(0..T)] \mid \text{eq. (1) holds for some } x_0 \in \mathcal{A}(x_0) \}.$$  \hspace{1cm} (11)

The fact that $[\tilde{X}(x_0)]$ is, in general, not a singleton has been proved for linear discrete–time autonomous systems by Lunze (1994). Its validity for the discrete–event setting will become obvious in the considerations below.

Fig. 3b illustrates the nondeterminism of the qualitative discrete-time behaviour. If the system starts in the qualitative state 1, the set of all trajectories has to be considered for which $[x_0] = 1$ holds. Elements of this set are depicted as circles in the figure. Obviously, at most of the time instances the system can assume different qualitative states. For example, for $k = 7, 8, 9$ both $[x_1] = [x_1]_5$ and $[x_1] = [x_1]_6$ may be true. The system has a nondeterministic behaviour. Note that the set of qualitative trajectories shown in part (b) of the figure includes the qualitative trajectory depicted in part (a).

The nondeterminism of discrete–event quantised systems is illustrated in Fig. 4. Part (a) shows a set of trajectories that start in the qualitative state 1. As the trajectories cross a different number of interval bounds, they generate different event sequences. In particular, for some trajectories the events $e_{55}$ and $e_{56}$ do not occur. Therefore, the qualitative trajectory becomes nondeterministic after the fifth event. Part (b) of the figure shows the event sequence. When comparing both parts of the figure one has to keep in mind that the events are enumerated by $k$.

Whether a given system is deterministic or not depends on the system properties and on the state–space quantisation. In the following, it will be investigated under which conditions a partitioning of the state–space can be found such that the system is deterministic.

4. A condition for deterministic partitionings

Obviously, the behaviour of the quantised system is deterministic if it maps any given qualitative state $\mathcal{A}(i)$ in one time step into a single other qualitative state $\mathcal{A}(j)$.

To investigate this property the functions $f_d$ and $f_e$ for the discrete-time or the discrete–event quantised system,
respectively, are introduced:

\[ f_\Delta(Q_i(t)) = \{ \bar{x} = A_\Delta \bar{x}, \bar{x} \in Q_i(0) \} \]  
(12)

\[ f_\gamma(Q_i(t)) = \{ \bar{x} = e^{A_\gamma t} \bar{x}, \forall \bar{x} \in Q_i(t_1, t_2) \}. \]  
(13)

The function \( f_\Delta \) determines all states that the system can reach from the set \( Q_i(t) \) in one time step \( T_\Delta \). \( f_\gamma \) generates all states that system (8) can assume between the next two successive events if it starts its movement within the set \( Q_i(t) \). Fig. 5 illustrates these functions.

The following definition is formulated simultaneously for both the discrete-time and the discrete-event dynamics of the quantised system by identifying \( f_\Delta \) and \( f_\gamma \) by the common symbol \( f \).

**Definition 2.** The dynamics of a quantised system is deterministic if there exists for each \( i \) exactly one \( j \) such that

\[ f(Q_i(t)) \subseteq Q_j(t) \]  
(14)

holds.

In the following, we will propose partitionings of the state space such that Eq. (14) holds. For their description we will use the bounds between the different qualitative states \( i \) and \( j \) which are denoted by

\[ \delta Q(i, j) = \delta Q(i) \cap \delta Q(j), \; i \neq j. \]  
(15)

5. Deterministic partitionings for discrete-time quantised systems

5.1. Determinism and cells containing unstable fixed points

Starting with the discrete-time system we first have to investigate unstable fixed points within the state space. Assuming that \( |\lambda_1| \neq 1 \) holds for all eigenvalues \( \lambda_1 \) of the system matrix \( A_\Delta \), the linear system has the finite fixed point \( x_{fix} = 0 \). If the system is stable this fixed point is stable and all trajectories end in this point. Hence, there always exists a bounded set \( \bar{x}_{fix} \) of points including \( x_{fix} \), e.g. every sphere centered in \( x_{fix} \), for which

\[ x(0) \in \bar{x}_{fix} \Rightarrow x(k) \in \bar{x}_{fix} \; \forall k > 0 \]  
(16)

holds. If we identify this set with the qualitative state \( i_{fix} \in N_{fix}, i_{fix} \neq 0 \), i.e. \( x_{fix}(t_{fix}) \) the system remains in this qualitative state for all times once it has reached it, i.e.

\[ [x(k)] = i_{fix} \Rightarrow [x(k + 1)] = i_{fix}. \]

For stable systems the unstable fixed point is infinity. In Section 2.1 we introduced a partitioning of the state space into \( N \) bounded and one unbounded set of quantitative states. In the following, we will call the unbounded set sink cell of the system. Due to the fact, that infinity is a fixed point there are always states staying in the sink cell. On the other hand, the system is stable and due to \( i_{fix} \neq 0 \) there are also states leaving the sink cell in one time step, moving towards the stable fixed point. Consequently, the sink cell behaves always nondeterministically.

If the system is unstable, the stable fixed point lies in the sink cell \( \bar{x}_{fix}(0) \) in which the system remains once it has reached it. The cell \( \bar{x}_{fix} \) containing the finite unstable fixed point \( x_{fix} \) will be left with certainty (except for the case that the initial state of the system is identical to this point), but not necessarily in one sampling time step. Hence, for a discrete–time system with finite sampling time \( T_\gamma \) it is always possible to stay in this qualitative state no matter how small the set \( \bar{x}_{fix}(t_{fix}) \) is chosen. Therefore, the quantised discrete-time system is always nondeterministic for unstable systems. A deterministic partitioning would require an infinite number of qualitative states.

For this reasons we will exclude the qualitative states containing unstable fixed points from the following discussion. We restrict the property of deterministic

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Fig. 5. Mapping of a qualitative state for discrete-time (a) and discrete-event (b) quantised systems.
behaviour to all qualitative states containing no unstable fixed points.

5.2. Natural partitioning

The following considerations are based on the idea of choosing an initial set of points and determining which points transit into the initially chosen set in one time step. For stable systems the initial set \( \mathcal{A}_{\text{fix}} \) must contain the fixed point and fulfills condition (16). For unstable systems the initial set is the sink cell.

Fig. 6 shows an example for a one-dimensional stable system. As the system behaviour is symmetric with respect to \( x = 0 \), only the positive part of the state space is depicted. The aim of the partitioning process is to find the sets of continuous states which map onto the succeeding set of states in one time step and onto the set \( \mathcal{A}_{\text{fix}} \) for large \( k \) (upper part of Fig. 6). For example,

\[
\mathcal{A}_2(2) = \{ \mathcal{A}_2(x) | x \in \mathcal{A}_1(3) \}
\]

has to hold. As for a linear system the transition function \( f_k(x) = A_k x \) is continuous, this aim can be reached if the bounds \( \delta 2(i,j) \) are found that are mapped onto the next bound closer to the fix point by applying \( f_k(x) \) once and onto \( \delta 2_{i+1} = \delta 2_0 \) with further iterations.

Since the system matrix \( A_0 \) is invertible, the bounds can be found by applying the mapping \( A_0^{-1} \) to \( \delta 2_0 \), iteratively. This procedure is shown in the lower part of Fig. 6.

For arbitrarily chosen interval \( \mathcal{A}_2(1) \) the bound \( \delta 2_0 \) of \( \mathcal{A}_2(1) \) is mapped by \( f_1(x) \) to obtain \( \delta 2_1 \)

\[
\delta 2_1 = \{ A_0^{-1} x | x \in \delta 2_0 \}.
\]

In an \( n \)-dimensional state space the natural partitioning is determined by iterating the mapping \( (n-1) \)-dimensional bound \( \delta 2_0 \) of the initial set with the inverse of the transition function \( f_1^{-1}(x) = A_0^{-1} x \).

Algorithm 1. A deterministic discrete-time partitioning can be derived by the following algorithm:

(1) Determine the bound \( \delta 2_0 \) of the set \( \mathcal{A}_{\text{fix}} \) that contains the finite fixed point and fulfills condition (16) if the system is stable or the bound of the sink cell \( \mathcal{A}_0(0) \) if the system is unstable, respectively.

(2) Iterate for \( \gamma = 0, 1, \ldots, N - 2 \):

\[
\delta 2_{\gamma+1} = f_1^{-1}(\delta 2_\gamma) = \{ A_0^{-1} x | x \in \delta 2_\gamma \}.
\]

Definition 3. A partitioning defined by the set of bounds \( \mathcal{A}_\gamma = \{ \delta 2_\gamma, \gamma = \{ 0, 1, 2, \ldots, N - 1 \} \} \) is called natural discrete-time partitioning.

With this partitioning each qualitative state, except for the qualitative state containing an unstable fixed point, has exactly one qualitative successor state. The graph of the resulting automaton is a chain of nodes connected in one direction (cf. Fig. 7b). If the underlying system is stable the chain starts in the initial state and ends in the stationary qualitative state. If the system is unstable the direction is vice versa.

Fig. 7a gives an example for a natural partitioning. The point \((0,0)\) is the stable fixed point. First a natural partitioning of the state space is generated. The initial set \( \mathcal{A}_{\text{fix}} \) with bound \( \delta 2_0 \) contains the fixed point and fulfills condition (16). By mapping \( \delta 2_0 \) with \( A_0^{-1} \) the bound \( \delta 2_1 \) is created. From \( \delta 2_1 \) we determine \( \delta 2_2 \) and so on, i.e. the bound \( \delta 2_0 \) is iteratively mapped outwards thereby defining a natural partitioning of the state space.

5.3. Granulated natural partitioning

In this section additional bounds will be introduced to a natural partitioning such that the resulting quantised
system remains deterministic. The idea behind granulated natural partitioning is that a “ring” \( \mathcal{D}_y \) given by two neighbouring bounds of a natural partitioning gets divided into an arbitrary number of subsets while the resulting partitioning still yields deterministic system behaviour if the additionally introduced bounds are mapped iteratively by \( f_{\mathcal{D}}^{-1} \).

Let \( \mathcal{D}_y \) be the set of points given by two neighbouring bounds of a natural partitioning \( \mathcal{D}_{\gamma} \) and \( \mathcal{D}_{\gamma+1} \), i.e.

\[
\mathcal{D}_y = \mathcal{D}_{\gamma} \cap \mathcal{D}_{\gamma+1},
\]

where \( \mathcal{D}_{\gamma} \) denotes the outer area of the closed contour \( \mathcal{D}_\gamma \), including infinity and \( \mathcal{D}_{\gamma+1} \) the inner area of the contour \( \mathcal{D}_\gamma \) including the contour itself.

**Algorithm 2.** An extended deterministic discrete-time partitioning can be derived by the following algorithm:

1. Determine a natural discrete-time partitioning consisting of a set of bounds \( \mathcal{D}_d \).
2. Add a set of bounds \( \mathcal{D}_b \) within any set of points \( \mathcal{D}_y \) with the property that these bounds introduce a partitioning of \( \mathcal{D}_y \) into an arbitrary number \( K_y \) of cells:

\[
\mathcal{D}_b: \mathcal{D}_y \rightarrow \{ \mathcal{D}_y(1), \mathcal{D}_y(2), \ldots, \mathcal{D}_y(K_y) \},
\]

\[
\bigcup_{j=1, K_y} \mathcal{D}_y(j) = \mathcal{D}_y,
\]

\[
\mathcal{D}_b = \{ \delta \mathcal{D}_y(i, j) = \delta \mathcal{D}_y(i) \cap \delta \mathcal{D}_y(j) \},
\]

(18)

(19)

(20)

3. Iterate for \( \zeta = 1, 2, \ldots, N - 1 \n\)

1. Determine \( f_{\mathcal{D}}^{-1} (\mathcal{D}_b(\zeta - 1)) = \{ f_{\mathcal{D}}^{-1}(x) \mid x \in \mathcal{D}_b(\zeta - 1) \} \)

2. Optionally add a set \( \mathcal{D}_b^+ \) of bounds such that

\[
\mathcal{D}_b^+: \mathcal{D}_b \rightarrow \{ \mathcal{D}_b^+(1), \mathcal{D}_b^+(2), \ldots, \mathcal{D}_b^+(K_y) \}, \quad K_y \geq K_{\zeta - 1},
\]

(21)

(22)

(23)

defines a partitioning of \( \mathcal{D}_b^+ \).

3. Let \( \mathcal{D}_b^+ = f_{\mathcal{D}}^{-1} (\mathcal{D}_b(\zeta - 1)) \cup \mathcal{D}_b^+ \).

**Definition 4.** A partitioning defined by the set of bounds \( \mathcal{D}_b = \bigcup \mathcal{D}_b \cup \mathcal{D}_d \) is called granulated natural discrete-time partitioning.

That is, a granulated natural partitioning can be generated by choosing a set of points \( \mathcal{D}_y \) from a natural partitioning, applying an arbitrary partitioning to \( \mathcal{D}_y \) and mapping the additionally introduced bounds \( \mathcal{D}_y \) iteratively with the inverse of the transition function \( f_{\mathcal{D}}^{-1}(x) \). Additionally, after each mapping further bounds \( \mathcal{D}_b^+ \) may be added to the mapped set \( \mathcal{D}_b(\zeta - 1) \).

6. Deterministic partitionings for discrete-event models

If a deterministic discrete-event model is to be found for the continuous-variable system one has to bear in mind that the transition times between succeeding events may vary arbitrarily, because the occurring events determine the sampling times. Therefore there exists an additional degree of freedom in finding a partitioning which yields a deterministic behaviour of the quantised system.
6.1. Singular points

First, the set \( \mathcal{A}_{\text{fix}} \) containing the fixed point of the continuous-variable system is investigated. If the system is stable, it approaches this set from any given initial state. Once this set has been entered, it will never be left again. Therefore, no further event will occur and the behaviour of the quantised system is given by some finite sequence of qualitative states. For the set \( \mathcal{A}_{\text{fix}} \) with its boundary \( \delta \mathcal{A}_{\text{fix}} \) the following condition holds:

\[
x(0) \in \mathcal{A}_{\text{fix}} \Rightarrow x(t) \in \mathcal{A}_{\text{fix}} \quad \forall t > 0
\]

\[
\forall x \in \delta \mathcal{A}_{\text{fix}}: \dot{x} \cdot \text{sign}(x) < 0,
\]

where

\[
\text{sign}(x_i) = \begin{cases} 
1 & \text{if } x_i > 0, \\
0 & \text{if } x_i = 0, \\
-1 & \text{if } x_i < 0,
\end{cases}
\]

i.e. for any point \( x = (x_1, \ldots, x_n)^T \) on the boundary \( \delta \mathcal{A}_{\text{fix}} \) the derivative points into \( \mathcal{A}_{\text{fix}} \).

Second, the sink cell has to be investigated. Stable systems will leave the sink cell with certainty although the time span until this event occurs might be infinitely long depending on the distance between \( \delta \mathcal{A}(0) \) and the initial state. As arbitrarily many qualitative states lie in the neighbourhood of the sink cell and therefore are possible successor states of the sink cell the dynamics in the sink cell is, in general, nondeterministic.

For an unstable system the behaviour of the quantised system in these states is vice versa: The system always leaves the sink cell with certainty although the time span until this event occurs might be infinitely long depending on the distance between \( \delta \mathcal{A}(0) \) and the initial state. As arbitrarily many qualitative states lie in the neighbourhood of the sink cell and therefore are possible successor states of the sink cell the dynamics in the sink cell is, in general, nondeterministic.

For an unstable system the behaviour of the quantised system in these states is vice versa: The system always leaves the sink cell with certainty although the time span until this event occurs might be infinitely long depending on the distance between \( \delta \mathcal{A}(0) \) and the initial state. As arbitrarily many qualitative states lie in the neighbourhood of the sink cell and therefore are possible successor states of the sink cell the dynamics in the sink cell is, in general, nondeterministic.

6.2. Natural partitioning

From Eq. (14) the following corollary can be derived for all qualitative states except the sink cell and the stationary state.

**Corollary 5.** The dynamics of a discrete-event quantised system is deterministic iff

\[
\forall i: \exists j \quad ([x(k)] = i) \Rightarrow \exists t \epsilon(t_{k+1}) = e_{ji}
\]

holds.

Corollary 5 implies that each set \( \mathcal{A}_i \) corresponding to a qualitative state has only one bound \( \delta \mathcal{A}(i, j) \) for which the vector \( \dot{x} \) in any point \( x \in \delta \mathcal{A}(i, j) \) points from the qualitative state \( i \) into the qualitative state \( j \). For all other points of the boundary \( \delta \mathcal{A}(i) \) the vector \( \dot{x} \) points into the state \( i \) or is parallel to \( \delta \mathcal{A}(i) \).

Like in the discrete-time case the natural partitioning in the discrete-event case is based on the choice of an initial set containing the fixed point of the underlying continuous-variable system. In the case of a stable system this will be the set \( \mathcal{A}_{\text{fix}} \) containing the finite fixed point and fulfilling condition (Eq. (25)). For unstable systems \( \mathcal{A}_{\text{fix}} \) is the sink cell. The partitioning is derived by mapping the bound \( \delta \mathcal{A}_0 \) of \( \mathcal{A}_{\text{fix}} \) with the inverse of the quantitative system equation \( f_i(x) = e^{A_i} \cdot x \). The partitioning principle resembles the discrete-time natural partitioning shown in Fig. 6 for a one-dimensional system.

The main difference to the discrete-time case is that all information about the transition time between two succeeding events is discarded by considering the order of the qualitative states within the state sequence, only. Therefore, the time span between two events can be chosen arbitrarily. As the time interval until the occurrence of the succeeding event can be different for each point \( x \) on the boundary \( \delta \mathcal{Q}_e \), for every defined partition \( \mathcal{Q}_e \) functions \( \delta t_i(x) \) can be chosen which describe the time span between the occurrence of two succeeding events for any point on the bounds \( \delta \mathcal{Q}_e \). Nevertheless these functions \( \delta t_i(x) \) have to be continuous in order to get smooth polytope-shaped boundaries.

**Algorithm 3.** A deterministic discrete-event partitioning can be derived by the following algorithm:

1. Determine the bound \( \delta \mathcal{A}_0, \delta \mathcal{A}_0 \) is the bound of the set \( \mathcal{A}_{\text{fix}} \) that contains the finite fixed point for stable systems and the bound of the sink cell \( \mathcal{A}_{\text{fix}} = \mathcal{A}_0 \) for unstable systems.
2. Iterate for \( \gamma = 0, 1, \ldots, N-2: \)

\[
\delta \mathcal{A}_{\gamma+1} = f_\gamma^{-1}(\delta \mathcal{A}_\gamma) = \{e^{-A_\gamma \delta t_i} \cdot x | x \in \delta \mathcal{A}_\gamma \},
\]

with arbitrary continuous functions \( \delta t_i(x) \).

**Definition 6.** A partitioning defined by the set of bounds \( \mathcal{Q}_e = \{\delta \mathcal{A}_\gamma, \gamma = \{0, 1, 2, \ldots, N-1\}\} \) is called natural discrete-event partitioning.

Fig. 9a shows part of a natural partitioning of the state space for a second-order oscillator. The dotted lines mark two continuous state trajectories. The bounds depicted introduce five qualitative states which are passed by the depicted and all other trajectories strating in \( \mathcal{A}_0 \) sequentially. The automaton graphs for the stable and unstable system are depicted in Fig. 9b.

Note that by introducing further constraints on the functions \( \delta t_i(x) \) the temporal behaviour of the quantised system can be influenced. As the crossing of the partition bounds by the continuous-variable system state leads to
the occurrence of an event, minimal and maximal bounds for the resting time of the quantised system within a qualitative state can be defined. Likewise, a dynamic behaviour with fixed transition times between two events could be obtained, if we choose $\delta t_i(x)$ as $\delta t_i(x) = T_e = \text{const}$. Thereby the temporal nondeterminism of the discrete-event behaviour is removed and the resulting partitioning is the same as in the discrete-time case.

### 6.3. Granulated natural partitioning

The derivation of further state-space partitionings yielding a deterministic behaviour of the quantised system is, like in the discrete-time case, based on a given natural partitioning. The sets $\mathcal{P}_i(i)$ of quantitative states defined by the natural partitioning are split up into several smaller sets. We will show which conditions the additional bounds have to satisfy in order to produce a deterministic quantised system.

**Algorithm 4.** An extended deterministic discrete-time partitioning can be derived by the following algorithm:

1. Determine a natural discrete-event partitioning consisting of a set of bounds $\mathcal{S}_e$.

(2) Iterate arbitrarily often:

(a) Add a set of bounds $\mathcal{S}_z$ fulfilling the condition

$$\mathcal{S}_z = \{ x_e^k(\mathcal{X}_0^k) \mid \mathcal{X}_0^k \in (\mathcal{S}_{b0} \cup \mathcal{S}_e) \}$$

with arbitrary closed set of starting points $\mathcal{X}_0^k$.

$$\delta(\delta \mathcal{P}(i,j)) \in (\mathcal{S}_{b1} \cup \mathcal{S}_e)$$

or

(b) Add a bound $\delta \mathcal{P}(i,j)$ which fulfills the condition

$$x^T \cdot n_x(x) < 0 \quad \forall x \in \delta \mathcal{P}(i,j)$$

or

$$x^T \cdot n_x(x) > 0 \quad \forall x \in \delta \mathcal{P}(i,j)$$

with the normal vector $n_x(x)$ on the bound $\delta \mathcal{P}(i,j)$.

(3) Derive the set $\mathcal{S}_{b0} = \bigcup \mathcal{S}_z$ from condition (a). The set of bounds fulfilling condition (b) constitutes the set $\mathcal{S}_{b1}$.

**Definition 7.** A partitioning defined by the set of bounds $\mathcal{S}_g = \mathcal{S}_e \cup \mathcal{S}_{b0} \cup \mathcal{S}_{b1}$ is called granulated natural discrete-event partitioning.

Each of the newly introduced bounds split up an existing qualitative state $k$ into two new states $i$ and $j$. The bounds contained in the set $\mathcal{S}_{b0}$ follow the trajectory of the inverse continuous-variable system between two succeeding bounds $\delta \mathcal{P}(k,j), \delta \mathcal{P}(l,k) \in (\mathcal{S}_{b1} \cup \mathcal{S}_e)$. As for each point

$$x \in \delta \mathcal{P}(i,j) \in \mathcal{S}_{b0}: \quad x^T \cdot n_x(x) = 0$$

holds, these bounds only split up sets $\mathcal{P}_i(i)$ introduced by the natural partitioning without having any effect on the dynamics of the quantised system. By introducing such a bound the automaton graph for the quantised system which previously consisted of a sequence of states gets split up into two branches (cf. Fig. 10b).

Each of the second type of bounds introduces an additional qualitative state. From condition (b) for the bounds contained in the set $\mathcal{S}_{b1}$ it follows that the equivalence class to be split up is bounded by two bounds from the set $\mathcal{S}_e \cup \mathcal{S}_{b1}$ and several bounds contained within the set $\mathcal{S}_{b0}$. The introduction of an additional bound replaces one of the states in the automaton graph by two sequential states, maintaining the deterministic behaviour.

Fig. 10 shows a granulated natural partitioning based on the natural partitioning from Fig. 9 and the resulting automaton graph. For this example the system was assumed to be stable so that $\mathcal{P}_i = \mathcal{P}_i(1)$. The additional bounds are shown as dashed lines. The two curved bounds belong to the set $\mathcal{S}_{b0}$ and the bound parting the qualitative states 7 and 8 is an element of $\mathcal{S}_{b1}$.
7. Example

In this section we will present state space partitionings of an autonomous two tank system resulting from the considerations in the previous sections. The plant shown in Fig. 11 has the linearized state-space model

$$\dot{x} = \begin{pmatrix} -0.08 & 0.08 \\ 0.08 & -0.13 \end{pmatrix} x + \begin{pmatrix} 4.5 \\ 0 \end{pmatrix} = A_c x + b_c. \quad (28)$$

The system is stable and has the fixed point $x_{fix} = - A_c^{-1} b_c = (92, 58)$.

In the discrete-time case the sampling time $T_s = 22s$ leads to the discrete-time model

$$x(k+1) = \begin{pmatrix} 0.41 & 0.30 \\ 0.30 & 0.24 \end{pmatrix} x(k) + \begin{pmatrix} 68 \\ 31 \end{pmatrix} = A_d x + b_d. \quad (29)$$

As initial set $Q_0$ we choose a circle with a radius of 15 around the fixed point. The contour of this set $\delta Q_0$ is mapped outwards with $A_d^{-1}$. This results in set 2 described by the contour $\delta Q_1$. Fig. 12 shows the resulting natural discrete-time partitioning of the state-space. Only the physically relevant area of the state space is depicted.

The natural discrete-event partitioning is shown in Fig. 12b. As initial set $Q_0$ we choose a circle with a radius of 15 around the fixed point. The contour of this set $\delta Q_0$ is mapped outwards with $A_d^{-1}$. This results in set 2 described by the contour $\delta Q_1$. Fig. 12 shows the resulting natural discrete-time partitioning of the state-space. Only the physically relevant area of the state space is depicted.

The natural discrete-event partitioning is shown in Fig. 12b. Like in the discrete-time case a circle with a radius of 15 was chosen as initial set $Q_0$ and mapped with the function $f^{-1}(x)$. For each mapping different functions $\delta t_i(x)$ were applied. After the third mapping the
minimum time in $\delta t_3(x)$ was chosen such that the resulting bound was completely outside of the physically relevant state space. With these partitionings the quantised system is deterministic.

8. Summary

In this paper we considered continuous–variable systems whose state can only be measured through a quantiser. Such systems can be seen from two different viewpoints. On the one hand, the time axis can be equidistantly sampled, on the other hand, the occurrence of events within the system determines the sampling times. We have shown that both viewpoints can be described by the same formalism and that in both cases the behaviour of the quantised system is in general nondeterministic. The main difficulties in finding an exact representation of the quantised system results from the nondeterministic dynamics. A deterministic behaviour, on the other hand, can be easily represented by an automaton.

In this paper we presented a sufficient condition for deterministic behaviour of the quantised system. We proved that deterministic behaviour can be achieved by an appropriate partitioning of the state space. The (granulated) natural state space partitioning was introduced as a method to construct partitionings yielding deterministic behaviour for autonomous systems.

A generalisation of the results obtained in this paper to systems with discrete input values is to find a different state-space partitioning for each of the discrete input values. These partitionings are then merged by superposition, whereby additional qualitative states are introduced. It can be stated that these merged partitionings in general do not lead to a deterministic behaviour of the quantised system. Further investigations in partitionings for nonautonomous systems have to be made.

References


