NLOS UV Communications Using M-ary Spectral-Amplitude-Coding

Mohammad Noshad, Student Member, IEEE, Maïté Brandt-Pearce, Senior Member, IEEE, and Stephen G. Wilson, Senior Member, IEEE

Abstract—We present an M-ary spectral amplitude code (SAC) modulation technique to improve the performance of free-space optical (FSO) communication systems. Although this approach can be used in any dispersive FSO system, in this paper we focus on non-line of sight (NLOS) ultraviolet (UV) systems relying on atmospheric scattering. Spectral amplitude encoding is applied on a broadband UV source using the same code families for the M-ary alphabet as used previously in SAC optical code division multiple access (OCDMA) systems. A differential structure using two photomultiplier tubes is utilized in conjunction with various demodulation algorithms to decode the received signal. Intersymbol interference (ISI), received beam divergence and shot noise are considered as the main factors limiting the system performance. An upper bound on the bit error probability is presented and compared with simulation results for various geometries and for different code parameters. The maximum bit rate for a fixed bit error probability is calculated in terms of the link length, and results for different alphabet sizes are shown. By sacrificing spectral efficiency without becoming more susceptible to ISI, the proposed system can support higher rates and longer distances for the same performance compared with on-off keying systems.

Index Terms—M-ary communication, ultraviolet communications, non-line of sight communication, spectral amplitude coding.

I. INTRODUCTION

T HE application of free space optics (FSO) for non-line of sight (NLOS) communications has attracted increasing interest because of its relative insensitivity to pointing errors and robustness against shadowing [1]. Atmospheric scattering in the UV band is higher than in other optical bands, and, consequently, in NLOS systems the receiver can receive more power from the transmitter than if it used another portion of the optical spectrum [2], making the UV band an interesting choice for NLOS communications [1]. Furthermore, background irradiance severely limits the performance of NLOS infrared and visible light optical communications, while the ultraviolet (UV) band (200-280 nm) has low background light because of sunlight filtering by the upper atmosphere [1]. Yet the strong scattering in the NLOS UV channel imposes a temporal dispersion on the transmitted optical pulses. Thus, both intersymbol interference (ISI) and low received power can be factors limiting the data rate in NLOS UV links [3].

In this paper we propose single-user M-ary modulation and demodulation techniques to combat the channel degradation and increase the distance-rate product of NLOS UV systems. The distance-rate product is an important measure for evaluating the performance of optical communication systems. It is defined as the product of the transmitter-receiver distance of the optical channel and its maximum achievable data rate for a predefined maximum acceptable bit error probability. This product is usually limited by either intersymbol interference (ISI) or low signal to noise ratio (SNR). Error correcting codes, equalization and M-ary transmission are the most common techniques for increasing the distance-rate product in both radio frequency (RF) and optical communications. The maximum data rate versus the distance between the transmitter and receiver has been presented in previous work on NLOS UV communications for on-off keying (OOK) and different system geometries [3], [4].

In [5] we propose a novel structure using spectral amplitude coding (SAC) for improving the performance of the single-user NLOS UV communication system. We utilize fixed cross-correlation codewords, shown to have optimal properties in [6], as modulation symbols in order to increase the bit rate of the NLOS UV communication system. In [5], an avalanche photodiode (APD) array is used for extracting the received signal in each wavelength. However, because of the low gain of the APD arrays in the UV range and the small aperture size of each element of the array, the system is not able to transmit data for distances longer than 100 m. Photomultiplier tubes (PMT) are a better choice for detecting weak signals. Due to the high gain of the PMTs, shot noise dominates and other noise sources, such as thermal noise, can be neglected. An experimental demonstration of a NLOS UV communication system using OOK and PMTs is reported in [7], but communications is still limited to 500 Kb/s and 200 m range.

In this paper we consider a different transmitter/receiver structure than in [5]. A broadband optical source is required for SAC, thus, since high-power broadband UV sources are scarce, widely-available broadband spectrum visible lasers are proposed as transmitter light sources. The encoding is done in the visible/infrared (IR) region, and then the encoded signal is frequency-doubled (or tripled) to the UV range. We use symmetric balanced incomplete block designs (BIBD) [8] to construct spectrally encoded symbols. At the receiver side, one or two PMTs can be used for decoding the received signal. The symbol time period, $T_s$, is divided into smaller
time slots and in each symbol time interval decoding is done using a symbol-by-symbol detector. Optimum single PMT, optimum dual PMT, and optimum linear dual PMT demodulation are considered as the three options for making a decision on the received signal. The performance of the linear detector is shown to be close to that of the optimum detector. The performance of the optimum linear dual PMT detector with various alphabet sizes is analyzed by considering the ISI, received beam divergence and shot noise. An analytical upper bound for the error probability is derived. Through our numerical results we show that \( M \)-ary SAC can achieve higher distance-rate products compared to OOK.

The rest of the paper is organized as follows. In Section II, the configuration of the transmitter is described. Section III describes the channel model. Several structures for the receiver using PMTs are presented in Section IV. The analytical bit error probability is calculated in Section V, using the upper bound. Then the system performance is analyzed using numerical results in Section VI. Finally, conclusions are made in Section VII.

II. TRANSMITTER STRUCTURE

Fig. 1 illustrates the structure of the transmitter using spectral encoding. The transmitter is composed of three parts: a wideband optical source in the visible or IR region, an optical encoder, and a wavelength converter, which transforms the encoded light to the UV range. A laser with wide spectrum (\( \sim 20 \) nm) and subpicosecond pulses, such as a Ti-Sapphire laser, is a good option for the source. These lasers are able to generate ultrashort light pulses with high repetition rate (\( \sim 80 \) M pulses/sec), much more frequent than the symbol rate considered here.

The output light from the wide spectrum source is collimated upon a diffraction grating. The diffraction grating reflects each wavelength with a specific angle, which depends on the physical parameters of the grating and the angle of the incident beam. A lens is placed in front of the grating so that the combination of the grating and lens decomposes the spectral content of the source light and focuses it on an encoder mask. The encoder mask consists of \( F \) elements, each of which passes or blocks its corresponding wavelength according to the desired codeword. The electrically-controlled encoder mask is changed at the start of each symbol period, depending on the symbol that has to be transmitted. After the encoder mask, another set of lens and diffraction grating are used to recombine the components of the encoded light. In this way, we obtain a coded output in the visible or IR light region and, since we apply amplitude encoding but no phase encoding on the ultrashort pulses, the encoded pulses still have sub-picosecond duration.

The last part of the transmitter is a wavelength converter, which uses harmonic generation in a nonlinear crystal, such as \( \beta \)-Barium Borate (BBO), \( \beta \)-BaB\(_2\)O\(_4\) [9], or Lithium Triborate (LBO), Li\(_3\)B\(_4\)O\(_5\) [10], to convert the encoded signal to UV light. The nonlinear crystal is placed inside an optical cavity in order to increase the conversion efficiency. Detailed configurations for using a wavelength converter to generate UV light are described in [11] and [12]. Due to the nonlinear effect and because of the cavity, the ultrashort pulses are broadened at the output of the wavelength converter, so that the rapid pulse-train now appears as an almost-rectangular pulse of duration \( T_s \). Transmit power levels on the order of 1 W are easily attainable using this technique.

In this paper, for our \( M \)-ary modulation we use the BIBD codes that have been proposed for expurgated pulse position modulation (EPPM) [6] and also for multiple access in SAC-OCDMA systems [13]. A BIBD code consists of \( M \) codewords of length \( F \), and its \( m \)th codeword is denoted by the vector \( c_m = [c_{m1}, c_{m2}, \ldots, c_{MF}] \), \( m = 1, 2, \ldots, M \), where \( c_{mj} \in \{0, 1\} \). We assume the weight of each codeword is \( w \), and the cross-correlation between each pair of codewords is \( \lambda \), which corresponds to a fixed cross-correlation. These two properties can be represented as [14]

\[
\sum_{i=1}^{F} c_{mi}c_{ni} = \begin{cases} w & ; m = n, \\ \lambda & ; m \neq n \end{cases}.
\]

Among all known fixed cross-correlation codes, the one with the lowest error probability has \( F = 2w + 1 \) and \( w = 2\lambda + 1 \) [6]. In order to make the control of the encoder and decoder masks simpler, we use cyclic codes, for which the codewords are cyclic shifts of each other [8]. In this case, the number of codewords in one code is equal to the code length (\( M = F \)), and the entire code can be characterized by any codeword. The transmitted vector at symbol time \( k \) is \( s_k = [s_{k1}, s_{k2}, \ldots, s_{kF}] \in \{c_1, c_2, \ldots, c_M\} \).

The linewidth of the source is divided into \( F \) wavelength bins by the encoding mask. We denote the transmitted pulse shape at wavelength bin \( f \), \( f = 1, 2, \ldots, F \), and in symbol interval \( k = 0 \) by \( F_f(t) \), resulting from the many temporally-dispersed ultrashort pulses in one symbol time. So the transmitted signal intensity at wavelength \( f \) is

\[
p_f(t) = \sum_{k=-\infty}^{\infty} s_{kf} F_f(t - kT_s),
\]

where \( T_s \) is the symbol time. The total optical intensity transmitted is the sum of these modulated signals at adjacent wavelengths.
III. CHANNEL MODEL

Analyzing the performance of NLOS UV communication systems requires a model of the path loss and the impulse response of the channel. Analytical [15], simulation [16], [17] and experimental [7] approaches have been used to obtain a precise estimate of the impulse response and to calculate the link loss. In this work we use the channel model presented in [17]. In this model, the common volume between the transmitter and receiver is divided into small cubic volumes, and then numerical integration is used to calculate the impulse response of the NLOS UV link.

Fig. 2 shows the geometry of the NLOS UV link that we use in this paper. The transmitter and receiver are located at distance $D$ from each other. We denote the transmitter beam full-width divergence by $\phi_1$, the receiver field of view (FOV) by $\phi_2$, the transmitter elevation angle by $\theta_1$, and the receiver elevation angle by $\theta_2$. The transmitter beam axis and receiver FOV axis are assumed coplanar for notational simplicity, but this assumption is not required for our approach.

In NLOS UV systems, the transmitted pulse is spread in the time domain due to the large scattering volume and the strong atmospheric scattering of the UV band. The received signal at wavelength $f$, $p^{(j)}_R(t)$, can be obtained by convolving the channel intensity impulse response in wavelength $f$, $h_f(t)$, with the transmitted signal in that wavelength, $p_f^{(j)}(t)$. So for the received signal intensity we have

$$p^{(j)}_R(t) = \sum_{k=-\infty}^{\infty} s_{kj} \left[ F_F(t - kT_s) \ast h_f(t) \right], \quad f = 1, 2, \ldots, F.$$  \hspace{1cm} (3)

IV. RECEIVER

The optimum receiver for SAC UV systems requires $F$ PMTs (one PMT for each wavelength), which, considering the size and price of PMTs, is impractical and not cost effective. Here, we propose a structure using one or two PMTs, which is sub-optimal but practical.

Our structure for the UV receiver using two PMTs (e.g. R1584: Hamamatsu\(^1\)) is shown in Fig. 3. We use a UV filter (e.g. Edmund Optics; 254 nm center wavelength, 40 nm FWHM bandwidth\(^2\)) to limit the background noise on the receiver. Consequently, and considering that the background radiation is extremely low in the UV range wavelengths of 250 nm - 280 nm, we neglect the effect of the background light in this paper.

A. Spatial Broadening Due to the Wide Receiver FOV

In the receiver, a UV diffraction grating (e.g., 3600 grooves/mm; 0.25 nm/mer dispersion at 250 nm; LaserComponents\(^3\)) and lens are used for decomposing the spectrum of the received beam. A decoder mask, composed of $F$ elements, is placed after the diffraction grating to decode the spectrum. As shown in Fig. 3, we assign the $z$-axis perpendicular to the grooves of the grating.

According to the receiver structure in Fig. 3, the diffraction grating reflects each wavelength with a specific angle. Thus, divergence of the received beam in the $z$ direction causes the signal of one wavelength to be reflected at a wider angle, causing interference between adjacent wavelengths. As a result, each point in the decoder mask plane receives power from more than one wavelength. This can be interpreted as a transformation of the divergence from the received beam angle into the wavelength domain after the diffraction grating. We denote this effect as spatial broadening, which looks like spectral broadening and causes interference between adjacent spectral bins, as illustrated in Fig. 4.

We assume that the normalized optical power distribution on the decoder mask plane at point $z$ from wavelength $f$ is denoted by $g(z - z_f)$, where $z_f$ is the center of the bin $f$, and $z - z_f$ is the distance from the center of the beam, assumed centered on the $f$th mask element. Furthermore, we assume that $g(z - z_f)$ is the same for all wavelengths. To describe this effect, we define the spatial broadening coefficients as

$$a_i = \frac{1}{\int_{-\infty}^{\infty} g(z)dz} \int_{-\infty}^{\infty} g(z)dz, \quad \Delta L,$$ \hspace{1cm} (4)

where $\Delta L$ is the width of one element of the decoder mask. These spatial broadening coefficients capture the interference that the optical signal in one wavelength bin introduces on its neighboring bins. We define the extent of spatial broadening by the integer $Z$ such that $\sum_{i=-Z}^{Z} a_i = 1$ and $a_i \approx 0$ for $\forall |i| > Z$.

\(^3\)http://www.lasercomponents.com/fileadmin/user_upload/home/Datasheets/optometr/gratings.pdf
Because of this spatial broadening, the optical power reflected from the diffraction grating in the angle that correspond to the wavelength \( f \) can be written as

\[
\tilde{p}^{(f)}(t) = \sum_{i=-Z}^{Z} a_i p^{(f-i)}(t).
\]  

According to (4), the \( a_i \)'s are functions of \( \Delta L = L/F \), where \( L \) is the length of the decoder mask and is assumed to be fixed. As the code length, \( F \), increases, \( \Delta L \) decreases and, consequently, \( Z \) increases. Thus, by increasing \( F \) the interference between the spectral bins worsens.

### B. Decision Variables

In order to capture as much of the received light as possible to decode the spectrally encoded signal, the decoder mask is a switch that modifies the beam’s direction of propagation in each element, unlike the amplitude (on-off) encoder mask used at the transmitter. The elements can either be micro-electromechanical-system (MEMS)-based mirrors with UV coating, or electro-optic crystals, such as BBO or LBO, which are transparent in the UV range. Each element can be in one of two different states, named \( u_1 \) and \( u_2 \), and its state can be changed between \( u_1 \) and \( u_2 \). By changing the state of each element, the beam of the corresponding wavelength can be directed to one of the two PMTs. The states of the elements of the decoder mask are determined by the codeword that the decoder controller sends to the mask.

As illustrated in Fig. 5, each symbol period with duration \( T_s \) is divided into \( M \) equal time slots and the decoder mask is changed in each time slot. Without loss of generality, we consider that the decoder mask matches the codeword that the \( m \)-th code symbol is transmitted in the \( m \)-th time slot if \( c_{mj} = 1 \) and is \( u_2 \) otherwise. Because of the non-rectangular temporal shape of the received pulse [7], the received energies in the time slots of one symbol period are not equal. Therefore, an interleaver is used to reorder the codes in the decoder mask, so that the average error probability becomes equal for all symbols.

A schematic of the receiver is shown in Fig. 6. According to (5) and the decoder function, the incident signal in wavelength \( f \) and time slot \( j \) upon PMT1 is \( c_{jj} \tilde{p}^{(f)}(t) \) and upon PMT2 is \( (1 - c_{jj}) \tilde{p}^{(f)}(t) \).

We denote the current generated by PMT1 and PMT2 as \( I_1^{(1)}(t) \) and \( I_1^{(2)}(t) \), and the additive noise at PMT1 and PMT2 as \( \nu^{(1)}(t) \) and \( \nu^{(2)}(t) \), respectively. Without loss of generality we consider the \( k = 0 \) symbol time. Let \( I_1^{(n)}(t) \) and \( \nu^{(n)}(t) \) respectively represent \( I_1^{(n)}(t) \) and \( \nu^{(n)}(t) \) in time slot \( j \) of symbol period \( k \), for \( n = 1, 2 \). Then according to [14], \( I_1^{(n)}(t) \) is given by

\[
I_1^{(n)}(t) = GR \sum_{f=1}^{F} (n - 1) - (-1)^n c_{jj} \tilde{p}^{(f)}(t) + \nu^{(n)}(t), \quad \frac{(j - 1)T_s}{M} \leq t < \frac{jT_s}{M},
\]  

where \( n = 1, 2, j = 1, 2, \ldots, M \), \( G \) is the gain and \( R \) is the responsivity of the PMTs.

An integrate-and-dump filter and a sampler are placed after each PMT, and a shift register of length \( M \) stores the output samples of all time slots. We represent the stored variables by vector \( r^{(n)} = (r_1^{(n)}, r_2^{(n)}, \ldots, r_M^{(n)}) \), \( n = 1, 2 \), where \( r_j^{(n)} \) is

\[
r_j^{(n)} = \int_{\frac{jT_s}{M}}^{\frac{(j+1)T_s}{M}} I_1^{(n)}(t) dt.
\]  

Using (2), (5) and (6), \( r_j^{(n)} \) becomes

\[
r_j^{(n)} = \left[ \sum_{i=-Z}^{Z} GR a_i \sum_{f=1}^{F} \left( (n - 1) - (-1)^n c_{jj} \right) \tilde{p}^{(f-i)}(t) \right] \times \sum_{m=\infty}^{\infty} s_{m(f-i)} \int_{\frac{jT_s}{M}}^{\frac{(j+1)T_s}{M}} \left[ \mathcal{F}_f(t - mT_s) * h_f(t) \right] dt + \int_{\frac{jT_s}{M}}^{\frac{(j+1)T_s}{M}} \nu_j^{(n)}(t) dt.
\]  

As mentioned above, an interleaver is used to change the order of the codewords of the decoder mask in each symbol period; therefore, for mathematical simplicity, we replace the term \( \int_{\frac{jT_s}{M}}^{\frac{(j+1)T_s}{M}} \left[ \mathcal{F}_f(t - mT_s) * h_f(t) \right] dt \) in (8) with its mean, \( \frac{1}{M} \int_{0}^{T_s} \left[ \mathcal{F}_f(t - mT_s) * h_f(t) \right] dt \). Here, \( \frac{1}{M} \int_{0}^{T_s} \left[ \mathcal{F}_f(t - mT_s) * h_f(t) \right] dt \) is the part of the optical energy transmitted in wavelength \( f \) of symbol time \( m \) and received during time slot \( j \) of the symbol at time \( k = 0 \).

Again for mathematical simplicity, we assume that the received pulse shape, i.e., \( \mathcal{F}_f(t) * h_f(t) \), is the same for all
wavelengths. We define the ISI coefficients, $\gamma_m$, as

$$\gamma_m = \int_0^{T_s} [f(t - mT_s) * h_f(t)] dt, \quad -K \leq m \leq K,$$

where $2K$ is the number of adjacent symbols influencing the desired symbol, i.e., $\gamma_i \approx 0$ for $|i| > K$. Using (9), (8) becomes

$$r_j^{(n)} = GR \sum_{m=-K}^{K} a_i \left( w(n-1) \right) - (-1)^n S(s_m, i) e_j^T \gamma_m M + \int_{\frac{mT_s}{2}}^{\frac{(m+1)T_s}{2}} \nu_j^{(n)}(t) dt,$$

where $S(s_m, i)$ is the $i$th right shift of vector $s_m$. One last approximation we make for analytical tractability is to overestimate the spatial interference on $s_m$ by using cyclic right shifts of $s_m$ instead of linear right shifts. For the remainder of this paper, $S(y, i)$ denotes the $i$th cyclic shift of vector $y$.

The mean of $r_j^{(1)}$, neglecting ISI, is given by

$$\mu_j^{(1)} = GR \gamma_0 M \sum_{i=-Z}^{Z} a_i S(s_0, i) e_j^T.$$

Let $m_k$ and $\hat{m}_k$ be the transmitted and estimated symbols in the $k$th symbol period, respectively. We define $\mu_j^{(1)}$ given $m_0 = \ell$ as $\mu_j^{(1)\ell} = GR \gamma_0 M \sum_{i=-Z}^{Z} a_i S(c_{\ell}, i) e_j^T$. Since the code is cyclic, we have $\mu_j^{(1)\ell} = S(\mu_j^{(1)}, \ell - 1)$, for all $\ell$, where the vector $\mu_j^{(1)} = [\mu_j^{(1)(1)}, \mu_j^{(1)(2)}, \ldots, \mu_j^{(1)(M)}]$, $\ell = 1, 2, \ldots, F$. From the receiver structure, letting $\mu_j^{(2)\ell}$ be the mean of $r_j^{(2)}$, we know $\mu_j^{(1)\ell} + \mu_j^{(2)\ell} = GR \gamma_0 M$; henceforth we use $GR \gamma_0 M - \mu_j^{(1)\ell}$ instead of $\mu_j^{(2)\ell}$.

Because of the high gain of the PMTs, shot noise dominates the thermal noise and, thus, the variance of $r_j^{(n)}$ in (10) given $m_0 = \ell$, neglecting other noise sources, can be approximated as

$$\text{Var}(r_j^{(n)}|\ell) = 2e\Delta f G \mu_j^{(n)\ell},$$

where $e$ is the electron charge ($e = 1.6 \times 10^{-19}$ C) and $\Delta f$ is the receiver equivalent noise bandwidth. Since in the receiver structure the symbol period is divided into $M$ equal time slots, $\Delta f$ is proportional to $M$ and increases as $M$ increases. In this study we assume the shot noise can be modeled as Gaussian distributed [18].

C. Symbol Detectors

At the end of the symbol period, the stored variables in the shift registers are fed into a symbol-by-symbol demodulator, which then estimates the symbol $\hat{m}_k$ for the $k$th time instant (ignoring ISI). Three different decision rules are considered:

1) Optimum Single-PMT Detector: In this case the decision is made based on a maximum likelihood (ML) rule using only the output of either PMT1 or PMT2. This is the simplest and least expensive option. The decision criterion for this detector can be written as

$$\hat{m}_0 = \arg \max_{1 \leq \ell \leq M} \Pr(r^{(n)}|m_0 = \ell),$$

where $n = 1$ or 2, $m_0$ is the symbol transmitted in the $k$ = 0 symbol period and $\Pr(r^{(n)}|m_0 = \ell)$ is the probability of receiving vector $r^{(n)}$, given $m_0 = \ell$.

2) Optimum Dual-PMT Detector: Given equally-likely transmitted symbols, the optimum detector can be implemented based on an ML rule using the outputs of both PMT1 and PMT2. For this detector, since the shot noise components in different time-slots have unequal errors, the decision rule in its most simple form is quadratic. Since the noise terms of PMT1 and PMT2 are independent, the decision criterion for this case can be written as

$$\hat{m}_0 = \arg \max_{1 \leq \ell \leq M} \Pr(r^{(1)}|m_0 = \ell)\Pr(r^{(2)}|m_0 = \ell).$$

3) Optimum Linear Dual-PMT Detector: In linear detectors, the decision statistic, written as a vector $x = [x_1, x_2, \ldots, x_M]$, is generated from $r^{(1)}$ and $r^{(2)}$ as

$$x = r^{(1)}Q + r^{(2)}Q',$$

where $Q = [Q_1, Q_2, \ldots, Q_M]$ and $Q' = [Q'_1, Q'_2, \ldots, Q'_M]$ are $M \times M$ matrices, formed by columns $Q_\ell = [q_{\ell 1}, q_{\ell 2}, \ldots, q_{\ell M}]^T$ and $Q'_\ell = [q'_{\ell 1}, q'_{\ell 2}, \ldots, q'_{\ell M}]^T$. Then the decision is made based on the following rule:

$$\hat{m}_0 = \arg \max_{1 \leq \ell \leq M} x_\ell.$$
For the optimal linear detector, $Q$ and $Q'$ are chosen to minimize the symbol error probability.

Note that if $a_0 = 1$ and $a_i = 0$ for $i \neq 0$, the three detectors described (the optimum dual-PMT detector, optimum linear dual-PMT detector and optimum single-PMT detector using PMT2) all reduce to

$$m_0 = \arg \min_{1 \leq \ell \leq M} r^{(2)}_\ell.$$

(17)

Since the performance of these detectors is not a function of the Hamming distances between the symbols, the fixed weight and fixed cross-correlation code families proposed here, further described in [6], may not be the optimum codes, i.e., those that have the best performance. Finding the optimum codes for each one of these detectors is beyond the scope of this paper.

V. ANALYTICAL BIT ERROR PROBABILITY CALCULATION

In this section, we calculate the optimum $Q$ and $Q'$ of the linear dual-PMT detector, and obtain expressions for the bit error probability of this receiver by considering the shot noise, temporal dispersion and spatial broadening effects. As shown below in Section VI, the symbol error rate of the linear dual-PMT detector is close to the error rate of the quadratic optimal dual-PMT detector, and acts as a bound on its performance. Therefore, and for mathematical simplicity, we analyze the bit error probability of only the linear dual-PMT detector in this section.

A. Optimal Linear Dual-PMT Detector for ISI-Free System

Generally, the symbol error probability can be written as

$$P_s = \sum_{\ell=1}^{M} \Pr(m_0 = \ell)(1 - \Pr(\ell|\ell)).$$

(18)

where we define $\Pr(\ell'|\ell) = \Pr(m_0 = \ell'|m_0 = \ell)$. Assuming the decision criterion to be as (16), an error arises if for at least one $\ell'$ ($\ell' \neq \ell$) we have $x_{\ell'} > x_{\ell}$, and therefore, $Pr(\ell|\ell) = Pr(x_{\ell'} > x_{\ell}|m_0 = \ell)$. The output samples in Fig. 6 at time slot $j$ are modeled as Gaussian distributed, and denoted as $r_{\ell}(j) \sim N(\mu_j^{(n)}|\ell), \text{Var}(r_{\ell}(j)|\ell)$. Thus, $x_{\ell'} - x_{\ell}$ is also a Gaussian random variable with mean $\mu_{(\ell'-\ell)} = \sum_{j=1}^{M} \mu_j^{(1)}(q_{j\ell} - q_{j\ell'}) + \mu_j^{(2)}(q_{j\ell} - q_{j\ell'})$ and variance $\text{Var}(r_{\ell}(j)|\ell) = \sum_{j=1}^{M} \text{Var}(r_{\ell}(j)|\ell)(q_{j\ell} - q_{j\ell'})^2 + \text{Var}(r_{j\ell}(\ell)|\ell)(q_{j\ell} - q_{j\ell'})^2$. Then

$$Pr(\ell|\ell) = \frac{1}{2} \text{erfc}\left(\frac{\mu_{(\ell'-\ell)}}{2\sqrt{\text{Var}(r_{\ell}(j)|\ell)}}\right).$$

(19)

Let $Q_\ell$ and $Q_\ell'$ be the solutions of the optimum detector in the ISI-free case. Since the symbols are cyclic shifts, $Q_\ell$ and $Q_\ell'$ are circulant matrices, so $Q_\ell' = S(Q_\ell, 1, -1)$ and $Q_\ell = S(Q_\ell', 1, -1)$, for all $\ell$. Therefore, $Pr(\ell|\ell)$, using (12), becomes

$$Pr(\ell|\ell) = \frac{1}{2} \text{erfc}\left(\frac{\mu^{(1)}(Q_\ell - Q_\ell') - \mu^{(2)}(Q_\ell' - Q_\ell)}{4\sigma jG^{1/2}}\sqrt{\mu^{(1)}(Q_\ell - Q_\ell') + (GRW_{\text{sun}} - \mu^{(1)}(Q_\ell - Q_\ell')) \xi_{\ell,\ell'}}\right).$$

(20)

where $Q_{\ell,\ell'} = [(q_{j\ell} - q_{j\ell'})^2]_{j=1}^{M}$ and $Q_{\ell,\ell'} = [(q_{j\ell} - q_{j\ell'})^2]_{j=1}^{M}$ are $M \times 1$ matrices. According to (19), adding a constant value to all $q_{j\ell}$’s or $q_{j\ell'}$’s does not change $Pr(\ell'|\ell)$. On the other hand, for the optimum linear detector $q_{j\ell} = q_{j\ell}$ and $q_{j\ell}' = q_{j\ell}$ if $\mu_j^{(n)|\ell} = \mu_j^{(n)|\ell}$. So, without loss of generality, for time slots $j$ such that $\mu_j^{(n)|\ell} = GR\frac{2\pi}{M} \lambda$, $q_{j\ell}^* = q_{j\ell}$ and $q_{j\ell}' = q_{j\ell}^*$ are chosen to be zero.

The union bound on the symbol error probability is given by

$$P_{s,UB} = \sum_{\ell=1}^{M} \sum_{\ell' = 1}^{M} \frac{1}{M} \Pr(\ell'|\ell).$$

(21)

Since the symbols and the channel action (after interleaving) are symmetric, (21) can be simplified to

$$P_{s,UB} = \sum_{\ell=1}^{M} \Pr(\ell|1).$$

(22)

For high SNRs the largest $Pr(\ell|1)$ becomes the dominant term. So, minimizing the maximum $Pr(\ell|1)$, over $2 \leq \ell' \leq M$, asymptotically minimizes the symbol error probability. Consequently, the solution to the following minimax problem gives us the $Q^*$ and $Q'^*$ of the asymptotically optimum linear detector:

$$(Q^*, Q'^*) = \arg \min_{Q,Q'} \max_{2 \leq \ell' \leq M} Pr(\ell|1).$$

(23)

In general, (23) is not an easy problem to solve. Here, we restrict ourselves to a specific case, which is valid for most practical systems. We assume that the spatial broadening is small enough to neglect $a_i$ for $|i| \geq 2$, i.e., the only nonzero $a_i$’s are $a_0, a_{-1}$ and $a_1$. We also assume that the spatial broadening is symmetric, i.e., $a_{-1} = a_{-1}$, and therefore, we have $a_0 + 2a_1 = 1$. In this case, since $\mu_j^{(1)|1} = GR\frac{2\pi}{M} \lambda$, for $j = 3, 4, \ldots, M - 1$, $q_{j\ell}' = q_{j\ell}$, and $q_{M}^* = q_{M}' = 0$ for $j = 3, 4, \ldots, M - 1$. We also have $\mu_j^{(1)|1} = GR\frac{2\pi}{M} (a_0w + 2a_1\lambda)$ and $\mu_j^{(2)|1} = GR\frac{2\pi}{M} (a_1w + (a_0 + a_1)\lambda)$, and thus, $Q_{21}^* = Q_{M1}^*$ and $Q_{21}' = Q_{M1}'$. Hence, (20) becomes as (24) at the top of the next page, where $\xi_1 = (a_0 - a_1)(w - \lambda)$, $\xi_2 = a_1(w - \lambda)$, $\xi_3 = a_0(w - \lambda)$, $\xi_4 = (1 - a_1)w + (1 + a_1)\lambda$, $\xi_5 = a_1w + (2 - a_1)\lambda$, and $\xi_6 = a_0w + (2 - a_0)\lambda$.

From Appendix A, the minimum of $Pr(2|1)$ over all $Q$ and $Q'$ is

$$1 \frac{1}{2} \text{erfc}\left(\frac{\sqrt{wR\frac{2\pi}{M} \lambda}}{2e^{2\xi_1}} + \frac{\sqrt{\xi_1^2(2w - \xi_1)}}{\xi_2(2w - \xi_2)}\right),$$

(25)

which is obtained for $q_{11} = \frac{\xi_1}{\xi_2} + \frac{\xi_2}{\xi_1}$ and $q_{22} = \frac{\xi_1}{\xi_2} + \frac{\xi_2}{\xi_1}$. Substituting these values in (24) we get $Pr(2|1) \geq Pr(\ell'|1)$ for $3 \leq \ell' \leq M - 1$. For any other $q_{11}, q_{22}, q_{11}'$ and $q_{22}'$, $\max_{2 \leq \ell' \leq M} Pr(\ell'|1)$ is larger than (25) and thus, (25) is the solution of the minimax problem in (23).

B. Symbol Error Probability

To calculate the bit error probability in the presence of ISI, first we obtain an upper bound on the symbol error probability and then we calculate a bound on the bit error probability.
\[ \Pr (\ell' | 1) = \begin{cases} 
\frac{1}{2} \text{erfc} \left( \sqrt{\frac{R_{\eta 0}}{4 \Delta f / M}} \frac{(q_{11} - q_{21}) \xi_1 + q_{21} \xi_2 + (q_{11} - q_{21}) \xi_1 - q_{21} \xi_2}{(q_{11} - q_{21})^2 + q_{21}^2 + (q_{11} - q_{21})^2 (2w - \xi_1) + q_{21}^2 (2w - \xi_2)} \right), & \ell' = 2, M; \\
\frac{1}{2} \text{erfc} \left( \sqrt{\frac{R_{\eta 0}}{4 \Delta f / M}} \frac{q_{11} \xi_1 + q_{21} \xi_2 - q_{21} \xi_1 - q_{21} \xi_2}{(q_{11} - q_{21})^2 + q_{21}^2 + (q_{11} - q_{21})^2 (2w - \xi_1) + q_{21}^2 (2w - \xi_2)} \right), & \ell' = 3, M - 1; \\
\frac{1}{2} \text{erfc} \left( \sqrt{\frac{R_{\eta 0}}{4 \Delta f / M}} \frac{q_{11} \xi_1 + 2q_{21} \xi_2 - q_{21} \xi_1 - 2q_{21} \xi_2}{(q_{11} - q_{21})^2 + q_{21}^2 + (q_{11} - q_{21})^2 (2w - \xi_1) + q_{21}^2 (2w - \xi_2)} \right), & \text{otherwise}. 
\end{cases} \]

\[ \Pr (\ell' | 1) = \frac{1}{2} \left( \frac{1}{M} \right)^{2K} \sum_{m-K=1}^{M} \sum_{m-1=1}^{M} \sum_{m=1}^{M} \sum_{m=K+1}^{M} \text{erfc} \left( \sqrt{\frac{1}{4 \Delta f/G}} \left( \sum_{i=-K}^{K} \frac{2a_i \mu^{(1)}(m_i, 1)}{2w} \left( \tilde{Q}_{1, \ell'} - \tilde{Q}_{1, \ell'} \right) + \left( GR_{\eta 0} \sum_{i=-K}^{K} \gamma_i \right) \| Q_{1, i}^{\ell'} - Q_{1, i}^{\ell'} \|^2 \right) \right). \]

We assume the detector used is the optimal linear detector ignoring ISI derived above. For the upper bound, we use the union bound in (22), where \( \Pr (\ell' | 1) \), considering ISI, becomes as (26), given at the top of the page.

### C. Bit Error Probability

Having bounded the symbol error probability, we still need the bit error probability in order to be able to compare the system performance with OOK systems. Assume \( b_\ell \), a \( \log_2 M \) digit binary number, is the binary sequence assigned to symbol \( \ell \). When \( m_0 = \ell \) is transmitted and \( m_0 = \ell' \) is received, \( d(b_\ell, b_{\ell'}) \) bits are decoded incorrectly, where \( d(b_\ell, b_{\ell'}) \) denotes the Hamming distance between the binary vectors \( b_\ell \) and \( b_{\ell'} \). According to [19], for equally likely symbols we have the following inequality for the bit error probability, \( P_b \),

\[ P_b \leq \frac{M}{2(M - 1)} P_{s, UB} = P_{b, UB}, \]

where \( P_{b, UB} \) is an upper bound on \( P_b \).

### VI. Numerical Results

In this section, numerical results are presented to compare the performance of the proposed \( M \)-ary transmission for different code-lengths with previously proposed OOK modulation. In this work, the main limiting factor on the bit-rate is inter-symbol interference, and other pulsed modulations, such as PPM, perform much worse than OOK, because of their short pulse duration. Hence, in our numerical results we only compare the performance with OOK.

We use Paley, projective geometry (PG) and twin prime power (TPP) difference sets [8] as code families in these results. For these code families \( F = 2w + 1 \) and \( w = 2\lambda + 1 \). For the bit-symbol mapping, since the code-lengths are in general not powers of two, we concatenate multiple symbols and assign longer binary sequences to them [20]. The transmitted power is assumed to be 1 W. The gain \( G \) of the PMTs is \( 10^7 \) and their responsivity \( R \) is 0.3 A/W. Three geometries are considered, described in Table I.

For simplicity, the transmitted signal is assumed to have a rectangular pulse shape in time, and the received pulse is obtained by convolving the transmitted signal with the impulse response of the channel, where the impulse response is computed using the numerical integration approach, presented in [17]. In addition, the spatial distribution of the transmitted beam is assumed to be a Gaussian function.

Fig. 7 compares the symbol error probability of different detectors using simulation results. These results are for case A in Table I and for \( F = 23 \). For these system parameters, \( a_0 = 0.78 \) and \( a_1 = a_{-1} = 0.11 \). The simulation result for the optimum detector neglecting spatial broadening, in (17), is also plotted. According to these results, for this code the optimum single detector using PMT2 has a better performance compared to the same detector using PMT1. This means that for the optimum single detector, the lowest error probability is achieved when the decoder mask matches the complements of the codewords. OOK using a single-PMT results in an error rate between the two. The difference between the error rates of the optimum single-PMT2 detector and the optimum dual-PMT detector shows the performance price that is paid for having one PMT instead of two. The performance of the optimum linear dual-PMT detector is close to that of the optimum dual-PMT detector, and since it has a lower complexity, we prefer to use this detector in the receiver. Henceforth all results assume this detector, and assume that similar conclusions hold if another detector is chosen.

Fig. 8 shows the \( P_b \) versus the code length, \( F \), for the three physical geometries in Table I. The analytical results of the upper bounds are compared with simulation results. The bit-rate is assumed to be fixed for all code lengths \( (R_\theta = 1 \text{ Mb/s}) \). For small \( F \)'s, ISI is the main limiting factor on system performance. For a fixed bit rate, however, by increasing the number of symbols \( (F) \), the symbol rate decreases, and, consequently, the ISI effect is reduced. On
Fig. 7. Simulated symbol error probability, $P_s$, versus the received power for OOK and $M$-ary SAC using an optimum dual PMT detector, optimum single PMT1 detector, optimum single PMT2 detector, optimum linear detector and optimum detector neglecting spatial broadening for case A and $F=23$.

Fig. 8. $P_b$ of the optimum linear dual-PMT detector versus the code length, $F$, for cases A, B and C defined in Table I.

For low transmitter and receiver inclination angles the optimum modulation is OOK, since the ISI is low for these geometries. In Fig. 8, for cases A and B, defined in Table I, the ISI limits the $P_b$ and the optimum values of $F$ are 15 and 19, respectively. In case C, since the elevation angle of both the receiver and the transmitter is small, the impulse response is shorter than the former cases and the ISI effect is smaller. Consequently, the optimum value of $F$ is 7. Also, according to these results the union bound is notably close to the simulation results and can be used as a good approximation for the $P_b$.

The maximum achievable bit rate to sustain a $P_b$ of at most $3 \times 10^{-5}$ is plotted in Fig. 9 versus the distance between the transmitter and receiver. The parameters of case B in Table I are assumed to obtain these results. For this $P_b$ and short distances the bit rate is limited by ISI for every constellation size. By increasing the distance, since the loss increases, the performance becomes limited by the SNR. For small distances, larger $F$s are able to achieve higher data rates compared to smaller $F$’s. But, as the distance increases, the SNR affects the $P_b$, and the optimum $F$ decreases. So for long ranges, small $F$’s or OOK become better choices for data transmission.

In the SNR limit, $P_b$ depends only on the received energy per symbol. According to (28) the received energy per symbol in NLOS UV links can be approximated as

$$E_s(D) = \frac{\alpha T_s}{D} e^{-\beta D},$$

where $\alpha$ and $\beta$ depend on the system and geometric parameters. Therefore, for longer ranges, $T_s$ needs to increase exponentially as $D$ increases for a fixed $E_s(D)$.

The maximum distance-rate product is presented in Fig. 10 in terms of the number of symbols, $F$, for $P_b = 3 \times 10^{-5}$. The optimum code length is larger for shorter distances and this maximum distance-rate product increases by decreasing the length of the link. Although for OOK and $F = 7$ this product is similar over a range of distances, for larger code lengths it varies considerably. This is because for smaller symbol sizes, the distance-rate product stays close to the ISI limit even for long ranges, while for a larger number of symbols
it separates from the ISI limit at short distances and then decreases exponentially. Using our SAC system increases the distance-rate product by a factor varying from 3 to 5 compared to OOK, depending on the transmission distance.

In Fig. 10 the maximum distance-rate product for the optimum linear dual-PMT detector is also plotted neglecting OOK, depending on the transmission distance.

Performance comparisons between systems utilizing the suggested technique and systems using an equalizer or error correcting code will be the subject of future work.

VIII. ACKNOWLEDGMENT

This research was funded by the National Science Foundation (NSF) under grant number ECCS-0901682.

APPENDIX A

In this appendix, in order to find the coefficients of the optimal linear detector in (23), we obtain the optimum values for \( y_1, y_2, \ldots, y_n \) to maximize

\[
H_n = \frac{(\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n)^2}{\beta_1 y_1^2 + \beta_2 y_2^2 + \cdots + \beta_n y_n^2}. \tag{A.1}
\]

Using mathematical induction, we prove that the maximum of \( H_n \) in (A.1) is given by

\[
\max_{y_1,\ldots,y_n} H_n = \sum_{i=1}^{n} \frac{\alpha_i^2}{\beta_i},
\]

and the solution is \( y_i = \frac{\alpha_i \beta_i}{\sum_{i=1}^{n} \alpha_i \beta_i}, i = 1, 2, \ldots, n \), where \( y_i \) can be any value. For the base step of the induction, we verify this statement for \( n = 2 \). For \( H_2 = (\alpha_1 y_1 + \alpha_2 y_2)^2 / (\beta_1 y_1^2 + \beta_2 y_2^2) \), the maximum is \( \frac{\alpha_1^2}{\beta_1} + \frac{\alpha_2^2}{\beta_2} \), and it is obtained for \( y_2 = \frac{\alpha_2^2}{\beta_2} y_1 \).

Our inductive assumption is that the statement is true for \( n = m - 1 \), and then we prove it is also true for \( n = m \). In order to maximize \( H_m \), the partial derivative of \( H_m \) with respect to \( y_m \) should be zero, i.e., \( \frac{\partial H_m}{\partial y_m} = 0 \). The solution of this equation is

\[
y_m = \frac{\alpha_1 (\beta_1 y_1^2 + \beta_2 y_2^2 + \cdots + \beta_{m-1} y_{m-1}^2)}{\beta_1 (\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_{m-1} y_{m-1})}, \tag{A.2}
\]

for which \( H_m \) becomes

\[
H_m = \frac{\alpha_m^2}{\beta_m} + H_{m-1}, \tag{A.3}
\]

where

\[
H_{m-1} = \left[ \frac{\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_{m-1} y_{m-1}}{\beta_1 y_1^2 + \beta_2 y_2^2 + \cdots + \beta_{m-1} y_{m-1}^2} \right]^2.
\]

By maximizing \( H_{m-1} \) we maximize \( H_m \) in (A.3). According to the inductive assumption, the maximum of \( H_{m-1} \) is

\[
\max_{y_1,\ldots,y_{m-1}} H_{m-1} = \sum_{i=1}^{m-1} \frac{\alpha_i^2}{\beta_i},
\]

which is obtained for \( y_i = \frac{\alpha_i \beta_i}{\sum_{i=1}^{m-1} \alpha_i \beta_i}, i = 1, 2, \ldots, m-1 \). Substituting these values in (A.2) and (A.3), we get \( y_{m-1} = \frac{\alpha_{m-1} \beta_{m-1}}{\beta_{m-1} \alpha_{m-1}} y_{m-1} \) and

\[
\max_{y_1,\ldots,y_m} H_m = \sum_{i=1}^{m} \frac{\alpha_i^2}{\beta_i},
\]

and this completes the inductive step. Using these results, (25) follows from (24).
REFERENCES


[14] M. Noshad and K. Jamshidi, “Bounds for the BER of codes with fixed different sources and corrupted by non-Gaussian noise. This interest has found applications in a variety of research projects including spread-spectrum multiple-access schemes, multimuser demodulation and detection, study of nonlinear effects on fiber-optic multuser/multichannel communications, optical networks subject to physical layer degradations, free-space optical multimuser communications, biomedical data processing, and radar signal processing and tracking of multiple targets. Dr. Brandt-Pearce has over 100 major journal and conference publications.

Stephen G. Wilson received degrees from Iowa State University, University of Michigan, and University of Washington, all in electrical engineering. He held a staff engineering position at Boeing (Seattle), before joining the faculty at the University of Virginia, where he has been since 1976. He is currently Professor of Electrical and Computer Engineering, and conducts research in digital modulation and coding, signal processing for communication, and applications of information theory. He is author of Digital Modulation and Coding (Pearson Prentice-Hall), and was associate editor for Coding Theory and Applications, IEEE TRANSACTIONS ON COMMUNICATIONS. Prof. Wilson has been recognized for outstanding teaching with the University of Virginia’s Distinguished Professor Award, and the State Council of Education in Virginia’s Outstanding Faculty Award.