

A Digest on Representation Theory of the Symmetric Group

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I. REPRESENTATION THEORY OF THE SYMMETRIC GROUP

In these notes I try to give an introduction to a number of (more or less) basic facts about symmetric functions and their relation to representations of the symmetric group. Most of the material is taken from Chapter I of [7] and from [6]. Omissions are intentional. Mistakes obviously not, and comments would be **most** welcome!

A. Partitions

A *partition* is a sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, \dots)$$

of non-negative integers in non-increasing order

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq \dots$$

and containing finitely many non-zero terms. The non-zero terms λ_i are called the *parts* of λ . The *length* of λ , denoted $l(\lambda)$, is the number of parts of λ . The *weight* of λ , denoted $|\lambda|$, is the sum of the parts:

$$|\lambda| = \sum_i \lambda_i.$$

A partition λ with weight $|\lambda| = n$ is also called a partition of n , and this is denoted $\lambda \vdash n$. We will also use the non-standard notation $\lambda \vdash n; r$, combining $\lambda \vdash n$ and $l(\lambda) \leq r$ in one statement.

For $\lambda \vdash n$, we use the shorthand

$$\bar{\lambda} := \lambda/n.$$

For $i \geq 1$, the i -th element of λ is denoted by λ_i . This element is a part if $i \leq l(\lambda)$, otherwise it is 0. It is frequently convenient to use a different notation that indicates the number of times each integer $i = 1, 2, \dots, |\lambda|$ occurs as a part, the so-called *multiplicity* m_i of i :

$$\lambda = (1^{m_1} 2^{m_2} \dots r^{m_r} \dots).$$

As a shorthand we shall use a superscripted index:

$$\lambda^i = m_i(\lambda).$$

Clearly, one has the relations

$$\sum_{i=1}^n \lambda^i = l(\lambda)$$

$$\sum_{i=1}^n i \lambda^i = |\lambda| = n.$$

When dealing with numerical calculations it is necessary to impose an ordering on the set of partitions. We will adhere here to the *reverse lexicographic ordering*, in which λ precedes μ , denoted $\lambda > \mu$, if and only if the first non-zero difference $\lambda_i - \mu_i$ is positive.

Example 1 Under this convention the partitions of 5 are ordered

$$(5), (41), (32), (31^2), (2^21), (21^3), (1^5).$$

B. Young frames and Young tableaux

Partitions can be graphically represented by *Young frames*, which are Young tableaux with empty boxes. The i -th part λ_i corresponds to the i -th row of the frame, consisting of λ_i boxes. Conversely, the Young frames of n boxes can be uniquely labelled by a partition $\lambda \vdash n$. We will therefore identify a Young frame with the partition labelling it.

A *Young tableau* (YT) of N objects and of shape $\lambda \vdash n$ is a Young frame λ in which the boxes are labelled by numbers $(1, \dots, N)$.

A *standard Young tableau* (SYT) of shape $\lambda \vdash n$ is a Young tableau of $N = n$ objects such that the labels appear *increasing* in every row from left to right, and *increasing* in every column downwards; hence every number occurs exactly once.

A *Semistandard Young tableau* (SSYT) of shape $\lambda \vdash n$ is a Young tableau such that the labels appear *non-decreasing* in every row from left to right, and *increasing* in every column downwards.

The number of SSYTs of N objects and of shape $\lambda \vdash n$ (imposing the condition $l(\lambda) \leq N$) is given by $s_\lambda(1^{\times N})$; see below for an explanation.

The number f^λ of SYTs of shape $\lambda \vdash n; r$ is ([6] p. 119)

$$f^\lambda = n! \frac{\Delta(\nu_1, \dots, \nu_r)}{\nu_1! \dots \nu_r!}, \quad r = l(\lambda), \quad (1)$$

where $\Delta(x_1, x_2, \dots, x_r)$ denotes the *difference product* of a non-increasing sequence

$$\Delta(x_1, x_2, \dots, x_r) := \prod_{i < j} (x_i - x_j),$$

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and the numbers $\nu_i = \nu_i(\lambda)$ are defined by

$$\nu_i(\lambda) := \lambda_i + l(\lambda) - i, \text{ for } i = 1, 2, \dots, l(\lambda).$$

Since the numbers ν_i always form a strictly decreasing sequence, the Δ factor is larger or equal to 1, giving a lower bound

$$f^\lambda \geq \frac{n!}{\nu_1! \dots \nu_r!}. \quad (2)$$

There exists an alternative formula for f^λ by Frame, Robinson and Thrall ([6], p.211):

$$f^\lambda = n! / \prod_{(i,j) \in \lambda} h_{ij}. \quad (3)$$

Here the boxes in the Young frame λ are designated by their coordinates, row i and column j , the *hook* at box (i, j) is the set of boxes $(i', j') \in \lambda$ with

$$(i' = i \wedge j' \geq j) \vee (i' \geq i \wedge j' = j),$$

and h_{ij} is the *hook-length* of the hook at box (i, j) , which is the number of boxes in the hook. In other words, h_{ij} equals the number of boxes to the right of the corner box (i, j) plus the number of boxes below it, plus 1 (the corner box itself). From this formula follows the upper bound, obtained by ignoring in every hook (i, j) the boxes below the corner box (i, j) :

$$f^\lambda \leq \frac{n!}{\lambda_1! \dots \lambda_r!}. \quad (4)$$

C. Permutations

The symmetric group S_n of order n is the group of all $n!$ permutations of n objects. We will denote an explicit permutation π as $[\pi(1) \pi(2) \dots \pi(n)]$, using square brackets to distinguish them from partitions with the same elements. The unit permutation $[1 2 \dots n]$ will be denoted by e . Every permutation $\pi \in S_n$ decomposes uniquely as a product of disjoint cycles ([6], p.28 ff).

Example 2 The permutation $[1 3 4 2 6 5]$ consists of the cycles $1 \rightarrow 1$, $5 \rightarrow 6 \rightarrow 5$, and $3 \rightarrow 4 \rightarrow 2 \rightarrow 3$, of order 1, 2 and 3.

The orders of the cycles, sorted in non-increasing order, determine the *cycle type* of the permutation. Evidently, the cycle type of a permutation $\pi \in S_n$ is a partition of n . We will denote the cycle type of a permutation $\pi \in S_n$ by $\rho = \rho(\pi) \vdash n$.

It turns out that the cycle type labels the *conjugacy classes* of S_n . Two permutations $\pi, \pi' \in S_n$ are conjugates if and only if there is a third permutation $\sigma \in S_n$ such that

$$\pi = \sigma \pi' \sigma^{-1}.$$

It is immediately clear that two permutations are conjugates if and only if they have the same cycle type; the action of

σ amounts simply to a relabelling of the objects in the cycle description. The relation “is conjugate to” is an equivalence relation, the induced equivalence classes are called the conjugacy classes of S_n , and they are, indeed, labelled by the cycle type ρ of their elements.

With a minor abuse of notation, we shall identify the conjugacy classes with their cycle type, and even write $\pi \in \rho$ for a permutation π with cycle type ρ .

The number of elements in a conjugacy class ρ of S_n , denoted h_ρ , is given by

$$h_\rho = n! z_\rho^{-1}, \quad (5)$$

where z_ρ is the integer

$$z_\rho = \rho^1! 1^{\rho^1} \rho^2! 2^{\rho^2} \dots \rho^n! n^{\rho^n}. \quad (6)$$

For example, $h_{(n)} = (n-1)!$ and $h_{(1^n)} = 1$. Obviously, we need to have

$$\sum_{\rho \vdash n} z_\rho^{-1} = 1.$$

D. Symmetric functions

The symmetric group S_n acts on the ring of polynomials in n independent variables x_1, x_2, \dots, x_n by permuting the variables, and a polynomial is *symmetric* if it is invariant under this action. In the theory of symmetric functions, it is often more convenient to work with symmetric functions in infinitely many variables. Because of the infinity of the number of variables, these functions cannot be called polynomials. Symmetric polynomials are obtained by setting all but a finite number of variables to 0.

There are a lot of special types of symmetric functions: the elementary symmetric functions are most widely known, but there also exist monomial symmetric functions, complete symmetric functions, power sum products, and Schur symmetric functions. We will briefly describe the types that will be most relevant for the purpose of these notes and refer to [7] for a more thorough description.

In all the cases described below, the symmetric functions are labelled by a partition λ of n , and are homogeneous in their variables of order n . That is for $\lambda \vdash n$, $f_\lambda(tx) = t^n f_\lambda(x)$. In the important case where k variables assume the value 1 and all others are 0, we denote the argument by $1^{\times k}$.

1. Monomial Symmetric Functions

The *monomial symmetric polynomials* m_μ in k variables, with $\mu \vdash n; k$, are given by:

$$m_\mu(x_1, \dots, x_k) = \sum_{\alpha} x_1^{\alpha_1} \dots x_k^{\alpha_k},$$

where the sum is over all *distinct* permutations α of $\mu = (\mu_1, \dots, \mu_k)$. Recall that, for $j > l(\mu)$, $\mu_j = 0$. The condition $k < l(\mu)$ must hold because otherwise $m_\mu(x_1, \dots, x_k) = 0$.

Example 3

$$m_{(2,1,1)}(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2.$$

The value of m_μ at $x = 1^{\times k}$ gives the number of terms in this sum and is equal to the multinomial coefficient

$$\binom{k}{\mu^1, \dots, \mu^n, k - l(\mu)}.$$

Any symmetric polynomial $P(x)$ can be written in terms of the m_μ by expanding P in its monomials and replacing the terms $x_1^{j_1} x_2^{j_2} \dots$ by $m_{(j_1, j_2, \dots)}(x)$ if $j_1 \geq j_2 \geq \dots$, and by 0 otherwise.

2. Products of Power Sums

For an integer $r \geq 1$, the r -th power sum in the variables x_i is

$$p_r = \sum_i x_i^r.$$

For a partition $\rho \vdash n$; r , the power sum products p_ρ are defined by

$$p_\rho = p_{\rho_1} p_{\rho_2} \dots p_{\rho_r}.$$

As a special case,

$$p_\rho(1^{\times k}) = k^r, \quad (7)$$

where $r = l(\rho)$ is nothing but the number of cycles in ρ .

3. Schur Functions

To define the Schur symmetric functions, or S-functions, it is best to start with the polynomial case, i.e. with a finite number k of variables x_1, \dots, x_k . The complete set of S-functions is obtained by letting k tend to infinity. The S-functions s_λ of k variables and of homogeneity order n are labelled by partitions $\lambda \vdash n$; k , and are defined by

$$s_\lambda(x_1, \dots, x_k) := \frac{\det(x_i^{\lambda_j + k - j})_{i,j=1}^k}{\det(x_i^{k-j})_{i,j=1}^k}$$

(recall again that for $j > l(\lambda)$, $\lambda_j = 0$). For $l(\lambda) > k$, one again has $s_\lambda(x_1, \dots, x_k) = 0$. If some variables assume equal values, a limit has to be taken, since both numerator and denominator vanish in that case.

The denominator in the definition of the S-function is a Vandermonde determinant and is thus equal to $\Delta(x_1, \dots, x_k)$. The numerator is divisible (in the ring of polynomials) by each of the differences $x_i - x_j$, and therefore also by the denominator; hence the S-functions in a finite number of variables really are polynomials.

Example 4 For $k = 2$ the non-zero Schur polynomials are

$$\begin{aligned} s_{(0)}(x, y) &= 1 \\ s_{(1)}(x, y) &= x + y \\ s_{(2)}(x, y) &= x^2 + xy + y^2 \\ s_{(1^2)}(x, y) &= xy \\ &\dots \\ s_{(p,q)}(x, y) &= \sum_{j=q}^p x^j y^{p+q-j}, \end{aligned}$$

with $p \geq q \geq 0$.

For the important case where all k variables assume the value 1 (i.e. giving the number of semistandard Young tableaux of k objects and of shape λ), we get ([6], p.201), for $l(\lambda) \leq k$:

$$s_\lambda(1^{\times k}) = \frac{\Delta(\lambda_1 + k - 1, \lambda_2 + k - 2, \dots, \lambda_k)}{\Delta(k - 1, k - 2, \dots, 0)}, \quad (8)$$

and, again, $s_\lambda(1^{\times k}) = 0$ for $l(\lambda) > k$. Note that

$$\Delta(k - 1, \dots, 0) = 1!2! \dots (k - 1)!$$

Example 5 For $\lambda = (n)$, one easily finds

$$s_{(n)}(1^{\times k}) = \binom{n + k - 1}{n}. \quad (9)$$

Also, if $k = 2$, then

$$s_{(p,q)}(1, 1) = p - q + 1.$$

An upper bound for $s_\lambda(1^{\times k})$ is [12]

$$s_\lambda(1^{\times k}) \leq (n + k)^{k(k-1)/2}, \quad (10)$$

showing that this value depends only polynomially on n . An improved bound, also by M. Christandl is

$$s_\lambda(1^{\times k}) \leq (n + 1)^{k(k-1)/2}. \quad (11)$$

This can be further improved to

$$s_\lambda(1^{\times k}) \leq \binom{n + k - 1}{n} f^\lambda. \quad (12)$$

This will be proven at the end of Section G.

The best lower bound on $s_\lambda(1^{\times k})$ that only takes n , $l(\lambda)$ and k into account is, somewhat surprisingly, for $l(\lambda) \leq k$: $s_\lambda(1^{\times k}) \geq 1$, with equality if all λ_i are equal and $l(\lambda) = k$. In the latter case, f^λ can be arbitrarily high.

4. Hall Inner Product and Transition Matrices

Each type of symmetric function mentioned above forms a basis of the space of symmetric functions. P. Hall introduced an inner product \langle, \rangle on this space, defined in an abstract way

by requiring that $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, where h_λ are the completely symmetric functions (we will not need their definition here). One can show

$$\begin{aligned}\langle p_\lambda, p_\mu \rangle &= z_\lambda \delta_{\lambda\mu} \\ \langle s_\lambda, s_\mu \rangle &= \delta_{\lambda\mu},\end{aligned}$$

which shows that the p_λ form an orthogonal basis of the symmetric functions, and the s_λ an orthonormal one.

Therefore, each different type of symmetric function can be expressed in terms of every other type (provided all have the same degree of homogeneity n), via a particular change of basis in the space of symmetric functions. A matrix expressing such a change of basis is called a *transition matrix* and its rows and columns are labelled by the different partitions of n , customarily ordered in reverse lexicographic ordering.

The transition matrix $K_{\lambda\mu}$ from m_μ to s_λ

$$s_\lambda = \sum_{\mu} K_{\lambda\mu} m_\mu, \quad (13)$$

is upper triangular and its entries are non-negative integers. These numbers are known as the Kostka numbers and have a combinatorial interpretation. The diagonal elements as well as the elements on the first row are all equal to one: $\forall \lambda \vdash n : K_{\lambda\lambda} = K_{(n),\lambda} = 1$. From this it follows, for example, that $s_{(n)}(x)$ is the sum of all degree- n monomials in the variables x (a fact which can, of course, also be derived directly from the definition of the Schur functions). In particular, then, we find that the number of all degree- n monomials in k variables is $s_{(n)}(1^{\times k}) = \binom{n+k-1}{n}$.

The transition matrix from p_ρ to m_μ is lower triangular and consists also of non-negative integers, again with a combinatorial interpretation [11]. The transition matrix from s_λ to p_ρ is the character table of S_n and will be discussed in the next section.

E. Characters of the symmetric group

The characters of a group are the traces of the representations of the group elements. As general representations decompose into direct sums of irreducible representations (irreps), the characters themselves also decompose into sums of simple characters. These simple characters form a cornerstone of representation theory and in fact determine the group completely. The simple characters can be arranged in a square table called the *character table*.

In the case of the symmetric group, the irreps are labelled by Young frames λ . The character of a permutation $\pi \in S_n$ in irrep λ is denoted $\chi^\lambda(\pi)$. Since characters are class functions, one only needs to find the characters of any representative of a conjugacy class, so that one can use the symbol χ_ρ^λ , with

$$\chi_\rho^\lambda = \chi^\lambda(\pi), \forall \pi \in \rho.$$

The character table is the matrix with elements χ_ρ^λ , where λ is the row index and ρ the column index (assuming reverse lexicographic ordering for both). As the conjugacy classes of

S_n are labelled by partitions of n , there are as many rows as columns, hence the character table is a square matrix.

Example 6 *The character table of S_3 is*

$\lambda \setminus \rho$	(3)	(2, 1)	(1 ³)
h_ρ	2	3	1
(3)	1	1	1
(2, 1)	-1	0	2
(1 ³)	1	-1	1

The character of the identity permutation e equals the degree of the representation in the given irrep. One can show that this degree is equal to the number of standard Young tableaux of shape λ ([6], p.170)

$$\chi^\lambda(e) = f^\lambda. \quad (14)$$

The characters in irrep $\lambda = (n)$ are all 1:

$$\chi_\rho^{(n)} = 1, \forall \rho \vdash n. \quad (15)$$

For ρ consisting of one cycle, $\rho = (n)$, the characters are ([6], p.208)

$$\chi_{(n)}^\lambda = \begin{cases} (-1)^k, & \lambda = (n-k, 1^k), 0 \leq k \leq n \\ 0, & \text{otherwise.} \end{cases} \quad (16)$$

The calculation of the other characters is rather involved. For numerical calculations one resorts to combinatorial rules such as the Murnaghan-Nakayama rule [6].

A very useful property of the simple characters is the orthogonality of the matrix $(z_\rho^{-1/2} \chi_\rho^\lambda)_{\lambda\rho}$. Thus one obtains two orthogonality relations:

$$(n!)^{-1} \sum_{\pi \in S_n} \chi^\lambda(\pi) \chi^{\lambda'}(\pi) = \sum_{\rho} z_\rho^{-1} \chi_\rho^\lambda \chi_\rho^{\lambda'} = \delta_{\lambda\lambda'} \quad (17)$$

and

$$\sum_{\lambda} \chi_\rho^\lambda \chi_{\rho'}^\lambda = z_\rho \delta_{\rho\rho'} \quad (18)$$

As a simple application of the latter orthogonality relation one obtains

$$\begin{aligned} \sum_{\lambda \vdash n} (f^\lambda)^2 &= \sum_{\lambda \vdash n} \chi^\lambda(e) \chi^\lambda(e) \\ &= z_{\rho(e)} = z_{(1^n)} = n! \end{aligned}$$

The numbers $(f^\lambda)^2/n!$ therefore form a distribution over the Young frames (called the *Plancherel measure of the symmetric group*).

We now briefly consider the irreducible polynomial representations of the full linear group $GL(d, \mathbb{C})$. These representations get their name from the fact that their matrix elements are polynomials in the elements of the represented matrix. Other kinds of representations exist, but they are not relevant in this context. Just like the irreps of the symmetric group, the

polynomial irreps of $GL(d, \mathbb{C})$ are labelled by Young frames. The conjugacy classes of $GL(d, \mathbb{C})$ consist of all matrices $A \in GL(d, \mathbb{C})$ having the same eigenvalues (a_1, \dots, a_d) and thus can be labelled by these eigenvalues. The simple characters (known, in this context, as *characteristics*) are denoted $\phi_\lambda(A) = \phi_\lambda(a_1, \dots, a_d)$. According to a famous result by Schur, these characters are the Schur functions (polynomials) of the eigenvalues ([6], p. 199):

$$\phi_\lambda(a_1, \dots, a_d) = s_\lambda(a_1, \dots, a_d).$$

F. Representations of S_n and $GL(d, \mathbb{C})$ on the tensor product space $(\mathbb{C}^d)^{\otimes n}$.

In this subsection we consider the space of rank- n tensors belonging to $\mathcal{H} = \mathbb{C}^d$ and take it as the representation space for representations of both the symmetric group S_n and of the full linear group $GL(d, \mathbb{C})$. Consider thereto an n -fold tensor product of vectors from \mathcal{H} : $\psi_1 \otimes \dots \otimes \psi_n$. We can let the symmetric group S_n act on this tensor by permuting the factors:

$$\pi : \psi_1 \otimes \dots \otimes \psi_n \rightarrow \psi_{\pi^{-1}(1)} \otimes \dots \otimes \psi_{\pi^{-1}(n)}.$$

This is a representation of S_n , in which a permutation π is represented by an *index permutation matrix* P_π : for any $\psi^{(n)} \in \mathcal{H}^{\otimes n}$:

$$\begin{aligned} \psi^{(n)} &\rightarrow \psi' = P_\pi \psi^{(n)} \\ \psi'_{i_{\pi(1)} i_{\pi(2)} \dots i_{\pi(n)}} &= \psi_{i_1 i_2 \dots i_n}. \end{aligned}$$

Since $P_\pi P_{\pi'} = P_{\pi\pi'}$, this is indeed a representation of S_n . To completely specify the matrix P_π one has to mention the underlying Hilbert space. Thus, if it is not clear from the context which space is meant, we will explicitly write $P_\pi(\mathcal{H})$.

The space of rank- n tensors can also serve as representation space for $GL(d, \mathbb{C})$. Let A be an element of $GL(d, \mathbb{C})$, i.e. a non-singular transformation on \mathcal{H} : $\psi \mapsto \psi' = A\psi$. This transformation induces in the tensor space the transformation

$$\psi^{(n)} \mapsto A^{\otimes n} \psi^{(n)}, \forall \psi^{(n)} \in \mathcal{H}^{\otimes n}$$

Because $A^{\otimes n} B^{\otimes n} = (AB)^{\otimes n}$, these n -fold tensor powers $A^{\otimes n}$ are representations of $GL(d, \mathbb{C})$. Specifically, they are *polynomial* representations, because the matrix elements of $A^{\otimes n}$ are polynomials in the matrix elements of A .

The two representations just mentioned turn out to be each others commutant. Any operator on $\mathcal{H}^{\otimes n}$ that commutes with all $A^{\otimes n}$, for all elements A of $GL(d, \mathbb{C})$, is a linear combination of index permutation matrices $P_\pi(\mathcal{H})$. Conversely, any operator commuting with all index permutation matrices P_π , for all $\pi \in S_n$, is a linear combination of tensor powers $A^{\otimes n}$ ([6], p. 148 and 150). This duality is called the *Weyl-Schur duality*. Consequently, the reductions of these two representations into irreducible components are strongly related to one another. In both cases, the components are labelled by Young frames $\lambda \vdash n$. The same basis that produces the decomposition of the P_π also produces the decomposition of the $A^{\otimes n}$.

In that basis, called the *Schur basis*, we get

$$\mathcal{H}^{\otimes n} = \bigoplus_{\lambda \vdash n} \mathcal{R}_\lambda \otimes \mathcal{S}_\lambda \quad (19)$$

$$P_\pi = \bigoplus_{\lambda \vdash n} \mathbb{1}_{r(\lambda)} \otimes S_\lambda(\pi) \quad (20)$$

$$A^{\otimes n} = \bigoplus_{\lambda \vdash n} R_\lambda(A) \otimes \mathbb{1}_{s(\lambda)}. \quad (21)$$

We also have

$$P_\pi A^{\otimes n} = A^{\otimes n} P_\pi = \bigoplus_{\lambda \vdash n} R_\lambda(A) \otimes S_\lambda(\pi).$$

The decomposition (19) is called the *Wedderburn decomposition* of $\mathcal{H}^{\otimes n}$. Here, we have denoted the dimension of the subspace \mathcal{R}_λ by $r(\lambda)$, and the dimension of \mathcal{S}_λ by $s(\lambda)$. The matrix $R_\lambda(A)$ is an irrep of $A \in GL(d, \mathbb{C})$ of degree $r(\lambda)$, operating on \mathcal{R}_λ . The matrix $S_\lambda(\pi)$ an irrep of $\pi \in S_n$ of degree $s(\lambda)$, operating on \mathcal{S}_λ . We can and we will take both irreps unitary.

Taking traces yields the corresponding simple characters

$$\text{Tr } R_\lambda(A) = s_\lambda(a_1, \dots, a_d) \quad (22)$$

$$\text{Tr } S_\lambda(\pi) = \chi^\lambda(\pi) = \chi_{\rho(\pi)}^\lambda, \quad (23)$$

where a_1, \dots, a_d are the eigenvalues of A . For the dimensions one finds ([6], p. 119 and 201)

$$r(\lambda) = \text{Tr } R_\lambda(\mathbb{1}_d) = s_\lambda(1^{\times d}) \quad (24)$$

$$s(\lambda) = \text{Tr } S_\lambda(e) = \chi^\lambda(e) = f^\lambda, \quad (25)$$

i.e. $r(\lambda)$ is the number of semistandard Young tableaux λ of d objects, and $s(\lambda)$ is the number of standard Young tableaux λ .

The presence of the identity matrices indicates that the irreducible representations occur a number of times in the decompositions. This number is called the *multiplicity*. So, while $r(\lambda)$ is the degree of irrep $R_\lambda(A)$, it is also the multiplicity of irrep $S_\lambda(\pi)$ in P_π . Likewise, $s(\lambda)$ is the degree of $S_\lambda(\pi)$ and also the multiplicity of $R_\lambda(A)$ in $A^{\otimes n}$.

In accordance with these decompositions, the tensor space $\mathcal{H}^{\otimes n}$ splits up into invariant subspaces. The subspaces $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$ are invariant under all $A^{\otimes n}$ and all P_π . They are further reducible into direct sums of $s(\lambda)$ subspaces of dimension $r(\lambda)$, invariant under the transformations $A^{\otimes n}$ but no longer invariant under index permutations P_π . These irreducible invariant subspaces are called the *symmetry classes* of the tensor space. They are labelled by standard Young tableaux of shape λ .

The orthogonality relations (17) and (18) follow from the so-called *Great Orthogonality Theorem* (GOT) about the matrix elements of the irreps of a group. For a group G of order g , with *unitary* irreps D^α and D^β of dimension d_α and d_β , we have

$$g^{-1} \sum_{x \in G} D_{ip}^\alpha(x) \overline{D_{jq}^\beta(x)} = d_\alpha^{-1} \delta_{\alpha\beta} \delta_{ij} \delta_{pq}.$$

For the irreps of the symmetric group S_n this yields

$$(n!)^{-1} \sum_{\pi \in S_n} S_\lambda(\pi)_{ip} S_\mu(\pi)_{jq} = (f^\lambda)^{-1} \delta_{\lambda\mu} \delta_{ij} \delta_{pq}. \quad (26)$$

Example 7 The best-known examples of symmetry classes are undoubtedly the totally symmetric class, corresponding to $\lambda = (n)$, and the totally antisymmetric class, corresponding to $\lambda = (1^n)$, which are used in the description of bosonic and fermionic quantum systems, respectively. The totally symmetric class appears with multiplicity $f^{(n)} = 1$ and has dimension $\binom{n+d-1}{d-1}$. The totally antisymmetric class also appears with multiplicity 1. Its dimension is $\binom{d}{n}$, if $d \geq n$, or 0 otherwise. In fact, as can be seen from (24), the dimension of any symmetry class λ is 0 if $l(\lambda) > d$. The irreps corresponding to these classes are given by

$$R_{(n)}(A) = A^{\vee n} \quad (27)$$

$$R_{(1^n)}(A) = A^{\wedge n}, \quad (28)$$

borrowing notation of ([3], Section I.5). The elements of these irreps are permanents and determinants of submatrices of A , resp.

Example 8 The simplest possible example is the case $n = d = 2$, for which the totally symmetric and totally antisymmetric class are the only symmetry classes occurring. The unitary rotation that exhibits the decomposition in this case is the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}.$$

For any 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$A^{\otimes 2} = \begin{pmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{pmatrix},$$

and

$$SA^{\otimes 2}S^\dagger = \begin{pmatrix} a^2 & \sqrt{2}ab & b^2 & 0 \\ \sqrt{2}ac & ad + bc & \sqrt{2}bd & 0 \\ c^2 & \sqrt{2}cd & d^2 & 0 \\ 0 & 0 & 0 & ad - bc \end{pmatrix}.$$

This exhibits the decomposition

$$SA^{\otimes 2}S^\dagger = R_{(2)}(A) \oplus R_{(1^2)}(A),$$

with the correct dimensions (3 and 1, resp.) and multiplicities (both 1). It is also a simple exercise to check the decomposition of the index permutations into two irreps of degree 1 and multiplicities 3 and 1, resp.:

$$SP_\pi S^\dagger = S_{(2)}(\pi)\mathbb{1}_3 \oplus S_{(1^2)}(\pi)\mathbb{1}_1,$$

with $S_{(2)}(\pi)$ identically one and $S_{(1^2)}(\pi)$ equal to the signature of π .

G. Projectors P^λ on the Invariant Subspaces

We now consider the invariant subspaces $\mathcal{R}_\lambda \otimes \mathcal{S}_\lambda$, corresponding to the Young frames λ . Their dimension is $f^\lambda s_\lambda(1^{\times d})$. We will denote the projectors on these subspaces by P^λ . They are the sum of the Young projectors corresponding to the standard Young tableaux λ . We will consider the Young projectors themselves in the next subsection. The projectors P^λ form an orthogonal set and add up to the identity on the full tensor space:

$$P^\lambda P^{\lambda'} = \delta_{\lambda\lambda'} P^\lambda \quad (29)$$

$$\sum_{\lambda \vdash n} P^\lambda = \mathbb{1} \quad (30)$$

$$\text{Tr} P^\lambda = f^\lambda s_\lambda(1^{\times d}). \quad (31)$$

Consider the conjugacy classes ρ of S_n with cycle type $\rho \vdash n$. We define the ‘‘class average’’ of all index permutation matrices with cycle type ρ as

$$P_\rho := h_\rho^{-1} \sum_{\pi \in \rho} P_\pi \quad (32)$$

Note the distinction between the notations P^λ , where the superscript λ labels an irrep, and P_ρ , where the subscript ρ labels a conjugacy class. Alternatively, we can write

$$P_\rho = (n!)^{-1} \sum_{\pi' \in S_n} P_{\pi'\pi\pi'^{-1}}. \quad (33)$$

The projectors P^λ can be expressed in terms of the index permutations P_π , according to a general relation ([8], eqn. (12.10)), as:

$$P^\lambda = (n!)^{-1} f^\lambda \sum_{\pi \in S_n} \chi^\lambda(\pi) P_\pi, \quad (34)$$

and in terms of P_ρ as:

$$P^\lambda = f^\lambda \sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^\lambda P_\rho. \quad (35)$$

Let A be a matrix with eigenvalues (a_1, \dots, a_d) . Taking the trace of one λ -term in (21) immediately yields (compare with Theorem 1 in [1])

$$\text{Tr}[P^\lambda A^{\otimes n}] = f^\lambda s_\lambda(a_1, \dots, a_d). \quad (36)$$

For $\pi \in \rho \vdash n$, it is easy to see that

$$\text{Tr}[P_\pi A^{\otimes n}] = \text{Tr}[P_\rho A^{\otimes n}] = p_\rho(a_1, \dots, a_d). \quad (37)$$

Combining this with (35) gives the famous Frobenius formula, relating the characteristics of the full linear group to the characters of the symmetric group ([6], p. 204):

$$s_\lambda(a_1, \dots, a_d) = \sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^\lambda p_\rho(a_1, \dots, a_d). \quad (38)$$

As this holds for any A , and thus for any set of values a_i of whatever dimension, it yields the transition matrix from the p_ρ symmetric functions to the S-functions ([7], Chapter I, eq. (7.10)):

$$s_\lambda = \sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^\lambda p_\rho. \quad (39)$$

The relations (35) and (39) can be inverted. Using the orthogonality relations of the characters, we find

$$P_\rho = \sum_{\lambda \vdash n} \chi_\rho^\lambda P^\lambda / f^\lambda \quad (40)$$

$$p_\rho = \sum_{\lambda \vdash n} \chi_\rho^\lambda s_\lambda. \quad (41)$$

The latter relation can also be derived directly by taking the trace of both sides of (21) and using (23) and (37) with $A = \mathbb{1}_d$. The former relation says that P_ρ has a much simpler structure than the index permutations P_π it is composed of, namely P_ρ acts as a multiple of identity on each of the invariant subspaces $\mathcal{R}_\lambda \otimes S_\lambda$. This also follows from Schur's lemma, since P_ρ commutes with every P_π .

Since S_n is a finite discrete group, its representations are normal, that is, unitary apart from equivalence ([6], p.84). In particular, as can be seen from (20), the irrep $S_\lambda(\pi)$ is equivalent to a real orthogonal matrix. Its eigenvalues are thus roots of unity, so its trace must be smaller in modulus than its dimension $f^\lambda = \chi^\lambda(e)$. As this trace is just the character $\text{Tr } S_\lambda(\pi) = \chi_\rho^\lambda$, and, for S_n , all characters are real, we find

$$\chi_\rho^\lambda \leq \chi^\lambda(e) = f^\lambda, \quad \forall \lambda, \rho \vdash n. \quad (42)$$

This allows us to prove the still outstanding bound (12). By the above inequality, by (15), (39), (9), and the positivity of z_ρ and $p_\rho(1^{\times d})$,

$$\begin{aligned} s_\lambda(1^{\times d}) &= \sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^\lambda p_\rho(1^{\times d}) \\ &\leq \sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^{(n)} f^\lambda p_\rho(1^{\times d}) \\ &= s_{(n)}(1^{\times d}) f^\lambda \\ &= \binom{n+d-1}{n} f^\lambda. \end{aligned}$$

H. Young Projectors

The Young projectors of $\mathcal{H}^{\otimes n}$ are the projectors on the various irreducible invariant subspaces $\mathcal{R}_{\lambda,a}$ of $\mathcal{H}^{\otimes n}$ making up the composite invariant subspaces $\bigoplus_a \mathcal{R}_{\lambda,a} = \mathcal{R}_\lambda \otimes S_\lambda$. Every invariant subspace corresponds to a standard Young tableau of shape $\lambda \vdash n$, of which there are f^λ . The index a labels these SYTs. A convenient labelling scheme uses the Yamanouchi symbols, which are explained in the next section.

The Young projectors are built up from the projectors $P^{(k)}$ and $P^{(1^k)}$ on the totally symmetric and totally antisymmetric

subspaces, resp., of $\mathcal{H}^{\otimes k}$, $1 \leq k \leq n$. We will use a customary shorthand here:

$$\begin{aligned} S^k &:= k! P^{(k)} = \sum_{\pi \in S_k} P_\pi \\ A^k &:= k! P^{(1^k)} = \sum_{\pi \in S_k} (-1)^{\text{sgn}(\pi)} P_\pi. \end{aligned}$$

Furthermore, we use the notation S_{i_1, i_2, \dots, i_k} and A_{i_1, i_2, \dots, i_k} for the operators acting as S^k and A^k on indices i_1, i_2, \dots, i_k of $\mathcal{H}^{\otimes n}$ and as identity on the other indices.

The Young projector $P^{\lambda,a}$ corresponding to the standard Young tableau (λ, a) is the product of three factors. The first factor is the product of one S^k per row of λ , where each S^k operates on the indices contained in the boxes in its row; this corresponds to symmetrising each row. The second factor is the product of one A^k per column of λ , where each A^k operates on the indices contained in the boxes in its column; each column is anti-symmetrised. Note that the different S^k in the first factor commute, and so do the A^k in the second factor. The third factor is the normalisation factor $f^\lambda/n!$. Using a self-explanatory notation, we get

$$P^{\lambda,a} = \frac{f^\lambda}{n!} \prod_{k \in \text{Col}(\lambda,a)} A^k \prod_{l \in \text{Row}(\lambda,a)} S^l. \quad (43)$$

Example 9 *The Young projector corresponding to the SYT*

1	2	5
3	4	

is

$$P\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array}\right) = \frac{f^{(3,2)}}{5!} (A_{1,3} A_{2,4}) (S_{1,2,5} S_{3,4}).$$

The Young projectors for $\mathcal{H}^{\otimes 3}$ are

$$\begin{aligned} P\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}\right) &= P^{(3)} \\ P\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}\right) &= \frac{2}{3!} (P_{[1,2,3]} + P_{[2,1,3]} - P_{[3,2,1]} - P_{[2,3,1]}) \\ P\left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}\right) &= \frac{2}{3!} (P_{[1,2,3]} + P_{[3,2,1]} - P_{[3,1,2]} - P_{[2,1,3]}) \\ P\left(\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}\right) &= P^{(1^3)} \end{aligned}$$

One readily checks that

$$P\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \hline \end{array}\right) + P\left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \hline \end{array}\right) = P^{(2,1)}.$$

The Young projectors can be written in terms of the representations of S_n , as follows. Define the following set of operators

$$P^{\lambda,ij} = \frac{f^\lambda}{n!} \sum_{\pi \in S_n} (S_\lambda(\pi))_{ij} P_\pi, \quad (44)$$

where indexes i and j run through the f^λ SYTs (Yamanouchi symbols) for frame λ . The quantities $(S_\lambda(\pi))_{ij}$ are the matrix elements of permutation π in irrep λ , which are calculated in the next Section.

In the Schur basis, these operators assume the simple form

$$P^{\lambda,ij} = \mathbb{1}_{r(\lambda)} \otimes e^{ij}, \quad (45)$$

which can easily be proven from (??) and the Great Orthogonality Theorem. Hence,

$$\text{Tr } A^{\otimes n} P^{\lambda,ij} = \delta_{ij} \text{Tr } R_\lambda(A) = \delta_{ij} s_\lambda(a).$$

The Young projectors $P^{\lambda,a}$ are nothing but the operators $P^{\lambda,ii}$, with $a = i$. As is clear from their Schur form, the $P^{\lambda,ii}$ are mutually orthogonal projectors:

$$P^{\lambda,ii} P^{\mu,jj} = \delta_{ij} \delta_{\lambda\mu} P^{\lambda,ii}.$$

The projectors P^λ are given by the sums

$$P^\lambda = \sum_i P^{\lambda,ii} = \frac{f^\lambda}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi) P_\pi.$$

I. Matrix Elements of the Irreps of S_n .

The standard Young tableaux of a given shape λ can be conveniently labelled. Since the labels $1, \dots, N$ occur exactly once, and within a row they have to appear in increasing order, every SYT of shape λ is uniquely identified by specifying in which row each label occurs. This gives the so-called *Yamanouchi symbol* $M = (M_1, M_2, \dots, M_n)$, where M_i is the row containing i . SYTs of given shape λ can thus be ordered by imposing the lexicographic ordering on the Yamanouchi symbols.

The *axial distance* in a frame λ from one box to another is defined in terms of the box coordinates (i, j) as $\rho_\lambda((i_1, j_1), (i_2, j_2)) = (i_2 - i_1) + (j_1 - j_2)$. I.e. the axial distance increases upon going down one box or one box to the left. Likewise, the axial distance from labels x to y in a SYT with Yamanouchi symbol M is $\rho_M((i_x, j_x), (i_y, j_y))$, where the coordinates are those of the boxes in which the labels occur (as given by M).

Example 10 *The Yamanouchi symbol of the SYT*

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array}$$

is $M = (11221)$.

The $f^\lambda = 5$ SYTs of shape $\lambda = (3, 2)$ are ordered $(11122), (11212), (11221), (12112), (12121)$; (12211) does not refer to an SYT because its second column decreases.

The axial distance from 5 to 3 in the above SYT $M = (11221)$ is $\rho_M(5, 3) = -\rho_M(3, 5) = 3$.

Every permutation can be decomposed as a product of *transpositions* $T_j := (j, j + 1)$, i.e. permutations in which

objects j and $j + 1$ are swapped and the other objects remain in the same position. It is therefore enough to just give the irrep matrices of these transpositions.

We now consider the so-called *Young's orthogonal form* for representing the irreps of S_n . In this form, the irreps are orthogonal and they are most easily expressed in a basis in which the i -th basis vector of irrep λ corresponds to the i -th SYT of shape λ . We will, of course, label each SYT using its Yamanouchi symbol. By adopting this basis, this partly fixes the basis of the representations of $A^{\otimes n}$ and P_π . Let the basis vectors thus be labelled by e_M .

The transposition T_j acts on the labels $1, 2, \dots, N$ by interchanging label j with $j + 1$. Likewise, it acts on a Yamanouchi symbol $M = (M_1, M_2, \dots, M_N)$ by interchanging element M_j with M_{j+1} , giving a new symbol $M' = (M_1, \dots, M_{j+1}, M_j, \dots, M_N)$ which we denote by $T_j(M)$. The Young tableau referred to by M' need no longer be a standard one.

Denote by $a_{j,M}$ the inverse of the axial distance from $j + 1$ to j in SYT M :

$$a_{j,M} = 1/\rho_M(j + 1, j).$$

Then the matrix elements of T_j in irrep λ can be found from:

$$S_\lambda(T_j)e_M = a_{j,M} e_M + \sqrt{1 - a_{j,M}^2} e_{T_j(M)}.$$

The second term is automatically 0 if $T_j(M)$ is not a SYT.

Example 11 *Consider irrep (31) of S_4 . The SYTs are*

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array}$$

with Yamanouchi symbols (1112) , (1121) , and (1211) . Transposition $T_1 = (1, 2)$ acts identically on the first two and replaces the third one with the non-standard symbol (2111) . The basis vectors are $e_{(1112)}$, $e_{(1121)}$, and $e_{(1211)}$. The matrix elements of T_1 can be found from

$$\begin{aligned} S_{(31)}(T_1)e_{(1112)} &= 1.e_{(1112)} + 0.e_{(1112)} \\ S_{(31)}(T_1)e_{(1121)} &= 1.e_{(1121)} + 0.e_{(1121)} \\ S_{(31)}(T_1)e_{(1211)} &= -1.e_{(1211)}. \end{aligned}$$

The three matrices of the generating transpositions are

$$\begin{aligned} S_{(31)}(T_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ S_{(31)}(T_2) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix} \\ S_{(31)}(T_3) &= \begin{pmatrix} -1/3 & \sqrt{8}/3 & 0 \\ \sqrt{8}/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The matrix of any other permutation π can be obtained by decomposing it as a product of generator transpositions and multiplying the matrices of the generators accordingly. Such a decomposition follows easily from its decomposition in disjoint cyclic permutations. Denote a cyclic permutation $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k \rightarrow i_1$ by (i_1, i_2, \dots, i_k) . The decomposition of π into generator transpositions can be obtained using the following two rules:

$$\begin{aligned} (i_1, i_2, \dots, i_k) &= (i_1, i_2)(i_2, i_3) \dots (i_{k-1}, i_k) \\ (i, j) &= (i, \dots, j-1)(j-1, j)(i, \dots, j-1)^{-1}, \end{aligned}$$

where the last equation applies for $i < j$.

Consider the conjugacy class ρ and its Young frame. The *representative permutation* π_ρ in ρ can be labelled by a Young diagram in which the numbers 1 to N are written in reading order. The decomposition in T_j is obtained by dropping the last box in every row and taking the product $T_{i_1} T_{i_2} \dots$, where i_j is the j -th label in the remaining diagram. In other words, π_ρ is obtained by erasing $T_{\rho_1}, T_{\rho_1+\rho_2}, \dots, T_{\rho_1+\rho_2+\dots+\rho_t}$ from the product $T_1 T_2 \dots T_N$.

J. Matrix Elements of the Irreps of $GL(d, \mathbb{C})$.

[TODO] Equivalently (as it is the same): Matrix Elements of the Irreps of $U(d)$.

Gel'fand-Zetlin basis: basis vectors labelled by SSYT's (rather than SYT's). This can be conveniently done using Gel'fand-Zetlin patterns.

K. Immanants

The next paragraphs are loosely based on Section 10.1 in [18].

From the discussion of the totally symmetric and antisymmetric class, one gets the impression that the matrix elements of $R_\lambda(A)$ for general λ must be related to *immanants* of A , which are generalisations of matrix determinant and permanent [17]. For $X \in M_n(\mathbb{C})$:

$$\text{Imm}_\lambda(X) := \sum_{\pi \in S_n} \chi^\lambda(\pi) X_{1,\pi(1)} X_{2,\pi(2)} \dots X_{n,\pi(n)}.$$

Consider the following generalisation of a submatrix. For I, J vectors of size n the elements of which are integers in the range $1, \dots, d$, and $A \in M_d(\mathbb{C})$, let $A[I, J]$ be the $n \times n$ matrix with elements

$$(A[I, J])_{i,j} = A_{I_i, J_j}.$$

If all elements of I (J) are distinct, this is a submatrix; if not, $A[I, J]$ contains some elements of A a number of times.

We now show that the elements of $P^\lambda A^{\otimes n}$ are proportional to certain immanants. Considering the above index vectors I and J as multi-indices over $\mathcal{H}^{\otimes n}$, we have

$$\begin{aligned} (A^{\otimes n})_{I,J} &= A_{I_1, J_1} A_{I_2, J_2} \dots A_{I_n, J_n} \\ &= (A[I, J])_{1,1} (A[I, J])_{2,2} \dots (A[I, J])_{n,n}. \end{aligned}$$

Hence,

$$\begin{aligned} (P^\lambda A^{\otimes n})_{I,J} &= \frac{f^\lambda}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi) (P_\pi A^{\otimes n})_{I,J} \\ &= \frac{f^\lambda}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi) \prod_{k=1}^n (A[I, J])_{\pi^{-1}(k), k} \\ &= \frac{f^\lambda}{n!} \text{Imm}_\lambda(A[I, J]). \end{aligned}$$

L. Branching Rule $GL(d) \rightarrow GL(d-1)$.

[TODO]

M. Reductions of Tensor Products of Representations.

[TODO] Littlewood Richardson Coefficients Clebsch Gordan

N. Symmetric Functions and Representations of Tensor Products

A property of index permutation matrices that is both simple and powerful is that index permutation matrices over tensor products of Hilbert spaces are tensor products themselves. With a minor abuse of notation we identify $(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\otimes n}$ with $\mathcal{H}_1^{\otimes n} \otimes \mathcal{H}_2^{\otimes n}$ and write

$$P_\pi(\mathcal{H}_1 \otimes \mathcal{H}_2) = P_\pi(\mathcal{H}_1) \otimes P_\pi(\mathcal{H}_2). \quad (46)$$

Here, $P_\pi(\mathcal{H}_1)$ acts on $\mathcal{H}_1^{\otimes n}$ and $P_\pi(\mathcal{H}_2)$ on $\mathcal{H}_2^{\otimes n}$.

This corresponds to considering symmetric functions of tensor products of variables. If $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ then their tensor product, which is customarily denoted xy rather than $x \otimes y$, consists of all possible products $x_i y_j$. For power product sums one immediately sees

$$p_\rho(xy) = p_\rho(x) p_\rho(y).$$

Using (39) and (41), this yields for Schur functions ([7], p.63)

$$s_\lambda(xy) = \sum_{\mu, \nu \vdash n} g_{\lambda\mu\nu} s_\mu(x) s_\nu(y), \quad (47)$$

where $g_{\lambda\mu\nu}$ are the so-called *Kronecker coefficients*

$$g_{\lambda\mu\nu} := (n!)^{-1} \sum_{\pi \in S_n} \chi^\lambda(\pi) \chi^\mu(\pi) \chi^\nu(\pi) \quad (48)$$

$$= \sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^\lambda \chi_\rho^\mu \chi_\rho^\nu. \quad (49)$$

From this definition it follows that $g_{\lambda\mu\nu}$ is symmetric in its indices. In the setting of representation theory one proves that these coefficients are non-negative integers. Remarkably, a fully combinatorial description of the Kronecker coefficients still has not been discovered.

One of the rare cases in which a closed formula can be given for the Kronecker coefficients, is $\lambda = (n)$. One finds, using (15) and (17),

$$g_{(n)\mu\nu} = \delta_{\mu\nu},$$

and

$$s_{(n)}(xy) = \sum_{\lambda \vdash n} s_\lambda(x) s_\lambda(y). \quad (50)$$

A consequence of (47) is that for A operating on \mathcal{H}_1 and for B operating on \mathcal{H}_2

$$\begin{aligned} & \text{Tr}\left[\frac{P^\lambda}{f^\lambda} (A \otimes B)^{\otimes n}\right] \\ &= \sum_{\mu, \nu \vdash n} g_{\lambda\mu\nu} \text{Tr}\left[\frac{P^\mu}{f^\mu} A^{\otimes n}\right] \text{Tr}\left[\frac{P^\nu}{f^\nu} B^{\otimes n}\right], \end{aligned} \quad (51)$$

where P^λ acts on $(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\otimes n}$, P^μ on $\mathcal{H}_1^{\otimes n}$, and P^ν on $\mathcal{H}_2^{\otimes n}$. In terms of the irreps of $\text{GL}(d, \mathbb{C})$ we have

$$R_\lambda(A \otimes B) \cong \bigoplus_{\mu, \nu \vdash n} g_{\lambda\mu\nu} R_\mu(A) \otimes R_\nu(B), \quad (52)$$

where $g_{\lambda\mu\nu}$ counts the number of copies of $R_\mu(A) \otimes R_\nu(B)$ in the direct sum. This statement gives the *content expansion* of irrep λ of $\text{GL}(d_1 d_2, \mathbb{C})$, restricted to the subgroup $\text{GL}(d_1, \mathbb{C}) \otimes \text{GL}(d_2, \mathbb{C})$. Taking the trace of (52) yields (51).

Example 12 Consider two diagonal matrices $A, B \in \text{GL}(2, \mathbb{C})$: $A = \text{diag}(a, b)$ and $B = \text{diag}(c, d)$. Any polynomial irrep of a diagonal matrix is again diagonal. For $n = 3$, $R_{(3)}(A) = \text{diag}(a^3, a^2b, ab^2, b^3)$ and $R_{(21)}(A) = \text{diag}(a^2b, ab^2)$, and the diagonal elements of $R_{(3)}(A \otimes B)$ are the degree-3 monomials in ac, ad, bc, bd . The statement $R_{(3)}(A \otimes B) \cong R_{(3)}(A) \otimes R_{(3)}(B) \oplus R_{(21)}(A) \otimes R_{(21)}(B)$ is then easily checked.

We can find a nice formula for the partial trace

$$\begin{aligned} & \text{Tr}_B[P^\lambda \cdot (\mathbb{1}_A \otimes P^\nu(B))] \\ &= f^\lambda (n!)^{-1} \sum_{\pi \in \mathcal{S}_n} \chi^\lambda(\pi) P_\pi(A) \text{Tr}[P^\nu(B) P_\pi(B)] \\ &= f^\lambda (n!)^{-1} \sum_{\pi \in \mathcal{S}_n} \chi^\lambda(\pi) P_\pi(A) \chi^\nu(\pi) s_\nu(1^{\times d_B}) \\ &= s_\nu(1^{\times d_B}) f^\lambda (n!)^{-1} \sum_{\pi \in \mathcal{S}_n} \chi^\lambda(\pi) \chi^\nu(\pi) P_\pi(A) \\ &= s_\nu(1^{\times d_B}) f^\lambda \sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^\lambda \chi_\rho^\nu P_\rho \\ &= s_\nu(1^{\times d_B}) f^\lambda \sum_{\rho \vdash n} z_\rho^{-1} \chi_\rho^\lambda \chi_\rho^\nu \sum_{\mu} \chi_\rho^\mu P^\mu / f^\mu \\ &= s_\nu(1^{\times d_B}) f^\lambda \sum_{\mu} \frac{g_{\lambda\mu\nu}}{f^\mu} P^\mu. \end{aligned} \quad (53)$$

Taking the trace with $P^\mu(A)$ then gives

$$\text{Tr}[P^\lambda \cdot P^\mu(A) \otimes P^\nu(B)] = f^\lambda g_{\lambda\mu\nu} s_\mu(1^{\times d_A}) s_\nu(1^{\times d_B}). \quad (54)$$

Summing over all $\lambda \vdash n$ yields

$$\begin{aligned} & \text{Tr}[\mathbb{1} \cdot P^\mu(A) \otimes P^\nu(B)] \\ &= \sum_{\lambda \vdash n} f^\lambda g_{\lambda\mu\nu} s_\mu(1^{\times d_A}) s_\nu(1^{\times d_B}). \end{aligned}$$

As the left-hand side equals $f^\mu f^\nu s_\mu(1^{\times d_A}) s_\nu(1^{\times d_B})$, this gives

$$\sum_{\lambda \vdash n} f^\lambda g_{\lambda\mu\nu} = f^\mu f^\nu. \quad (55)$$

O. Integration over the Unitary Group

Collins and Sniady [13] considered integrals of polynomial functions on the unitary group $U(d)$. In the context of quantum information theory, their Proposition 2.3 can be formulated as giving an explicit expression for the Choi matrix of a d -dimensional n -particle twirl $\mathcal{I}_{d,n}$.

Proposition 1 (Collins-Sniady) Let \mathcal{H}_{in} and \mathcal{H}_{out} be two copies of the Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$. Let $|I\rangle$ be the (non-normalised) state vector on $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$ given by $\sum_{i=1}^{d^n} |ii\rangle$. Let $P^{(n)} = P^{(n)}(\mathbb{C}^d \otimes \mathbb{C}^d)$ be the projector on the totally symmetric subspace of $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$.

Then the integral $\mathcal{I}_{d,n}$ over the unitary group w.r.t. the normalised Haar measure,

$$\mathcal{I}_{d,n} := \int_{U(d)} d\mu(U) \mathbb{1} \otimes U^{\otimes n} \cdot |I\rangle\langle I| \cdot \mathbb{1} \otimes U^{\dagger, \otimes n},$$

is equal to

$$\mathcal{I}_{d,n} = P^{(n)} \cdot \mathbb{1} \otimes (\text{Tr}_{\text{in}} P^{(n)})^{-1}. \quad (56)$$

An explicit form of $(\text{Tr}_{\text{in}} P^{(n)})^{-1}$ is given by

$$(\text{Tr}_{\text{in}} P^{(n)})^{-1} = \sum_{\lambda \vdash n; d} \frac{f^\lambda}{s_\lambda(1^{\times d})} P^\lambda.$$

Indeed, from (53) follows

$$\text{Tr}_{\text{in}} P^{(n)} = \sum_{\lambda \vdash n} s_\lambda(1^{\times d}) P^\lambda(\mathbb{C}^d) / f^\lambda.$$

The inverse of this matrix is then readily seen to be the one given above.

Corollary 1 Let A be a matrix over $(\mathbb{C}^d)^{\otimes n}$. Then the integral $I(A)$ over the unitary group $U(d)$ w.r.t. the Haar measure,

$$I(A) := \int_{U(d)} d\mu(U) U^{\otimes n} A U^{\dagger, \otimes n},$$

is equal to

$$I(A) = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \text{Tr}[A P_\pi] P_{\pi^{-1}} \sum_{\lambda \vdash n; d} \frac{f^\lambda}{s_\lambda(1^{\times d})} P^\lambda. \quad (57)$$

It is a straightforward exercise to simplify (57) in the particular case that A is a tensor power $A = a^{\otimes n}$:

$$\begin{aligned} I(a^{\otimes n}) &= \sum_{\lambda \vdash n; d} \frac{s_\lambda(a)}{s_\lambda(1^{\times d})} P^\lambda \\ &= \sum_{\lambda \vdash n; d} \frac{\text{Tr}[P^\lambda a^{\otimes n}]}{\text{Tr}[P^\lambda]} P^\lambda. \end{aligned}$$

II. APPLICATION: THE KEYL-WERNER THEOREM

Let ρ be a density matrix. Since the projectors P^λ add up to the identity, the sequence of numbers $(\text{Tr}[P^\lambda \rho^{\otimes n}])_{\lambda \vdash n}$ forms a distribution over the set of partitions of n . From the Frobenius-Schur results we know that $\text{Tr}[P^\lambda \rho^{\otimes n}] = f^\lambda s_\lambda(r_1, \dots, r_d)$, where $r = (r_1, \dots, r_d)$ are the eigenvalues of ρ ; this has been rediscovered by Alicki et al in [1]. The second result in [1] was that for large n this distribution tends to a multinomial distribution with mean value $\lambda = nr^\downarrow$ (where r^\downarrow equals r , rearranged in non-ascending order). Consequently, for very large n , the distribution tends to the point distribution at nr^\downarrow . This second result was then rediscovered in turn by Keyl and Werner [10], who gave an estimate for the pointwise rate of convergence. Let Σ be the closed set

$$\Sigma := \{x \in \mathbb{R}^{+,d} : x_1 \geq x_2 \geq \dots \geq x_d \geq 0, \sum_i x_i = 1\}.$$

Theorem 1 (Keyl-Werner) *For a continuous function g on Σ and a state ρ with eigenvalues $r = (r_1, \dots, r_d)$,*

$$g(r^\downarrow) = \lim_{n \rightarrow \infty} \sum_{\lambda \vdash n} g(\bar{\lambda}) \text{Tr}[P^\lambda \rho^{\otimes n}], \quad (58)$$

with uniform convergence over the whole set Σ .

Hayashi and Matsumoto [15] gave a shorter proof of their results, based on the upper bound

$$\text{Tr}[P^\lambda \rho^{\otimes n}] \leq s_\lambda(1^{\times d}) \exp(-nS(\bar{\lambda}||r^\downarrow)), \quad (59)$$

where $S(r||s) := \sum_i r_i(\log r_i - \log s_i)$ is the classical (Kullback-Leibler) relative entropy between distributions r and s .

To prove this bound, we need the following bounds on $s_\lambda(x)$.

Proposition 2 *Let the variables x_i be non-negative and sorted in non-ascending order, then for $\lambda \vdash n$*

$$x_1^{\lambda_1} \dots x_k^{\lambda_k} \leq s_\lambda(x_1, \dots, x_k) \leq s_\lambda(1^{\times k}) x_1^{\lambda_1} \dots x_k^{\lambda_k}. \quad (60)$$

The essence of this pair of bounds is that for any non-negative vector x , $s_\lambda(x) \approx \prod_i (x_i^\downarrow)^{\lambda_i}$. Typical proofs proceed using representation-theoretical arguments on $\text{Tr}[P^\lambda X^{\otimes n}]$. Here we show that these bounds can be derived from combinatorial arguments only. Specifically, we use the properties of the Kostka numbers $K_{\lambda\mu}$.

Proof of Proposition 2. Recall (13):

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu.$$

First note that for the $x \in \mathbb{R}^{+,k}$ obeying the conditions of the Proposition,

$$m_\mu(x) \leq m_\mu(1^{\times k}) x_1^{\mu_1} x_2^{\mu_2} \dots x_k^{\mu_k}.$$

Here, the factor $m_\mu(1^{\times k})$ counts the number of terms present in m_μ . Second, since the Kostka matrix is upper-triangular we only need to look at $\mu \leq \lambda$ (in the reverse lexicographic ordering). For such μ ,

$$x_1^{\mu_1} \dots x_k^{\mu_k} \leq x_1^{\lambda_1} \dots x_k^{\lambda_k}.$$

Inserting this in (13) and noting that $K_{\lambda\mu} \geq 0$ directly yields the upper bound.

The lower bound follows from restricting the sum in (13) to the term with $\lambda = \mu$ (this can be done by positivity of all terms). That is, $s_\lambda \geq K_{\lambda\lambda} m_\lambda = m_\lambda$. Furthermore, for $x \in \mathbb{R}^{+,k}$,

$$m_\mu(x) \geq x_1^{\lambda_1} \dots x_k^{\lambda_k},$$

which follows by retaining only one term in $m_\lambda(x)$, namely the largest one. This proves the lower bound. \square

We also recall the approximation of the multinomial coefficients that follows from Stirling's formula:

$$\frac{n!}{\lambda_1! \dots \lambda_r!} = \kappa_\lambda^{-1/2} \exp(nS(\bar{\lambda})),$$

where S is the entropy function, $S(x) = -\sum_i x_i \log x_i$, and κ_λ is a polynomial correction factor given by (apart from higher-order corrections)

$$\kappa_\lambda \approx \left(\prod_i \bar{\lambda}_i \right) (2\pi n)^{r-1}.$$

For large enough values of n , $\kappa_\lambda \geq 1$, and can therefore be left out in upper bounds. Likewise, we find the approximation

$$\frac{n!}{\lambda_1! \dots \lambda_r!} r_1^{\lambda_1} \dots r_d^{\lambda_d} = \kappa_\lambda^{-1/2} \exp(-nS(\bar{\lambda}||r)).$$

The bound (59) now follows directly from (4), (60), (36) and the approximation just mentioned. Using (2) and the lower bound on $s_\lambda(r)$ in (60), we also immediately get a lower bound

$$\text{Tr}[P^\lambda \rho^{\otimes n}] \geq \kappa_\lambda^{-1/2} \frac{\nu_1! \dots \nu_r!}{\lambda_1! \dots \lambda_r!} \exp(-nS(\bar{\lambda}||r^\downarrow)). \quad (61)$$

The factors in front of the exponential in both bounds, (59) and (61), are of polynomial order in n . Hence, from combining these two bounds a variation on Keyl-Werner follows immediately.

Theorem 2 Let ρ be a density matrix with spectrum r^\perp . For a sequence of partitions $\lambda_{(n)} \vdash n$ such that $\lambda_{(n)}$ tends to the distribution s^\perp , one gets

$$\lim_{n \rightarrow \infty} \text{Tr}[P^{\lambda_{(n)}} \rho^{\otimes n}]^{1/n} = \exp(-S(s^\perp || r^\perp)). \quad (62)$$

Proof of Theorem 1. Equip Σ with the trace norm distance $D(x, y) = \sum_i |x_i - y_i|$. For every $\lambda \vdash n; d$ such that $\bar{\lambda}$ is outside an ϵ -ball around r^\perp it is clear that $\exp(-nS(\bar{\lambda} || r^\perp))$ goes to 0 exponentially with n . The factor $s_\lambda(1^{\times d})$ is polynomial in n . Eq. (59) thus gives that $\text{Tr}[P^\lambda \rho^{\otimes n}]$ also tends to 0 exponentially for $\bar{\lambda}$ outside the ϵ -ball. The total number of partitions $\lambda \vdash n; d$ is polynomial in n , hence the sum of $\text{Tr}[P^\lambda \rho^{\otimes n}]$ over $\bar{\lambda}$ outside the ϵ -ball also tends to 0 exponentially. As the projectors P^λ sum up to the identity, the total sum of $\text{Tr}[P^\lambda \rho^{\otimes n}]$ over *all* $\bar{\lambda}$ is 1. Hence, for any $\epsilon > 0$ the sum over all $\bar{\lambda}$ *inside* the ϵ -ball tends to 1. Multiplying in the factor $g(\bar{\lambda})$ and assuming continuity of g yields the statement of the Theorem. Uniformity of convergence follows since the convergence rate of $\exp(-nS(\bar{\lambda} || r^\perp))$ does essentially depend on $D(\bar{\lambda}, r^\perp)$ only, i.e. on the chosen value of ϵ , irrespective of the spectrum r . \square

The Keyl-Werner Theorem applies in particular to matrix norms of positive semidefinite matrices, and by restricting to diagonal matrices also to vector norms of non-negative real vectors. In fact, A. Barvinok independently showed that any vector norm for arbitrary real vectors can be approximated by a polynomial [5].

III. PARTIAL TRACES

A. Partial trace of P_ρ

Just like the index permutation matrices P_π , the class averages P_ρ operate on the tensor space $\mathcal{H}^{\otimes n} = (\mathbb{C}^d)^{\otimes n}$. Here we describe what happens when one or more of the n tensor factors is traced out. The calculations involved are feasible because a partial trace of an index permutation matrix is again an index permutation matrix (times a certain power of d).

The basic idea is that tracing out a single tensor factor in P_ρ corresponds to deleting a box at random from the Young frame associated to the class type $\rho \vdash n$. More precisely, if the boxes in a Young frame are linearly ordered and assigned the labels $i = 1, \dots, n$, and the result of deleting box i in Young frame ρ is denoted by $\rho \setminus i$, then

$$\text{Tr}_1[P_\rho] = (n!)^{-1} \sum_{i=1}^n P_{\rho \setminus i}. \quad (63)$$

(I'm not sure anymore about the factor $1/n!$)

This can be seen easily when one realises that for any representative $\pi \in \rho$, one has $P_\rho = (n!)^{-1} \sum_{\pi' \in S_n} P_{\pi' \pi \pi'^{-1}}$. Furthermore, if we choose the representative π such that the cycles appear in standard order (consecutively, and with non-increasing cycle order) then tracing out the first tensor factor

corresponds to deleting box 1. Conjugation with π' corresponds to permuting the box labels, hence (63) follows.

As it stands now, however, (63) is not correct. First of all, if a box is deleted from a row of a Young frame, then some rows might have to be swapped, to ensure that the row lengths are still in non-increasing order. We will not denote this re-ordering explicitly, but assume it implicit in the notation $\rho \setminus i$. Secondly, and more importantly, consider what happens when box i is the only box in its row, so that deleting it generates an empty row. Tracing out a cycle of length one in an index permutation P_π incurs an additional factor of d . For example, if $\pi = (1)$, then $P_\pi = \mathbb{1}$ and $\text{Tr} \mathbb{1} = d$. We can amend this shortcoming by adding an extra factor $d^{(\rho \setminus i)^0}$ to (63), where we denote the number of rows in $\rho \setminus i$ that have been set to 0 by $(\rho \setminus i)^0$. Thus:

$$(\rho \setminus i)^0 = l(\rho) - l(\rho \setminus i).$$

We can rephrase (63) in a more convenient way. The Young frames that are obtained from $\rho \setminus i$ are $(\rho_1 - 1, \rho_2, \dots)$, $(\rho_1, \rho_2 - 1, \dots)$, etc. As there are ρ_1 out of n possibilities to pick a box from the first row, ρ_2 out of n to pick one from the second row, etc, we have

$$\text{Tr}_1[P_\rho] = \sum_{j=1}^{l(\rho)} (\rho_j/n) d^{(\dots, \rho_{j-1}, \dots)^0} P_{(\dots, \rho_{j-1}, \dots)}.$$

This can be rewritten even more explicitly in terms of the exponents ρ^k . To simplify the typography we will here write the cycle type as an argument of P rather than as a subscript:

$$\begin{aligned} \text{Tr}_1[P_\rho] &= \sum_{k=2}^n \rho^k (k/n) P(1^{\rho^1} \dots (k-1)^{\rho^{k-1}+1} k^{\rho^k-1} \dots) \\ &\quad + \rho^1 (1/n) d P(1^{\rho^1-1} 2^{\rho^2} \dots). \end{aligned} \quad (64)$$

This formula can be used recursively to obtain expressions for traces $\text{Tr}_J P_\rho$ over any number J of copies of \mathcal{H} . However, it is much more convenient to derive an expression directly, in a similar way as has been done above for the single copy trace. The central operation in calculating $\text{Tr}_J P_\rho$ is deleting J boxes from the Young frame ρ . This amounts to deleting j_1 boxes from the first row, j_2 from the second row, etc, where the numbers $j := (j_1, j_2, \dots)$ are such that $0 \leq j_k \leq \rho_k$ and $\sum_k j_k = J$. As there are $\binom{n}{j}$ ways to delete J boxes from a Young frame with n boxes, and there are $C_{\rho_k}^{j_k}$ ways to delete j_k boxes from a row of length ρ_k , the multiplication factor (which was ρ_k/n in the single copy case) is now

$$\alpha_{j, \rho} := \frac{C_{\rho_1}^{j_1} C_{\rho_2}^{j_2} \dots}{C_n^J}.$$

We get

$$\text{Tr}_J[P_\rho] = \sum_j \alpha_{j, \rho} d^{(\rho-j)^0} P_{\rho-j},$$

where the sum is over all tuples j obeying the requirements stated above, and $\rho - j$ is the sequence (not a partition)

$(\rho_i - j_i)_i$. Introducing the sequence of $l(\rho)$ non-negative integers $\sigma := \rho - j$, these requirements translate to $|\sigma| := \sum_k \sigma_k = n - J$ and $\sigma \leq \rho$, which means that $\sigma_k \leq \rho_k$ for all k . There is clearly no harm in writing $P_{\rho-j} = P_\sigma$ since a class type can equally well be described by an unsorted sequence of numbers. If we further define $l(\sigma)$ as the number of non-zero elements in the sequence σ , then $(\rho - j)^0 = l(\rho) - l(\sigma)$. The partial trace now takes the form

$$\mathrm{Tr}_J[P_\rho] = \frac{1}{C_n^J} \sum_{\substack{\sigma \leq \rho \\ |\sigma| = n - J}} C_{\rho_1}^{\sigma_1} C_{\rho_2}^{\sigma_2} \dots d^{l(\rho) - l(\sigma)} P_\sigma.$$

This can be reformulated in terms of partitions rather than sequences. To do this, first note that the explicit condition $\sigma \leq \rho$ can be dropped if we adopt the rule that $C_n^k = 0$ whenever $k < 0$ or $k > n$. Second, for different reorderings of σ , P_σ is still the same thing. Hence, in terms of partitions σ , we finally arrive at

Theorem 3 (Partial Trace of P_ρ) For a cycle type $\rho \vdash n$, the partial trace over $J \leq n$ copies of $\mathcal{H} = \mathbb{C}^d$ of P_ρ is

$$\mathrm{Tr}_J[P_\rho] = \frac{1}{C_n^J} \sum_{\sigma \vdash n - J} d^{l(\rho) - l(\sigma)} G_\sigma^\rho P_\sigma, \quad (65)$$

with

$$G_\sigma^\rho := \sum_{\alpha} C_{\rho_1}^{\alpha_1} C_{\rho_2}^{\alpha_2} \dots \quad (66)$$

Here the sum is over all sequences α that are distinct permutations of σ zero-padded to length $l(\rho)$, and we adopt the convention that $C_n^k = 0$ whenever $k < 0$ or $k > n$. The number of these distinct permutations is $l(\rho)! / (l(\rho) - l(\sigma))! \sigma_1! \sigma_2! \dots$.

Note that G_σ^ρ is a non-negative integer for which

$$\sum_{\sigma \vdash n - j} G_\sigma^\rho = C_n^j.$$

Consider for example the special case where all parts of ρ are equal. Replace n by qn and put $\rho = (q^n)$. Then no part of σ may exceed q and

$$\mathrm{Tr}_J[P_{(q^n)}] = \frac{1}{C_{qn}^J} \sum_{\substack{\sigma \vdash qn - j; n \\ \sigma_1 \leq q}} d^{n - l(\sigma)} G_\sigma^{(q^n)} P_\sigma. \quad (67)$$

with

$$G_\sigma^{(q^n)} = \frac{n! C_q^{\sigma_1} C_q^{\sigma_2} \dots C_q^{\sigma_l}}{(n - l(\sigma))! \sigma_1! \sigma_2! \dots \sigma_l!}.$$

B. Partial trace of P^λ

A direct consequence of the Theorem is that a similar statement must hold for the projectors P^λ . There must exist constants $G_{\lambda'}^\lambda$, depending on d , such that for $\lambda \vdash n$:

$$\mathrm{Tr}_1[P^\lambda] = \frac{1}{n} \sum_{\lambda' \vdash n - 1} G_{\lambda'}^\lambda P^{\lambda'}. \quad (68)$$

To determine these constants, consider the matrix $(A + t\mathbb{1})^{\otimes n}$, where A has eigenvalues a . Expansion of the sum gives

$$(A + t\mathbb{1})^{\otimes n} = \sum_{j=0}^n t^j (\mathbb{1}^{\otimes j} \otimes A^{\otimes n-j} + \text{d.p.}),$$

where ‘‘d.p.’’ stands for the distinct permutations of the tensor factors, of which there are $\binom{n}{j}$.

Taking the trace of the product of this matrix with P^λ yields

$$\begin{aligned} f^\lambda s_\lambda(a + t) &= \sum_{j=0}^n t^j \binom{n}{j} \mathrm{Tr}[\mathrm{Tr}_j[P^\lambda] A^{\otimes n-j}] \\ &= \sum_{j=0}^n t^j \sum_{\lambda' \vdash n-j} G_{\lambda'}^\lambda f^{\lambda'} s_{\lambda'}(a), \end{aligned}$$

Taking the first derivative w.r.t. t in $t = 0$ gives

$$\left. \frac{\partial}{\partial t} \right|_{t=0} s_\lambda(a + t) = \frac{1}{f^\lambda} \sum_{\lambda' \vdash n-1} G_{\lambda'}^\lambda f^{\lambda'} s_{\lambda'}(a).$$

We will now explicitly calculate this derivative.

Proposition 3

$$\begin{aligned} \left. \frac{\partial}{\partial t} \right|_{t=0} s_\lambda(a_1 + t, \dots, a_d + t) &= \sum_{\substack{j=1 \\ \lambda_j > \lambda_{j+1}}}^{l(\lambda)} (d + \lambda_j - j) s_{\lambda - e^j}(a_1, \dots, a_d), \quad (69) \end{aligned}$$

where $e_i^j := \delta_{i,j}$.

The condition $\lambda_j > \lambda_{j+1}$ ensures that the sum only includes those terms for which $\lambda - e^j$ is a genuine Young frame.

The proof will be based upon the representation of a Schur polynomial s_λ as the determinant

$$s_\lambda = \det (h_{\lambda_i + j - i})_{i,j=1}^{l(\lambda)},$$

which is the so-called Jacobi-Trudi formula [7]. The matrix elements are the symmetric functions h_r , defined as the sum of all monomials of total degree r . One takes $h_0 = 1$, and $h_r = 0$ for $r < 0$.

The first derivative of h_r is supplied by the following lemma.

Lemma 1

$$\begin{aligned} \left. \frac{\partial}{\partial t} \right|_{t=0} h_r(a_1 + t, \dots, a_d + t) &= (r + d - 1) h_{r-1}(a_1, \dots, a_d). \quad (70) \end{aligned}$$

Proof of the Lemma. The generating polynomial of the functions h_r is [7]

$$H(s; x) = \sum_{r=0}^{\infty} h_r(x) s^r = \prod_{i=1}^d (1 - x_i s)^{-1}.$$

Taking the derivative of $\log H(s; x+t)$ w.r.t. t yields

$$\left. \frac{\partial}{\partial t} \right|_{t=0} H(s; x+t) = sH(s; x) \sum_{i=1}^d (1-x_i s)^{-1}.$$

Taking the derivative of $\log H(s; x)$ w.r.t. s yields

$$\frac{\partial}{\partial s} H(s; x) = H(s; x) \sum_{i=1}^d x_i (1-x_i s)^{-1}.$$

Since

$$(1-x_i s)^{-1} = 1 + x_i s (1-x_i s)^{-1},$$

we find

$$\left. \frac{\partial}{\partial t} \right|_{t=0} H(s; x+t) = dsH(s; x) + s^2 \frac{\partial}{\partial s} H(s; x).$$

Equating the coefficients of s^r on both sides yields the statement of the Lemma. \square

Proof of the Proposition. We will make abundant use of the linearity of the determinant in any of its rows.

From the Jacobi-Trudi formula,

$$\left. \frac{\partial}{\partial t} \right|_{t=0} s_\lambda(a_1+t, \dots, a_d+t) = \sum_{k=1}^{l(\lambda)} \det(G_k),$$

where the matrix G_k is obtained from the matrix $G_\lambda := (h_{\lambda_i+j-i})_{i,j=1}^{l(\lambda)}$ by differentiating the elements on the k -th row w.r.t. t . Denoting $\left. \frac{\partial}{\partial t} \right|_{t=0} h_r(a_1+t, \dots, a_d+t)$ by h'_r and $l(\lambda)$ by l , the k -th row of G_k is

$$(h'_{\lambda_k-k+1}, h'_{\lambda_k-k+2}, \dots, h'_{\lambda_k-k+l})$$

By the Lemma, this is equal to

$$\begin{aligned} & ((\lambda_k + 1 - k + d - 1)h_{\lambda_k-k}, \\ & (\lambda_k + 2 - k + d - 1)h_{\lambda_k-k+1}, \dots, \\ & (\lambda_k + l - k + d - 1)h_{\lambda_k-k+l-1}) \\ &= (\lambda_k - k + d) \cdot (h_{\lambda_k-k}, h_{\lambda_k-k+1}, \dots, h_{\lambda_k-k+l-1}) \\ & \quad + (0, h_{\lambda_k-k+1}, \dots, (l-1)h_{\lambda_k-k+l-1}). \end{aligned}$$

We can thus write

$$\det(G_k) = (\lambda_k - k + d) \det(G'_k) + \det(G''_k),$$

where G'_k and G''_k are obtained from G_λ by replacing its k -th row by $(h_{\lambda_k-k}, h_{\lambda_k-k+1}, \dots, h_{\lambda_k+l-k-1})$ and $(0, h_{\lambda_k-k+1}, \dots, (l-1)h_{\lambda_k+l-k-1})$, respectively.

It is easy to see that $G'_k = G_{\lambda-e^k}$. Furthermore, if $\lambda_k = \lambda_{k+1}$, $\det(G'_k) = 0$ because then the k -th row in G'_k is equal to its $(k+1)$ -th one.

The sum of the second terms $\sum_{k=1}^l \det(G''_k)$ is identically zero. This can be seen as follows. Let t be an independent scalar variable. Let G'_λ be the matrix whose p -th column is equal to the p -th column of G_λ plus $(p-1)t$ times the first

column of G_λ . As G'_λ can be obtained from G_λ by elementary column operations (the first column is unchanged and added to the other columns), $\det G'_\lambda = \det G_\lambda$ for any value of t . This implies that the first derivative of $\det G'_\lambda$ w.r.t. t is zero. We find

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \det G'_\lambda = \sum_{k=1}^l \det(G''_k),$$

which shows that the right-hand side is indeed zero.

Therefore,

$$\begin{aligned} & \left. \frac{\partial}{\partial t} \right|_{t=0} s_\lambda(a_1+t, \dots, a_d+t) \\ &= \sum_{k=1}^l \det(G_k) \\ &= \sum_{\substack{k=1 \\ \lambda_k > \lambda_{k+1}}}^l (\lambda_k - k + d) \det(G_{\lambda-e^k}), \end{aligned}$$

which is the statement of the Proposition. \square

As a consequence, we get a simple formula for the constants $G_{\lambda'}^\lambda$ in case $|\lambda'| = |\lambda| - 1$. The only values of λ' for which $G_{\lambda'}^\lambda$ is non-zero are $\lambda - e^j$, for those j where $\lambda_j > \lambda_{j+1}$. In those cases:

$$\begin{aligned} G_{\lambda-e^j}^\lambda &= (d + \lambda_j - j) \frac{f^\lambda}{f^{\lambda-e^j}} \\ &= n \frac{s_\lambda(1^{\times d})}{s_{\lambda-e^j}(1^{\times d})}. \end{aligned}$$

We obtain two formulas for the partial trace of P^λ :

Proposition 4 (Partial Trace of P^λ)

$$\frac{\text{Tr}_1[P^\lambda]}{f^\lambda} = \frac{1}{n} \sum_{\substack{j=1 \\ \lambda_j > \lambda_{j+1}}}^{l(\lambda)} (d + \lambda_j - j) \frac{P^{\lambda-e^j}}{f^{\lambda-e^j}} \quad (71)$$

and

$$\frac{\text{Tr}_1[P^\lambda]}{s_\lambda(1^{\times d})} = \sum_{\substack{j=1 \\ \lambda_j > \lambda_{j+1}}}^{l(\lambda)} \frac{P^{\lambda-e^j}}{s_{\lambda-e^j}(1^{\times d})}. \quad (72)$$

Taking the trace of these equations yields:

$$s_\lambda(1^{\times d}) = \frac{1}{n} \sum_{\substack{j=1 \\ \lambda_j > \lambda_{j+1}}}^{l(\lambda)} (d + \lambda_j - j) s_{\lambda-e^j}(1^{\times d}) \quad (73)$$

and

$$f^\lambda = \sum_{\substack{j=1 \\ \lambda_j > \lambda_{j+1}}}^{l(\lambda)} f^{\lambda-e^j}. \quad (74)$$

The latter formula is just a mental leap away from a proof that f^λ is the number of SYTs of shape λ .

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