

# UNIFICATION OF BESSEL FUNCTIONS OF DIFFERENT ORDERS

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19 june 1999

## **Abstract**

We investigate the internal space of Bessel functions which is associated to the group  $Z$  of positive and negative integers defining their orders. As a result we propose and prove a new unifying formula ( to be added to the huge literature on Bessel functions ) generating Bessel functions of real orders out of integer order one's. The unifying formula is expected to be of great use in applied mathematics. Some applications of the formula are given for illustration

*Published in International Journal of Theoretical Physics*  
*IJTP vol,39,No,4,2000*  
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Very early studies [e.g., [1],[2]] proposed a unifying scheme for special functions showing that some of these functions may originate from the same structure .For Bessel functions of concern here their generating functions are representation "states" of the derivative and /or integral operators of arbitrary orders.More precisely we have the "inner" structure

$$\begin{aligned}
 m &\in Z \\
 \partial_{|m|} &= \frac{\partial}{z\partial z} \cdot \frac{\partial}{z\partial z} \cdots \frac{\partial}{z\partial z} \\
 \partial_{-|m|} &= \int z dz \cdot \int z dz \cdots \int z dz \\
 \partial_m \Phi(z, t) &= (-t)^{-m} \Phi(z), m \in Z \\
 \Phi(z, t) &= \sum_{n=-\infty}^{n=\infty} \phi_n(z) t^n
 \end{aligned} \tag{1}$$

Where  $\int z dz$  is the " truncated " primitive ,ie in defining the integral we omit the constant of integration  $\int \frac{df}{dz} dz = f$  . For the polynomials such as Hermite and Laguerre for instance ,the generating functions only involve the realization of the set N of positive integers,with slight modifications of the derivative operators to account for the conventions used in defining these polynomials.It is important to note that although this common structure only set up the z-dependance of the generating functions ,it is the "dynamical" part of the scheme so to say.The t dependence is simply set by imposing some given desired properties.For Bessel functions we require a "symmetry" between positive and negative indices that is  $J_{-n} = (-1)^n J_n$  ,while for the polynomials it is the natural property of orthonormality that is invoked.In this paper we unify Bessel functions of integer orders with those of real orders as a complement to the unification of special functions initiated in the above cited papers and subsequent papers.To our knowledge there is no such relation , between integer and real orders ,known to exist.Both functions are usually independently defined either by their own differential equations or by their expansions in the form of a series or integral .The mechanism underlying the unification we propose is of a different nature than the one described above,but makes however use of the operators  $\partial_m m \in Z$  which served to define Bessel functions of integer orders.

Let us first introduce the method in a simple example to see how the mechanism works to convert an integer into a real.Suppose we are given an abstract state  $| n \rangle n \in Z$  and a set of raising  $\Pi(m), m > 0$  and lowering  $m < 0$  operators .Then it is easy to show , given that data that the state  $| n + \lambda \rangle$  is related to the state  $| n \rangle$  through the following formula ( Provided the involved series converges which is the case here ).

$$| n + \lambda \rangle = \exp(-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}) | n \rangle \tag{2}$$

Fourier transforming the  $| n \rangle$  state as

$$| n \rangle = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{in\theta} | \theta \rangle \tag{3}$$

Where the  $\Pi(m)$  operators acts on the  $| \theta \rangle$  state by simple multiplication by the factor  $e^{in\theta}$  .We get

$$\begin{aligned}
|n + \lambda\rangle &= \int_{-\pi}^{\pi} \exp(-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}) e^{in\theta} |\theta\rangle \frac{d\theta}{2\pi} \\
&= \int_{-\pi}^{\pi} e^{i(n+\lambda)\theta} |\theta\rangle \frac{d\theta}{2\pi}
\end{aligned} \tag{4}$$

In deriving the last line use have been made of the known formula  $\sum_{m=1}^{\infty} (-1)^m \frac{\sin m\theta}{m} = -\frac{\theta}{2}$ ,  $-\pi < \theta < \pi$ .

It is essential that the  $|n\rangle$  states have the appropriate weights for the Fourier components otherwise the series defining the transformation simply diverges. Reduced Bessel functions  $\phi_n(z) = \frac{J_n(z)}{z^n}$  fit into the scheme. We indeed have a set of raising and lowering operators  $\Pi(m) = (-1)^m \partial_m$   $m \in Z$  [3] [4]

$$(-1)^m \frac{d^m}{(zdz)^m} \phi_n(z) = \phi_{n+m}(z) \quad n \in Z \tag{5}$$

and the involved series does converge as it has almost the same structure as for the example above. We then propose and prove the unifying formula.

$$\frac{J_{n+\lambda}(z)}{z^{n+\lambda}} = \exp(-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}) \frac{J_n(z)}{z^n} \tag{6}$$

The operators act on the  $\phi_n$  as

$$\Pi(m)\phi_n = \phi_{n+m} \tag{7}$$

To check the above formula, we could naively express the Bessel functions  $\phi_{n+m}$  as entire series and then perform the sum over  $m$ . This is however a cumbersome procedure. A simple and illuminating way to proceed is to use the integral representation of  $\phi_n$ , the "analog" of the  $|n\rangle$  state Fourier decomposition.

$$\phi_n(z) = \frac{1}{2\pi i} \int_{l_0} \left(\frac{1}{2}\right)^n \tau^{-n-1} \exp\left(\tau - \frac{z^2}{4\tau}\right) d\tau \tag{8}$$

Where  $\tau$  is a complex variable and  $l_0$  is a positively oriented closed path encircling the origin one time. Expanding the exponential in 6 as

$$\begin{aligned}
\phi_{n+\lambda}(z) &= \exp(-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}) \phi_n(z) \\
&= \sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!} \left( \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m} \right)^p \phi_n(z).
\end{aligned} \tag{9}$$

And using the property that  $\Pi(m_1) \dots \Pi(m_p) = \Pi(m_1 + \dots + m_p)$ , the term of order  $\lambda^p$  takes the form.

$$\frac{(-\lambda)^p}{p!} \sum_{m_1} \dots \sum_{m_p} \frac{(-1)^{m_1 + \dots + m_p}}{m_1 \dots m_p} \phi_{m_1 + \dots + m_p} \tag{10}$$

Inserting the  $\phi_n$  integral representation into 10 we get .

$$\begin{aligned} \frac{1}{2\pi i} \frac{(-\lambda)^p}{p!} \int_{l_0} \sum_{m_1} \dots \sum_{m_p} \frac{(-1)^{m_1+\dots+m_p}}{m_1 \dots m_p} \left(\frac{1}{2}\right)^{m_1+\dots+m_p} \tau^{-(m_1+\dots+m_p+n)-1} \exp\left(\tau - \frac{z^2}{4\tau}\right) d\tau. \\ = \frac{1}{2\pi i} \frac{(-\lambda)^p}{p!} \int_{l_0} \left(\frac{1}{2}\right)^n \tau^{-n-1} \left(\sum_m (-1)^m \frac{(2\tau)^{-m}}{m}\right)^p \exp\left(\tau - \frac{z^2}{4\tau}\right) d\tau \end{aligned} \quad (11)$$

The relevant term to sum is .

$$\sum_{m \in Z/(0)} (-1)^m \frac{(2\tau)^{-m}}{m} \quad (12)$$

It is to be noted that the above series diverges on the whole complex plane except on the circle centring the origin and with half unity radius, on which it converges uniformly. It is only on that circle do we have the right to commute the signs  $\sum$  and  $\int$  when we insert 8 into 10 .Now the miracle happens .To make the series converge we just deform the path  $l_0$  down to the above circle as the integrand in 8 has only essential singularities at  $\tau = 0$  and  $\tau = \infty$ . To compute the relevant sum ,put  $2\tau = e^{i\theta}$  ,the series then converges to .

$$\begin{aligned} \sum_{m \in Z/(0)} (-1)^m \frac{(2\tau)^{-m}}{m} &= -2i \sum_{m=1}^{m=\infty} (-1)^m \frac{\sin m\theta}{m} \\ &= i\theta \quad -\pi < \theta < \pi \\ &= \ln 2\tau \end{aligned} \quad (13)$$

Where the branch cut of the logarithm is taken along the negative real axis as  $\theta$  ranges from  $-\pi$  to  $\pi$  .Inserting the result into 11 ,the latter becomes.

$$\frac{1}{2\pi i} \frac{(-\lambda)^p}{p!} \int_l \left(\frac{1}{2}\right)^n \tau^{-(n+\lambda)-1} (\ln 2\tau)^p \exp\left(\tau - \frac{z^2}{4\tau}\right) d\tau. \quad (14)$$

Where the new positively oriented path  $l$  is now getting round the cut lying on the negative real axis. The above expression in 14 is however nothing than the  $\lambda^p$  order term of the real order Bessel function  $\phi_{n+\lambda}$  when it is written in terms of its integral representation.

$$\phi_{n+\lambda}(z) = \frac{1}{2\pi i} \int_{l_0} \left(\frac{1}{2}\right)^{n+\lambda} \tau^{-(n+\lambda)-1} \exp\left(\tau - \frac{z^2}{4\tau}\right) d\tau \quad (15)$$

To see this ,rewrite  $(2\tau)^{-\lambda} = \exp(-\lambda \ln 2\tau)$  and take the  $\lambda^p$  order which is  $\frac{1}{p!} (-\lambda)^p (\ln 2\tau)^p$  .We thus have succeed to prove the proposed unifying formula. Now we propose two applications of this formula for illustration . The first immediate application is this .Bessel functions enjoy the essential property

$$(-1)^m \frac{d^m}{(zdz)^m} \phi_n(z) = \phi_{n+m}(z) \quad n \in Z \quad (16)$$

This formula is easily generalized to  $\phi_{n+\lambda}$  with  $\lambda$  real since the operator  $\frac{d^m}{(zdz)^m}$  appearing in 16 commutes with the operator defining  $\phi_{n+\lambda}$  in 6 The second application is the

that we know the generating function of the reduced integer order Bessel function .That is we want to indirectly compute

$$\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z) t^n \quad (17)$$

Where t is a complex parameter .To perform the sum we use the unifying formula.

$$\begin{aligned} \sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z) t^n &= \\ &= \sum_{n=-\infty}^{n=\infty} \exp(-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}) \phi_n t^n \\ &= \exp(-\lambda \sum_{m \in Z/(0)} (-1)^m \frac{\Pi(m)}{m}) \Phi(z, t) \end{aligned} \quad (18)$$

Where  $\Phi(z, t)$  is the generating function for Bessel functions of integer orders. That is  $\Phi(z, t) = \sum_{n=-\infty}^{\infty} \phi_n(z) t^n$  .The action of the  $\Pi(m)$  operators are simply .

$$\Pi(m)\Phi = t^{-m}\Phi \quad (19)$$

Then the third line in 18 involves the following sum we got used to.

$$\sum_{m \in Z/(0)} (-1)^m \frac{t^{-m}}{m} \quad (20)$$

This series is only convergent on the circle of radius unity ,so that we put  $t = e^{i\theta}$  and find.

$$\begin{aligned} \sum_{m \in Z/(0)} (-1)^m \frac{e^{-im\theta}}{m} &= i\theta - \pi < \theta < \pi \\ &= \ln t \end{aligned} \quad (21)$$

Putting this into 18 we get the end result that the generating function for  $\frac{J_{n+\lambda}}{z^{n+\lambda}}$  as defined above is shown to be  $t^{-\lambda}$  times the generating function for Bessel functions of integer orders. Our formula shows that this is true on the circle of radius unity except on the the point -1 .

$$\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z) t^n = t^{-\lambda} \sum_{n=-\infty}^{n=\infty} \phi_n(z) t^n = t^{-\lambda} \exp\left(\frac{t}{2} - \frac{z^2}{2t}\right). \quad (22)$$

Note that our formula only predict the result on the circle .If we assume in addition that the series is regular on the complex plane except on the branch cut on the negative axis, then the result we found is valid outside the circle and cover the whole domain of regularity. The unifying formula will be used in a future paper on the unitarity conditions.

a direct method .We will use the integral representation of the reduced Bessel function ,otherwise the computation is simply awful.

$$\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z) t^n = \frac{t^{-\lambda}}{2\pi i} \sum_{n=-\infty}^{n=\infty} \int_l \left(\frac{t}{2\tau}\right)^{(n+\lambda)} \exp\left(\tau - \frac{z^2}{4\tau}\right) \frac{d\tau}{\tau} \quad (23)$$

Making the change of variable  $\frac{t}{2\tau} = e^{i\theta}$ .We rewrite the above expression as

$$\sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z) t^n = \frac{t^{-\lambda}}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{n=\infty} e^{in\theta} e^{i\lambda\theta} \exp\left(\frac{t}{2}e^{-i\theta} - \frac{z^2}{2t}e^{i\theta}\right) d\theta \quad (24)$$

And knowing that the involved series converges to the sum of Dirac distributions.

$$\sum_{n=-\infty}^{n=\infty} e^{in\theta} = \sum_{p=-\infty}^{p=\infty} 2\pi\delta(\theta - 2\pi p) \quad (25)$$

We get after insertion of the above result .

$$\begin{aligned} \sum_{n=-\infty}^{n=\infty} \phi_{n+\lambda}(z) t^n &= t^{-\lambda} \int_{-\pi}^{\pi} \sum_{p=-\infty}^{p=\infty} \delta(\theta - 2\pi p) e^{i\lambda\theta} \exp\left(\frac{t}{2}e^{-i\theta} - \frac{z^2}{2t}e^{i\theta}\right) d\theta \\ &= t^{-\lambda} \exp\left(\frac{t}{2} - \frac{z^2}{2t}\right) \end{aligned} \quad (26)$$

Note that only the p=0 term in the above sum contributes since the path of integration encircles the origin only one time.Here again we get the same result with the advantage however that the result is valid on the whole complex t plane except the cut .This is because we can always deform the  $\tau$  contour so to make  $|\frac{t}{2t}| = 1$  and hence to sum the series.Let us end up with the following worth remarks.First ,we have shown that Bessel functions of real orders can be obtained from integer orders ones through the mechanism we described above,and from this point of view integer order Bessel functions are more "fundamental".Second,there is some very known functions which appear in mathematical physics but are simply constructed from various combinations of the linairly independant  $J_p(z)$  and  $J_{-p}(z)$  , where p is real .These are Hankel  $H_p^1(z)$ ,  $H_p^2(z)$  and Neumann functions  $N_p(z)$  which are expressed as

$$\begin{aligned} N_p(z) &= \frac{J_p(z)\cos p\pi - J_{-p}}{\sin p\pi} \\ H^1(z) &= J_p(z) + iN_p(z) \\ H^2(z) &= J_p(z) - iN_p(z) \end{aligned} \quad (27)$$

These functions are also concerned with the above unification but to give the formula which relates integer orders to real one's is not a straightforward matter although they are simply linear combinations of Bessel functions.The reason is that the unifying formula acts on  $J_n(z)/z^n$  and not directly on  $J_n(z)$  and therefore we cannot naively apply it to the defining formulas 27 with p integer .We managed however to unify Neumann's functions of different orders using our unifying formula but following an indirect way .The result will appear in a forthcoming paper.

**Acknowledgement** I would like to thank the head of the high energy section at the **abdus salam** international centre for theoretical physics Dr S.Randjbar -Daemi for the continuous help and encouragements during my stay as guest scientist at the centre.

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