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Theta function parameterization and fusion for 3-D integrable Boltzmann weights

G. von Gehlen[a,1], S. Pakuliak[b,c,2], S. Sergeev[b,d,e,3]

[a] *Physikalisches Institut der Universität Bonn, Nussallee 12, D-53115 Bonn, Germany*
[b] *Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Moscow region, Russia*
[c] *Institute of Theoretical and Experimental Physics, Moscow 117259, Russia*
[d] *Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany*
[e] *Research School of Physical Sciences and Engineering, The Australian National University, Oliphant Building 60, Canberra ACT 0200, Australia*

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Abstract

We report progress in constructing Boltzmann weights for integrable 3-dimensional lattice spin models. We show that a large class of vertex solutions to the modified tetrahedron equation can be conveniently parameterized in terms of N-th roots of theta-functions on the Jacobian of a compact algebraic curve. Fay's identity guarantees the Fermat relations and the classical equations of motion for the parameters determining the Boltzmann weights. Our parameterization allows to write a simple formula for fused Boltzmann weights $\mathfrak{R}$ which describe the partition function of an arbitrary open box and which also obey the modified tetrahedron equation. Imposing periodic boundary conditions we observe that the $\mathfrak{R}$ satisfy the normal tetrahedron equation. The scheme described contains the Zamolodchikov-Baxter-Bazhanov model and the Chessboard model as special cases.

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Introduction

The tetrahedron equation is the three dimensional generalization of the Yang-Baxter equation which guarantees the existence of commuting transfer matrices. The importance of Yang-Baxter equations for modern mathematics and for mathematical physics is well known. However, the nature of the tetrahedron equation is much less understood, this mainly because it is a much more complicated equation.

Given the physical interest to understand the nature of the singularities which give rise to 3-D phase transitions, any effort which gets us closer to analytic results for 3-D

[1]e-mail: gehlen@th.physik.uni-bonn.de; [2]pakuliak@thsun1.jinr.ru; [3]sergeev@thsun1.jinr.ru

statistical systems seems worthwhile. What has been achieved recently, is to construct large classes of 3-D solvable models with $\mathbb{Z}_N$ spin variables and to streamline the otherwise complicated formalism. The goal of an analytic calculation of partition functions and order parameters is not yet close by. The only available result from Baxter [26] does not lend itself to generalizations.

The first solution of the tetrahedron equation was obtained in 1980 by A. Zamolodchikov [1, 2] and then generalized by R. Baxter and V. Bazhanov [3] and others [4]. These models have $\mathbb{Z}_N$-spin variables and solve the IRC (Interaction Round a Cube) version of the tetrahedron equation. However, later also the solution of the dual equation, the vertex tetrahedron equation, was obtained [5], generalizing several vertex solutions known previously [6, 7]. Here we shall consider only vertex-type solutions, which are usually denoted by $\mathbf{R}$, the symmetry will include a $\mathbb{Z}_N$. In general these $\mathbf{R}$-matrices obey the so called "simple modified tetrahedron equation" which recently has been investigated in [9]. The modified tetrahedron equation (MTE) allows us to obtain the ordinary tetrahedron equation for composite weights or vertices. In the IRC formulation this has been shown in [10, 11], while the most simple vertex case was considered in [12].

In this paper we shall introduce a new convenient theta-function parameterization of general $\mathbf{R}$ operators. This parameterization will allows us to define fused weights $\mathfrak{R}$, which are partition functions of open cubes of size M^3, and which obey a MTE. In special cases the $\mathfrak{R}$ solve an ordinary tetrahedron equation.

The vertex matrix $\mathfrak{R} \in \mathrm{End}\,(\mathbb{C}^{3\,N\,M^2})$ is parameterized in the terms of N-th roots of theta-functions on the Jacobian of a genus $g = (M-1)^2$ compact algebraic curve Γ_g. The divisors of three meromorphic functions on Γ_g play the role of the spectral parameters for $\mathfrak{R}$. An additional parameter of $\mathfrak{R}$ is an arbitrary $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$. The tetrahedron equation for $\mathfrak{R}$ holds due to M^4 simple modified tetrahedron equations. In the case when $M = 1$ and therefore $\Gamma_g = S_2$, the solution of the simple tetrahedron equation of [5] is reproduced.

This paper is organized as follows. In Sec.1 we recall the definitions of the vertex formulation of 3-D integrable ZBB model and sketch the derivation of the matrix operator $\mathbf{R}_{ijk}$ from a current conservation principle and Z-invariance. It satisfies the MTE and can be parameterized by quadrangle line-sections. In Sec.2 we introduce the parameterization of $\mathbf{R}_{ijk}$ in terms of theta functions. The Fermat relation and the Hirota equations are written as Fay identities. In Sec.3 we show that a theta function parameterization allows a compact formulation of the fusion of many $\mathbf{R}_{ijk}$ to the Boltzmann weight $\mathfrak{R}$ of a whole open cube which satisfies a MTE. Finally, in Sec.4 we first consider the special case of vanishing Jacobi transforms, in which $\mathfrak{R}$ satisfies a simple TE. Then we discuss the rational case and the relation to Chessboard models.

1 The R-matrix and its parameterization

1.1 The R-matrix of the vertex ZBB model

We start recalling the vertex formulation of the BBZ-model [5]. We consider a 3-dimensional lattice with the elementary cell defined by three non-coplanar vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and general vertices

$$\mathbf{n} \;=\; n_1\,\mathbf{e}_1 + n_2\,\mathbf{e}_2 + n_3\,\mathbf{e}_3, \qquad\qquad n_1,\; n_2,\; n_3 \in\; \mathbb{Z}. \tag{1}$$

We label the directed link along $\mathbf{e}_j$ starting from $\mathbf{n}$ by $(j,\ \mathbf{n})$. On these links there are spin variables $\sigma_{j,\,\mathbf{n}}$ which take values in $\mathbb{Z}_N$. The partition function is defined by

$$Z \;=\; \sum_{\{\sigma\}} \prod_{\mathbf{n}} \langle \sigma_{1,\mathbf{n}}, \sigma_{2,\mathbf{n}+\mathbf{e}_2}, \sigma_{3,\mathbf{n}} |\, \mathbf{R}\, | \sigma_{1,\mathbf{n}+\mathbf{e}_1}, \sigma_{2,\mathbf{n}}, \sigma_{3,\mathbf{n}+\mathbf{e}_3} \rangle \;. \tag{2}$$

where $\mathbf{R}$ is an operator (which in the ZBB model is independent of $\mathbf{n}$) mapping the initial three spin variables to the the three final ones, so that

$$R_{\sigma_1,\sigma_2,\sigma_3}^{\sigma_1',\sigma_2',\sigma_3'} \;=\; \langle \sigma_1, \sigma_2, \sigma_3 | \mathbf{R} | \sigma_1', \sigma_2', \sigma_3' \rangle \tag{3}$$

is a $N^3 \times N^3$ matrix independent of $\mathbf{n}$.

For the vertex ZBB-model, (3) can be expressed as a kind of cross ratio of four cyclic functions $W_p(n)$. Introduce a two component vector $p = (x, y)$ which is restricted to the Fermat curve

$$x^N \,+\, y^N \;=\; 1\,. \tag{4}$$

Then define the function $W_p(n)$ by

$$W_p(0) \;=\; 1, \qquad W_p(n) \;=\; \prod_{\nu=1}^{n} \frac{y}{1 \,-\, q^{\nu}\, x} \quad \text{for } n > 0. \tag{5}$$

where

$$q \;=\; e^{2\pi\, i/N} \tag{6}$$

is the primitive $N\!-\!th$ root of unity. Because of the Fermat curve restriction, $W_p(n)$ is cyclic in n:

$$W_p(n+N) \;=\; W_p(n).$$

Now $\mathbf{R} \;=\; \mathbf{R}(p_1, p_2, p_3, p_4)$ is defined by the following matrix function of four Fermat points p_1, p_2, p_3, p_4

$$R_{\sigma_1,\sigma_2,\sigma_3}^{\sigma_1',\sigma_2',\sigma_3'} \;\stackrel{\text{def}}{=}\; \delta_{\sigma_2+\sigma_3,\sigma_2'+\sigma_3'}\; q^{(\sigma_1'-\sigma_1)\,\sigma_3'}\; \frac{W_{p_1}(\sigma_2-\sigma_1)\, W_{p_2}(\sigma_2'-\sigma_1')}{W_{p_3}(\sigma_2'-\sigma_1)\, W_{p_4}(\sigma_2-\sigma_1')}\;, \tag{7}$$

where x-coordinates of four Fermat curve points in (7) are identically related by

$$x_1\; x_2 \;=\; q\;\; x_3\; x_4\;. \tag{8}$$

So the matrix elements $R_{\sigma_1,\sigma_2,\sigma_3}^{\sigma_1',\sigma_2',\sigma_3'}$ depend on three complex numbers. These correspond to Zamolodchikov's spherical angles in the IRC-formulation of the BBZ-model [5].

The structure of the indices of the matrix (7) allows one to consider $\mathbf{R}$ as the operator acting in the tensor product of three vector spaces

$$\mathcal{V} \;=\; \mathbb{C}^N\;, \quad \mathbf{R} \,\in\, \mathrm{End}\,(\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}) \tag{9}$$

It is conventional to enumerate naturally the components of the tensor product of several vector spaces, so that (7) are the matrix elements of $\mathbf{R} \;=\; \mathbf{R}_{123}$. Of course, $\mathbf{R}_{123}$ acts trivially on the all other vector spaces if one considers $\mathcal{V}^{\otimes\Delta}$ for some arbitrary Δ.

(7) is known as the **R**-matrix of the Zamolodchikov - Bazhanov - Baxter model, see [5]. The proof that (7) satisfies the Tetrahedron Equation

$$\sum_{j_1\cdots j_6} R^{j_1,j_2,j_3}_{i_1,i_2,i_3}(p^{(1)})\, R^{k_1,j_4,j_5}_{j_1,i_4,i_5}(p^{(2)})\, R^{k_2,k_4,j_6}_{j_2,j_4,i_6}(p^{(3)})\, R^{k_3,k_5,k_6}_{j_3,j_5,j_6}(p^{(4)})$$

$$= \sum_{j_1\cdots j_6} R^{j_3,j_5,j_6}_{i_3,i_5,i_6}(p^{(4)})\, R^{j_2,j_4,k_6}_{i_2,i_4,j_6}(p^{(3)})\, R^{j_1,k_4,k_5}_{i_1,j_4,j_5}(p^{(2)})\, R^{k_1,k_2,k_3}_{j_1,j_2,j_3}(p^{(1)})\,. \qquad (10)$$

is rather tedious [5]. In (10) the arguments $p^{(j)}$ $(j = 1,\ldots,4)$ stand for four Fermat curve points $(p_1^{(j)}, p_2^{(j)}, p_3^{(j)}, p_4^{(j)})$ each. These 16 points depend on five independent parameters expressible in terms of spherical angles, see [5]. Note that here on the left and right hand side the same $p^{(j)}$ appear. This will not be the case in the generalizations which will be discussed soon.

Baxter and Forrester [19] have studied whether this model describes phase transitions. They used variational and numerical methods and found strong evidence that for the parameter values for which (10) is satisfied, the ZBB-model is just at criticality [19]. So, in order to get a chance to describe phase transitions while staying integrable (recall that also for the 2D Potts model this is a problem), one should enlarge the framework and define more general Boltzmann weights and introduce less restrictive Tetrahedron equations. Less restrictive and still powerful generalized equations can be used, as has been shown by Mangazeev and Stroganov [10]: they introduced Modified Tetrahedron Equations which guarantee commuting layer-to-layer transfer matrices. Further work along this line has been done in [11, 12].

1.2 R-matrix satisfying the Modified Tetrahedron Equation

Since in the above-mentioned work the proof that particular Boltzmann weights satisfy a particular MTE has been rather tedious, here we shall follow the approach introduced in [21] in which there is no need for an explicit check of the MTE. The Boltzmann weights are constructed from "physical" principles which guarantee the validity of the MTE and nevertheless leave much freedom to obtain a broad class of integrable 3D-models. We give a short summary of the argument.

One starts with an *oriented* 3-D basic lattice. The dynamic variables living on the links i of this lattice are taken to be elements $u_i, w_i \in \mathfrak{w}_i$ an ultralocal Weyl algebra $\mathfrak{w} = \bigotimes \mathfrak{w}_i$ at the primitive *N-th* root of unity: $u_i\, w_j = q^{\delta_{ij}}\, w_j\, u_i$, ($q$ as in eq.(6)), which generalize the $\mathbb{Z}_N$ spin variables of (2) and (3). The Weyl elements are represented by standard $N\times N$ raising and diagonal matrices. The *N-th* powers of the Weyl variables are centers of the algebra and so are scalar variables.

The main object constructed is an invertible canonical mapping $\mathcal{R}_{ijk}$ in the space of a triple Weyl algebra. $\mathcal{R}_{ijk}$ operates at the vertices of the 3D lattice, mapping the three Weyl elements on the "incoming" links onto those on the "outgoing" links, see Fig. 1.

The construction of $\mathcal{R}_{ijk}$ is based on two postulates, a Kirchhoff-like current conservation and a Baxter Z-invariance, and gives a unique explicit result: a canonical and invertible rational mapping operator. Since q is a root of unity, $\mathcal{R}_{ijk}$ decomposes into a matrix conjugation $\mathbf{R}_{ijk}$, and a purely functional mapping $\mathcal{R}^{(f)}_{ijk}$ which acts on the scalar parameters (the Weyl centers). So, for any rational function Φ on $\mathfrak{w}$:

$$\mathcal{R}_{123} \circ \Phi \;=\; \mathbf{R}_{123}(\mathcal{R}^{(f)}_{123} \circ \Phi)\mathbf{R}^{-1}_{123}. \tag{11}$$

It turns out that the matrix $\mathbf{R}_{ijk}$ has the form (7) where the four Fermat curve parameters, again constrained by (8), are rational functions of the scalar Weyl center parameters.

Next consider an auxiliary plane which cuts the three incoming links near a vertex, and a second auxiliary plane cutting through the outgoing links, see Fig. 1. We take the six Weyl dynamic variables to sit on the six intersection points of the auxiliary planes. $\mathcal{R}_{123}$ can be regarded as the mapping of the ingoing auxiliary plane to the parallel shifted outgoing auxiliary plane. Now consider the vertices of the basic lattice to be formed as the intersection points of three sets of non-parallel planes. The three planes which form the vertex A of Fig. 1 intersect the auxiliary planes in the lines X, Y, Z shown in Fig. 2. Seen from the moving auxiliary plane, $\mathcal{R}_{123}$ shifts the line X through the vertex with index 1 or Y through the vertex 2 etc. We attach variables b_1, $b_2, \ldots,$ d_3 to each section of the lines X, Y, Z as shown in Fig. 2. It is convenient to parameterize the two scalar

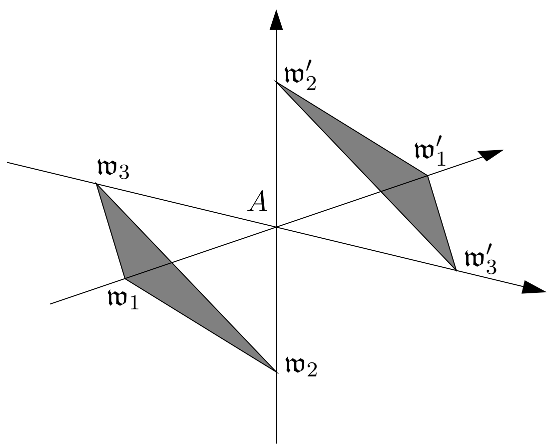

Figure 1: The six links of the basic lattice intersecting in the vertex A, intersected by auxiliary planes (shaded) in two different positions: first passing through $\mathfrak{w}_1$, $\mathfrak{w}_2$, $\mathfrak{w}_3$ and second through $\mathfrak{w}'_1$, $\mathfrak{w}'_2$, $\mathfrak{w}'_3$. The second position is obtained from the first by moving the auxiliary plane parallel through the vertex A. The Weyl variables, elements of $\mathfrak{w}_i$, $\mathfrak{w}'_i$ live on the links of the basic lattice. $\mathcal{R}$ maps the left auxiliary triangle onto the upper right one.

variables associated with the incoming dynamic Weyl variable $\mathfrak{w}_1$ (corresponding to u_1^N and w_1^N in usual notation) by the ratios c_2^N/c_3^N and d_3^N/d_2^N. Analogously e.g. those for $\mathfrak{w}'_2$ are defined as $b_1^N/{b'_2}^N$ and ${d'_2}^N/d_1^N$ etc. Details of the rule to parameterize the scalar variables in terms of "line-section" variables $b_1, \ldots, d_3$ etc. are explained in [9]. However, these will not be essential here, since one of the aims of this paper is to introduce and use another parameterization. Just observe that $\mathcal{R}^{(f)}_{123}$ changes only three of the line-section parameters: b_2, c_2, d_2. From the explicit form of the canonical operator $\mathbf{R}_{123}$ (see [9]) one finds that the functional mapping is rational in the N-th powers of the line sections:

$$\begin{aligned}
{b_2}'^N &= \frac{b_1^N c_3^N d_2^N + b_2^N c_3^N d_3^N + \kappa_1^N b_3^N c_2^N d_3^N}{c_2^N d_2^N}; \\
{c_2}'^N &= \frac{\kappa_1^N b_3^N c_1^N d_2^N + \kappa_3^N b_2^N c_3^N d_1^N + \kappa_1^N \kappa_3^N b_3^N c_2^N d_1^N}{\kappa_2^N b_2^N d_2^N};
\end{aligned}$$

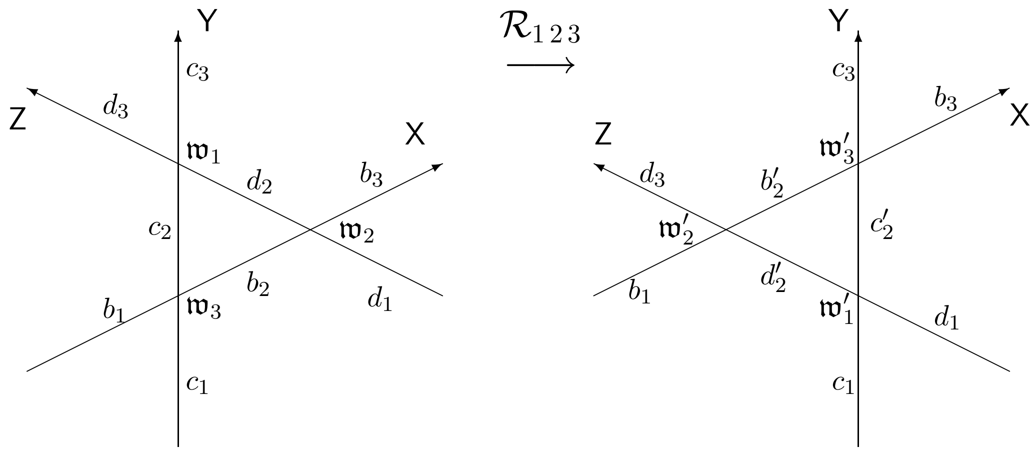

Figure 2: The canonical invertible mapping $\mathcal{R}_{123}$ shown in the auxiliary planes passing through the incoming (left) and outgoing (right) dynamic variables which are elements of $\mathfrak{w}_i$ resp. $\mathfrak{w}'_i$. The directed lines X, Y, Z are the intersections of the three planes forming the vertex A of Fig. 1. Their sections are labeled by the line-section parameters $b_1, \ldots, d_3$. Note, that orientation of the lines is not unique. But the choosen one corresponds to the numbering (44) and (45) of the fused vertex considered in the Section 3.

$$d_2{'}^N \;=\; \frac{b_2^N c_1^N d_3^N + b_1^N c_1^N d_2^N + \kappa_3^N b_1^N c_2^N d_1^N}{b_2^N c_2^N}. \tag{12}$$

Here $\kappa_1,\, \kappa_2,\, \kappa_3$ are fixed parameters ("coupling constants") of the mapping $\mathcal{R}_{123}$. In the line-section parameterization the three independent Fermat curve parameters of $\mathbf{R}_{123}$ are

$$x_1 = \frac{b_2 c_3}{\kappa_1 b_3 c_2}; \qquad x_2 = \frac{\kappa_2 b_1 c'_2}{b'_2 c_1}; \qquad x_3 = \frac{b_1 c_3}{\sqrt{q}\, b'_2 c_2}. \tag{13}$$

The matrix elements of $\mathbf{R}_{ijk}$ will be used as the Boltzmann weights of integrable 3D lattice models of statistical mechanics with the partition function defined in analogy to (2).

Via the physical assumptions made in constructing $\mathcal{R}_{ijk}$ the validity of the TE is built in. Simply considering two different sequences of Z-invariance shifts in a geometric figure formed by *four* intersecting straight lines ("quadrangle"), one concludes that (see e.g. [9])

$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} \;\sim\; \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123}\,, \tag{14}$$

i.e. that $\mathcal{R}_{ijk}$ satisfies the Tetrahedron equation. Inserting (11) into (14) and choosing various phases of *N-th* roots leads to the MTE for the matrix operator $\mathbf{R}_{ijk}$:

$$\begin{aligned} &\mathbf{R}_{123} \cdot \Big(\mathcal{R}^{(f)}_{123} \circ \mathbf{R}_{145}\Big) \cdot \Big(\mathcal{R}^{(f)}_{123}\mathcal{R}^{(f)}_{145} \circ \mathbf{R}_{246}\Big) \cdot \Big(\mathcal{R}^{(f)}_{123}\mathcal{R}^{(f)}_{145}\mathcal{R}^{(f)}_{246} \circ \mathbf{R}_{356}\Big) \\ &\sim\; \mathbf{R}_{356} \cdot \Big(\mathcal{R}^{(f)}_{356} \circ \mathbf{R}_{246}\Big) \cdot \Big(\mathcal{R}^{(f)}_{356}\mathcal{R}^{(f)}_{246} \circ \mathbf{R}_{145}\Big) \cdot \Big(\mathcal{R}^{(f)}_{356}\mathcal{R}^{(f)}_{246}\mathcal{R}^{(f)}_{145} \circ \mathbf{R}_{123}\Big)\,. \end{aligned} \tag{15}$$

Via the Fermat points $\mathbf{R}_{ijk}$ depends on the scalar variables, see e.g. (13). The scalar variables which appear in the matrices $\mathbf{R}_{ijk}$ are to be transformed by the functional transformations $\mathcal{R}^{(f)}_{ijk}$. Let us write shorthand

$$\mathbf{R}^{(1)} \;=\; \mathbf{R}_{123}; \qquad \mathbf{R}^{(2)} \;=\; \mathcal{R}^{(f)}_{123} \circ \mathbf{R}_{145}; \qquad \mathbf{R}^{(3)} \;=\; \mathcal{R}^{(f)}_{123}\mathcal{R}^{(f)}_{145} \circ \mathbf{R}_{246};$$

$$
\begin{array}{rclrcl}
\mathbf{R}^{(4)} &=& \mathcal{R}^{(f)}_{123}\mathcal{R}^{(f)}_{145}\mathcal{R}^{(f)}_{246}\circ\mathbf{R}_{356}; & \mathbf{R}^{(5)} &=& \mathcal{R}^{(f)}_{356}\mathcal{R}^{(f)}_{246}\mathcal{R}^{(f)}_{145}\circ\mathbf{R}_{123}; \\
\mathbf{R}^{(6)} &=& \mathcal{R}^{(f)}_{356}\mathcal{R}^{(f)}_{246}\circ\mathbf{R}_{145}; \qquad \mathbf{R}^{(7)} = \mathcal{R}^{(f)}_{356}\circ\mathbf{R}_{246}; & \mathbf{R}^{(8)} &=& \mathbf{R}_{356}.
\end{array}
\tag{16}
$$

Then (20) becomes

$$
\mathbf{R}^{(1)}\,\mathbf{R}^{(2)}\,\mathbf{R}^{(3)}\,\mathbf{R}^{(4)} \;=\; \rho\,\mathbf{R}^{(8)}\,\mathbf{R}^{(7)}\,\mathbf{R}^{(6)}\,\mathbf{R}^{(5)},
\tag{17}
$$

where each $\mathbf{R}^{(j)}$ acts non-trivially in only three of the six spaces $\mathcal{V} = \mathbb{C}^N$. ρ is a scalar density factor which comes in when passing from mappings to matrix equations.

The parameters which determine the $\mathbf{R}^{(j)}$ are the corresponding Fermat curve coordinates. Taking into account the functional transformations in (16) in terms of the line-section parameters we find (for details see [9]): $\mathbf{R}^{(j)} = \mathbf{R}(x_1^{(j)}, x_2^{(j)}, x_3^{(j)})$ with

$$
\begin{pmatrix}
x_1^{(1)} & x_2^{(1)} & x_3^{(1)} \\
x_1^{(2)} & x_2^{(2)} & x_3^{(2)} \\
\vdots & \vdots & \vdots \\
x_1^{(7)} & x_2^{(7)} & x_3^{(7)} \\
x_1^{(8)} & x_2^{(8)} & x_3^{(8)}
\end{pmatrix}
= q^{-1/2}
\begin{pmatrix}
\dfrac{b_2c_3}{\kappa_1 b_3 c_2} & \dfrac{\kappa_2 b_1 c_2'}{b_2' c_1} & \dfrac{b_1 c_3}{q^{1/2}\, b_2' c_2} \\
\dfrac{a_2 c_2'}{\kappa_1 a_3 c_1} & \dfrac{\kappa_4 a_1 c_1''}{a_2'' c_0} & \dfrac{a_1 c_2'}{q^{1/2}\, a_2'' c_1} \\
\vdots & \vdots & \vdots \\
\dfrac{a_2 b_3}{\kappa_2 a_3 b_2} & \dfrac{\kappa_4 a_1^{\dagger} b_2^{\dagger\dagger}}{a_2^{\dagger\dagger} b_1^{\dagger}} & \dfrac{a_1^{\dagger} b_3}{q^{1/2}\, a_2^{\dagger\dagger} b_2} \\
\dfrac{a_1 b_2}{\kappa_3 a_2 b_1} & \dfrac{\kappa_5 a_0 b_1^{\dagger}}{b_0 a_1^{\dagger}} & \dfrac{a_0 b_2}{q^{1/2}\, a_1^{\dagger} b_1}
\end{pmatrix}.
\tag{18}
$$

The once or multiply transformed parameters like c_1'', $b_1^{\dagger}$ follow from the iteration of equations like (12). Altogether, since there are eight matrices $\mathbf{R}^{(j)}$ appearing in the MTEs, and as seen in Fig. 2, each transformation changes three line-section parameters, we have 24 equations for 32 different line-section parameters (these parameters can seen in Table 1 below). This is a set of classical integrable equations which conveniently are written in Hirota form:

$$
\begin{array}{rcl}
{b_2'}^N c_2^N d_2^N &=& b_1^N c_3^N d_2^N + b_2^N c_3^N d_3^N + \kappa_1^N b_3^N c_2^N d_3^N; \\
\kappa_2^N b_2^N {c_2'}^N d_2^N &=& \kappa_1^N b_3^N c_1^N d_2^N + \kappa_3^N b_2^N c_3^N d_1^N + \kappa_1^N \kappa_3^N b_3^N c_2^N d_1^N; \\
b_2^N c_2^N {d_2'}^N &=& b_2^N c_1^N d_3^N + b_1^N c_1^N d_2^N + \kappa_3^N b_1^N c_2^N d_1^N; \\[1em]
{a_2''}^N c_1^N d_1^N &=& a_1^N {c_2'}^N d_1^N + a_2^N {c_2'}^N {d_2'}^N + \kappa_1^N a_3^N c_1^N {d_2'}^N; \\
\kappa_4^N a_2^N {c_1''}^N d_1^N &=& \kappa_1^N a_3^N c_0^N d_1^N + \kappa_5^N a_2^N {c_2'}^N d_0^N + \kappa_1^N \kappa_5^N a_3^N c_1^N d_0^N; \\
a_2^N c_1^N {d_1''}^N &=& a_2^N c_0^N {d_2'}^N + a_1^N c_0^N d_1^N + \kappa_5^N a_1^N c_1^N d_0^N; \\[1em]
{a_1'''}^N b_1^N {d_2'}^N &=& a_0^N {b_2'}^N {d_2'}^N + d_3^N a_1^N {b_2'}^N + \kappa_2^N d_3^N b_1^N {a_2''}^N; \\
\kappa_4^N a_1^N {b_1'''}^N {d_2'}^N &=& \kappa_2^N b_0^N {a_2''}^N {d_2'}^N + \kappa_6^N a_1^N {b_2'}^N {d_1''}^N + \kappa_2^N \kappa_6^N {a_2''}^N b_1^N {d_1''}^N; \\
a_1^N b_1^N {d_2'''}^N &=& b_0^N d_3^N a_1^N + a_0^N b_0^N {d_2'}^N + \kappa_6^N a_0^N b_1^N {d_1''}^N; \\[1em]
\vdots & \vdots & \\[1em]
{b_1^{t}}^N {c_1^{\dagger}}^N {d_1^{\dagger\dagger}}^N &=& b_0^N {c_2^{\dagger\dagger\dagger}}^N {d_1^{\dagger\dagger}}^N + {b_1^{\dagger}}^N {c_2^{\dagger\dagger\dagger}}^N {d_2^{\dagger\dagger\dagger}}^N + \kappa_1^N {b_2^{\dagger\dagger}}^N {c_1^{\dagger}}^N {d_2^{\dagger\dagger\dagger}}^N; \\
\kappa_2^N {b_1^{\dagger}}^N {c_1^{t}}^N {d_1^{\dagger\dagger}}^N &=& \kappa_1^N c_0^N {b_2^{\dagger\dagger}}^N {d_1^{\dagger\dagger}}^N + \kappa_3^N d_0^N {b_1^{\dagger}}^N {c_2^{\dagger\dagger\dagger}}^N + \kappa_1^N \kappa_3^N d_0^N {b_2^{\dagger\dagger}}^N {c_1^{\dagger}}^N; \\
{b_1^{\dagger}}^N {c_1^{\dagger}}^N {d_1^{t}}^N &=& c_0^N {b_1^{\dagger}}^N {d_2^{\dagger\dagger\dagger}}^N + b_0^N c_0^N {d_1^{\dagger\dagger}}^N + \kappa_3^N b_0^N d_0^N {c_1^{\dagger}}^N.
\end{array}
\tag{19}
$$

The first three of these equations are just (12), defining $\mathcal{R}^{(f)}_{123}$, i.e. ${b'_2}^N$, ${c'_2}^N$, ${d'_2}^N$ in terms of the unprimed $b_1, \ldots, d_3$. The first six equations together (e.g. express in the fourth eq. on the right hand side c'^N_2 and d'^N_2 from the first and third equations) define $\mathcal{R}^{(f)}_{123} \circ \mathcal{R}^{(f)}_{145}$, etc. The complete expressions for (18) and (19) can be found in [9].

Straightforward (best done by e.g. Maple) combination of the first twelve eqs. of (19) on one hand, and of the last twelve of eqs.(19) on the other hand, shows that the functional mappings given in (19) automatically satisfy the functional TE:

$$\mathcal{R}^{(f)}_{123} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{356} \;=\; \mathcal{R}^{(f)}_{356} \cdot \mathcal{R}^{(f)}_{246} \cdot \mathcal{R}^{(f)}_{145} \cdot \mathcal{R}^{(f)}_{123} \tag{20}$$

where for the superposition of two operators acting on a function Φ we use the notation $((\mathcal{A} \cdot \mathcal{B}) \cdot \Phi) \stackrel{def}{=} (\mathcal{A} \cdot (\mathcal{B} \cdot \Phi))$. Of course, the validity of (20) is a consequence of the physical rules used when constructing $\mathcal{R}_{ijk}$. In the line-section parameterization the relation between the first, second etc. lines in both (18) and in (19) is not transparent. Introducing a new parameterization in the next subsection will make these relations simple and explicit.

2 Parameterization using concepts of algebraic geometry

2.1 Theta functions

It is well-known [14, 15, 16, 17, 18] that Hirota-type equations can be identically satisfied by a parameterization in terms of theta functions on an algebraic curve. We shall now introduce such a parameterization in order to write (18) and eqs.(19) in a more systematic way which also allows us to formulate fusion in a transparent manner. For the notations of algebraic geometry see e.g. [13].

Let Γ_g be an abstract generic algebraic curve of the genus g with $\boldsymbol{\omega}$ being the canonical g-dimensional vector of the homomorphic differentials. For any two points $X, Y \in \Gamma_g$ let $\mathbf{I}^X_Y \;:\; \Gamma_g^2 \mapsto \mathrm{Jac}(\Gamma_g)$ be

$$\mathbf{I}^X_Y \;\stackrel{\mathrm{def}}{=}\; \int_Y^X \boldsymbol{\omega}\;. \tag{21}$$

Let further $E(X, Y) \;=\; -\,E(Y, X)$ be the prime form on Γ_g^2, and $\Theta(\mathbf{v})$ be the theta-function on $\mathrm{Jac}(\Gamma_g)$.

It is well known, the theta-functions on the Jacobian of an algebraic curve obey the Fay identity

$$\begin{aligned}\Theta(\mathbf{v})\,\Theta(\mathbf{v} + \mathbf{I}^A_B + \mathbf{I}^C_D) \;\;&=\;\; \Theta(\mathbf{v} + \mathbf{I}^A_D)\,\Theta(\mathbf{v} + \mathbf{I}^C_B)\,\frac{E(A,B)\,E(D,C)}{E(A,C)\,E(D,B)}\\ &+\;\; \Theta(\mathbf{v} + \mathbf{I}^A_B)\,\Theta(\mathbf{v} + \mathbf{I}^C_D)\,\frac{E(A,D)\,E(C,B)}{E(A,C)\,E(D,B)}\,,\end{aligned} \tag{22}$$

which involves four points $A, B, C, D \in \Gamma_g$ and a $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$. We shall show that in the parameterization to be introduced below, the Fermat relations become just Fay-identities. The Fay identity involves only cross ratios of prime forms, and these ratios have a simple expression in terms of non-singular odd characteristic theta functions:

$$\left[\begin{array}{cc} X & X' \\ Y & Y' \end{array}\right] \;\stackrel{\mathrm{def}}{=}\; \frac{E(X,Y)\,E(X',Y')}{E(X,Y')\,E(X',Y)} \;=\; \frac{\Theta_{\epsilon_{odd}}(\,\mathbf{I}^Y_X\,)\,\Theta_{\epsilon_{odd}}(\,\mathbf{I}^{Y'}_{X'}\,)}{\Theta_{\epsilon_{odd}}(\,\mathbf{I}^{Y'}_X\,)\,\Theta_{\epsilon_{odd}}(\,\mathbf{I}^{Y}_{X'}\,)}\;. \tag{23}$$

For solving the 24 trilinear equations we shall need an identity with more arguments obtained by combining two Fay identities:

$$\begin{aligned}
&\Theta(\mathbf{v}+\mathbf{I}^Q_X)\,\Theta(\mathbf{v}+\mathbf{I}^Q_Y+\mathbf{I}^{Z'}_Z)\,\Theta(\mathbf{v}+\mathbf{I}^Q_Z+\mathbf{I}^{Y'}_Y) \\
&-\quad \Theta(\mathbf{v}+\mathbf{I}^Q_X+\mathbf{I}^{Z'}_Z)\,\Theta(\mathbf{v}+\mathbf{I}^Q_Y)\,\Theta(\mathbf{v}+\mathbf{I}^Q_Z+\mathbf{I}^{Y'}_Y)\left[\begin{array}{cc} X & Y \\ Z & Z' \end{array}\right] \\
&-\quad \Theta(\mathbf{v}+\mathbf{I}^Q_X+\mathbf{I}^{Y'}_Y)\,\Theta(\mathbf{v}+\mathbf{I}^Q_Y+\mathbf{I}^{Z'}_Z)\,\Theta(\mathbf{v}+\mathbf{I}^Q_Z)\left[\begin{array}{cc} X & Z \\ Y & Y' \end{array}\right] \\
&+\quad \Theta(\mathbf{v}+\mathbf{I}^Q_X+\mathbf{I}^{Y'}_Y+\mathbf{I}^{Z'}_Z)\,\Theta(\mathbf{v}+\mathbf{I}^Q_Y)\,\Theta(\mathbf{v}+\mathbf{I}^Q_Z)\left[\begin{array}{cc} X & Y \\ Z & Z' \end{array}\right]\left[\begin{array}{cc} X & Z' \\ Y & Y' \end{array}\right] \;=\; 0\,.
\end{aligned} \tag{24}$$

Further we will need the N-th roots of theta-functions and prime forms. Define $e(X,Y)$ and $\theta(\mathbf{v})$ by

$$e(X,Y)^N \;=\; \Theta_{\epsilon_{odd}}(\mathbf{I}^X_Y) \;\sim\; E(X,Y)\,; \qquad \theta(\mathbf{v})^N \;=\; \Theta(\mathbf{v})\,. \tag{25}$$

Since in the following we shall have to write many equations involving theta-functions, it is convenient to introduce special abbreviations. For ($Q, A, B_1, B'_1, \ldots \in \Gamma_g$) we define:

$$\begin{aligned}
&(A,\;B_1+B_2+\ldots+B_n) \;\equiv\; \Theta(\mathbf{v}+\mathbf{I}^Q_A+\sum_{j=1}^{n}\mathbf{I}^{B_j{}'}_{B_j})\,; \qquad \langle A,B\rangle \;\equiv\; E(A,B)\,; \\
&[A,\;B_1+B_2+\ldots+B_n\,] \;\equiv\; \theta(\mathbf{v}+\mathbf{I}^Q_A+\sum_{j=1}^{n}\mathbf{I}^{B_j{}'}_{B_j}) \\
&\{A,B\} \;\equiv\; -\,q^{-1/2}\,e\,(A,B')\,/\,e(A,B)\,; \qquad \overline{\{A,B\}} \;\equiv\; -e(A',B)\,/\,e(A,B)\,.
\end{aligned} \tag{26}$$

Note that the two brackets introduced here do not indicate explicitly the dependence on the variables $\mathbf{v},\ Q,\ B'_1,\ldots B'_n$ since these always come in the same form.

We also introduce, using these notations:

$$\begin{aligned}
&\mathbf{F}\,(\mathbf{v};\,X,Y',Y,Z',Z) \;\stackrel{\text{def}}{=} \\
&(X)(Y,Z)(Z,Y)\,\langle Y,Z\rangle\,\langle X,Z'\rangle\,\langle X,Y'\rangle \,-\,(X,Z)(Y)(Z,Y)\,\langle X,Z\rangle\,\langle Y,Z'\rangle\,\langle X,Y'\rangle \\
&-(X,Y)(Y,Z)(Z)\,\langle X,Y\rangle\,\langle Y',Z\rangle\,\langle X,Z'\rangle \,+\,(X,Y+Z)(Y)(Z)\,\langle X,Z\rangle\,\langle X,Y\rangle\,\langle Y',Z'\rangle\,,
\end{aligned} \tag{27}$$

so that the Double-Fay-identity (24) is

$$\mathbf{F}\,(\mathbf{v};\,X,Y',Y,Z',Z) \;=\; 0. \tag{28}$$

The dependence on Q is trivial since it appears only in the combination $\mathbf{v}+\mathbf{I}^Q_{\ldots}$. So Q is not an independent variable.

2.2 Re-parameterization of R

Let us introduce the new parameterization of the matrix (7). As we illustrated in Fig. 2, in the auxiliary plane the mapping $\mathcal{R}_{123}$ can be considered as a relative shift of three directed lines X, Y, Z with respect to each other. Now, for the given algebraic curve Γ_g and $\mathbf{v} \in \mathbf{C}^g$, we introduce three pairs of points on Γ_g:

$$X',\ X,\ Y',\ Y,\ Z',\ Z\ \in\ \Gamma_g. \tag{29}$$

Another point $Q \in \Gamma_g$ will just serve as a trivial normalization. Then let

$$\mathbf{R}\ =\ \mathbf{R}(p_1,p_2,p_3,p_4)\quad \Longleftrightarrow \quad \mathbf{R}\ =\ \mathbf{R}(\mathbf{v};X',X;Y',Y;Z',Z) \tag{30}$$

with, using the shorthand notations (26) and $p_j = (x_j, y_j)$:

$$\begin{aligned}
x_1 &= \frac{1}{q}\frac{\{X,Z\}}{\{Y',Z\}}\frac{[X,Y]\,[Y,Z]}{[X,Y+Z]\,[Y]}; & y_1 &= \frac{e(Z,Z')\,e(X,Y')}{e(X,Z)\,e(Y',Z')}\,\frac{[Z,Y]\,\theta(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'})}{[X,Y+Z]\,[Y]}\\
x_2 &= \frac{\{X',Z\}}{\{Y,Z\}}\frac{[X]\,[Y,X+Z]}{[X,Z]\,[Y,X]}; & y_2 &= q\,\frac{e(Z,Z')\,e(X',Y)}{e(X',Z)\,e(Y,Z')}\,\frac{[Z,X]\,\theta(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'})}{[X,Z]\,[Y,X]}\\
x_3 &= \frac{1}{q}\frac{\{X,Z\}}{\{Y,Z\}}\frac{[X]\,[Y,Z]}{[X,Z]\,[Y]}; & y_3 &= q\,\frac{e(Z,Z')\,e(X,Y)}{e(X,Z)\,e(Y,Z')}\,\frac{[Z]\,\theta(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'})}{[X,Z]\,[Y]}\\
x_4 &= \frac{1}{q}\frac{\{X',Z\}}{\{Y',Z\}}\frac{[X,Y]\,[Y,X+Z]}{[X,Y+Z]\,[Y,X]}; & y_4 &= \frac{e(Z,Z')\,e(X',Y')}{e(X',Z)\,e(Y',Z')}\,\frac{[Z,X+Y]\,\theta(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'})}{[X,Y+Z]\,[Y,X]}.
\end{aligned} \tag{31}$$

Actually, see (5), for defining $\mathbf{R}_{123}$ we don't need the y_i themselves but only their three ratios

$$\begin{aligned}
\frac{y_3}{y_1} &= q\,\frac{\{\overline{Y,Z'}\}}{\{X,Y\}}\frac{[Z]\,[X,Y+Z]}{[X,Z]\,[Z,Y]}; & \frac{y_3}{y_2} &= \frac{e(X,Y)\,e(X',Z)}{e(X,Z)\,e(X',Y)}\,\frac{[Z]\,[Y,X]}{[Y]\,[Z,X]};\\
\frac{y_4}{y_1} &= \frac{\{\overline{X,Y'}\}}{\{\overline{X,Z}\}}\frac{[Z,X+Y]\,[Y]}{[Z,Y]\,[Y,X]}.
\end{aligned} \tag{32}$$

from which $e(Z,Z')$ and $\theta(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'})$ drop out. These three ratios cannot be expressed all in the same form.

Note that for this parameterization we used a generic algebraic curve and generic points on it, and a generic point on its Jacobian, all in order to parameterize just three independent complex numbers x_1, x_2, x_3. In (31) all $x_k,\ y_k$ are the periodical functions of $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$.

The parameterization (31) is suggested by a few assumptions: First: The prime forms shall appear in the x_i only in the form of N-*th* roots of (23):

$$\frac{\{X,Z\}}{\{Y,Z\}}\ =\ \begin{bmatrix} X & Y \\ Z & Z' \end{bmatrix}^{1/N}. \tag{33}$$

Second, considering (13), we demand that the line-section parameters $b_1,\ b_2,\ b_3,\ b_2'$ (sections of the line X in Fig. 2) should be proportional to N-*th* roots of theta functions of the form $[X,\ldots]$ defined in (26). Analogously, the sections $c_1,\ \ldots,\ c_2'$ of the line Y are assumed to be proportional to $[Y,\ldots]$. Finally, we consider that we want to use Fay identities to satisfy the Fermat relations, and later, the Hirota equations.

The merit of this parameterization will be seen in several places: when we consider the transformed mappings $\mathbf{R}^{(2)}, \ldots, \mathbf{R}^{(6)}$, when we re-write eqs.(19) and when we construct composite weights in Sec. 3.

We now must verify that (31), and its generalization to the other Fermat points in the MTE, give a consistent parameterization of the relevant equations (4), (18) and (19). We first check that (31) satisfies the Fermat relations

$$x_j^N + y_j^N = 1. \tag{34}$$

Indeed, these are true due to the Fay identity, which for $(A, B, C, D \in \Gamma_g)$ we write as

$$- \langle A, C\rangle\langle D, B\rangle\, \Theta(\mathbf{v})\, \Theta(\mathbf{v} + \mathbf{I}_B^A + \mathbf{I}_D^C) + \langle A, B\rangle\langle D, C\rangle\, \Theta(\mathbf{v} + \mathbf{I}_D^A)\, \Theta(\mathbf{v} + \mathbf{I}_B^C) \\ + \langle A, D\rangle\langle C, B\rangle\, \Theta(\mathbf{v} + \mathbf{I}_B^A)\, \Theta(\mathbf{v} + \mathbf{I}_D^C) = 0. \tag{35}$$

For $j = 1$ put in (35) $(A,\ B,\ C,\ D) \to (Y',\ X,\ Z',\ Z)$ and $\mathbf{v} \to \mathbf{v}' = \mathbf{v} + \mathbf{I}_Y^Q$, giving

$$\langle Z, X\rangle\langle Y', Z'\rangle(Y)(X, Y + Z) - \langle Z, Z'\rangle\langle Y', X\rangle(Z, Y)\Theta(\mathbf{v}' + \mathbf{I}_X^{Z'}) \\ - \langle Z', X\rangle\langle Y', Z\rangle(X, Y)(Y, Z) = 0,$$

for $j = 2$ put in (35) $(A,\ B,\ C,\ D) \to (X',\ Y,\ Z',\ Z)$, $\mathbf{v} \to \mathbf{v}'' = \mathbf{v} + \mathbf{I}_X^Q$, giving

$$\langle Z', Y\rangle\langle X', Z\rangle(Y, X)(X, Z) - \langle Z, Y\rangle\langle X', Z'\rangle(X)(Y, X + Z) \\ + \langle Z, Z'\rangle\langle X', Y\rangle(Z, X)\Theta(\mathbf{v}'' + \mathbf{I}_Y^{Z'}) = 0,$$

for $j = 3$ put in (35) $(A,\ B,\ C,\ D) \to (Z,\ X,\ Z',\ Y)$, $\mathbf{v} \to \mathbf{v}^+ = \mathbf{v} + \mathbf{I}_Z^Q$:

$$\langle X, Z'\rangle\langle Y, Z\rangle(X)(Y, Z) - \langle X, Z\rangle\langle Y, Z'\rangle(Y)(X, Z) + \langle Z, Z'\rangle\langle X, Y\rangle(Z)\Theta(\mathbf{v}+\mathbf{I}_Y^Q+\mathbf{I}_X^{Z'}) = 0,$$

or

$$\langle X, Z'\rangle\langle Y, Z\rangle\Theta(\mathbf{v}^+ + \mathbf{I}_X^Z)\Theta(\mathbf{v}^+ + \mathbf{I}_Y^{Z'}) - \langle X, Z\rangle\langle Y, Z'\rangle\Theta(\mathbf{v}^+ + \mathbf{I}_Y^Z)\Theta(\mathbf{v}^+ + \mathbf{I}_X^{Z'}) \\ + \langle Z, Z'\rangle\langle X, Y\rangle\Theta(\mathbf{v}^+)\Theta(\mathbf{v}^+ + \mathbf{I}_X^Z + \mathbf{I}_Y^{Z'}) = 0.$$

For $j = 4$ put in (35) $(A,\ B,\ C,\ D) \to (X',\ Z,\ Y',\ Z')$, $\mathbf{v} \to \mathbf{v}^* = \mathbf{v} + \mathbf{I}_X^Q + \mathbf{I}_Y^{Z'}$.

2.3 Line-section parameters and Hirota equations in terms of theta functions

For writing the MTE in our new parameterization and to check (18) and (19) we have to consider three more spaces $\mathcal{V} = \mathbb{C}^N$, corresponding to the indices 4, 5, 6. In Fig. 2 the first three spaces were located at the intersection points of the lines X, Y, Z. To include the other three spaces, consider the "quadrangle" formed by *four* lines shown in Fig. 3. Corresponding to the new line U we introduce another pair of points $U', U \in \Gamma_g$.

Looking at Fig. 3 we see that, instead of labeling the spaces by the vertices $1, \ldots, 6$ of the quadrangle, we can as well label them by the pair of lines which intersect in these vertices, so identifying

$$1 \sim \mathsf{YZ}; \qquad 2 \sim \mathsf{XZ}; \qquad 3 \sim \mathsf{XY};$$
$$4 \sim \mathsf{UZ}; \qquad 5 \sim \mathsf{UY}; \qquad 6 \sim \mathsf{UX}. \tag{36}$$

a_0	$= [U]\{U,X\}\{U,Y\}\{U,Z\}$	b_1	$= [X]\{X,Y\}\{X,Z\}\{\overline{U,X}\}$
a_1	$= [U,X]\{U,Y\}\{U,Z\}$	b_2	$= [X,Y]\{X,Z\}\{\overline{U,X}\}$
$a_1^\dagger$	$= [U,Y]\{U,Z\}\{U,X\}$	b_2'	$= [X,Z]\{X,Y\}\{\overline{U,X}\}$
$a_1''' = a_1^{\dagger\dagger\dagger}$	$= [U,Z]\{U,X\}\{U,Y\}$	b_0	$= [X,U]\{X,Y\}\{X,Z\}$
a_2	$= [U,X+Y]\{U,Z\}$	b_3	$= [X,Y+Z]\{\overline{U,X}\}$
$a_2^{\dagger\dagger} = a_2^T$	$= [U,Y+Z]\{U,X\}$	$b_1''' = b_1^t$	$= [X,U+Z]\{X,Y\}$
a_2''	$= [U,X+Z]\{U,Y\}$	$b_1^\dagger$	$= [X,U+Y]\{X,Z\}$
a_3	$= [U,X+Y+Z]$	$b_2^{\dagger\dagger} = b_2^T$	$= [X,U+Y+Z]$
c_2	$= [Y]\{Y,Z\}\{\overline{U,Y}\}\ \{\overline{X,Y}\}$	d_3	$= [Z]\{\overline{U,Z}\}\ \{\overline{X,Z}\}\ \{\overline{Y,Z}\}$
$c_1^\dagger$	$= [Y,U]\{Y,Z\}\{\overline{X,Y}\}$	$d_2''' = d_2^{\dagger\dagger\dagger}$	$= [Z,U]\{\overline{X,Z}\}\ \{\overline{Y,Z}\}$
c_3	$= [Y,Z]\{\overline{U,Y}\}\ \{\overline{X,Y}\}$	d_2'	$= [Z,X]\{\overline{Y,Z}\}\ \{\overline{U,Z}\}$
c_1	$= [Y,X]\ \{Y,Z\}\{\overline{U,Y}\}$	d_2	$= [Z,Y]\{\overline{U,Z}\}\ \{\overline{X,Z}\}$
c_0	$= [Y,U+X]\{Y,Z\}$	$d_1'' = d_1^t$	$= [Z,U+X]\{\overline{Y,Z}\}$
$c_2^{\dagger\dagger\dagger} = c_2^T$	$= [Y,Z+U]\{\overline{X,Y}\}$	d_1	$= [Z,X+Y]\{\overline{U,Z}\}$
c_2'	$= [Y,X+Z]\{\overline{U,Y}\}$	$d_1^{\dagger\dagger}$	$= [Z,U+Y]\{\overline{X,Z}\}$
$c_1'' = c_1^t$	$= [Y,Z+U+X]$	d_0	$= [Z,U+X+Y]$

Table 1: The 32 line section parameters appearing in eqs.(19), expressed in terms of theta functions and prime factor ratios, using the abbreviations (26). Observe that in the prime factor brackets, the points come always in the order U, X, Y, Z (without and with primes).

Note that ordering of lines pair in the identification (36) is important, since it distinguishes the orientation we used and mirror reflected one.

Next we assume that we can write the κ_j all in the form (33). Then from (36) it is suggestive to build e.g. κ_1 from the points Y', Y, Z', Z only, etc. and put (factors $q^{1/2}$ are inserted to produce correct signs when forming N-th powers for (19)):

$$\begin{aligned} \kappa_1 &= q^{1/2}\frac{\{Y',Z\}}{\{Y,Z\}}; \qquad \kappa_2 = q^{1/2}\frac{\{X',Z\}}{\{X,Z\}}; \qquad \kappa_3 = q^{1/2}\frac{\{X',Y\}}{\{X,Y\}}; \\ \kappa_4 &= q^{1/2}\frac{\{U',Z\}}{\{U,Z\}}; \qquad \kappa_5 = q^{1/2}\frac{\{U',Y\}}{\{U,Y\}}; \qquad \kappa_6 = q^{1/2}\frac{\{U',X\}}{\{U,X\}}. \end{aligned} \tag{37}$$

Now we consider (31) and (37) and assume that the line-section parameters a_0, $a_1, \ldots$ and d_0, $d_1, \ldots$ follow the same scheme as postulated for $b_0, \ldots,$ $c_0, \ldots$ above after (33): $a_i \sim [U, \ldots]$, $d_i \sim [Z, \ldots]$. So eqs.(18) lead us to the expressions for all line-section parameters shown in Table 1. We make ample use of the short-hand notations (26).

Apart from Q which always comes with $\mathbf{v}$, we use eight arbitrary points X', X, Y', Y,

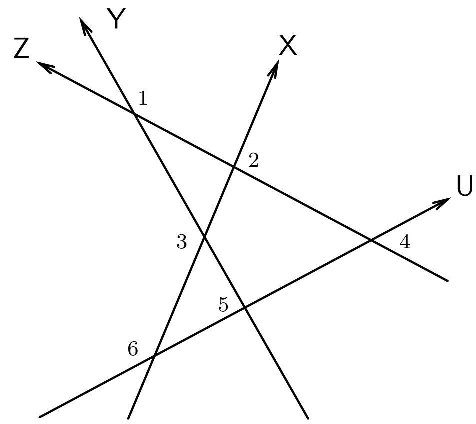

Figure 3: Quadrangle in the auxiliary plane formed by the directed intersection lines of four oriented lattice planes. The six spaces $\mathcal{V}$ in which the MTE operates are considered to be located at the six intersection points.

$Z', Z, U', U \in \Gamma_g$. The κ_j may as well be written in terms of the bared brackets using $\{\overline{A, B'}\}\{A, B\} = \{\overline{A, B}\}\{A', B\}$. From Table 1 we see that the a_i don't depend on U', the b_i not on X' etc.

Finally, using the results of Table 1 and (37), we can re-write all the Hirota equations (19) in terms of theta functions on Γ_g and prime form cross ratios. Not very surprisingly in view of [14, 17, 18], it turns out that these all have the form of the double-Fay identity. Also, as expected from Fig. 3, and the meaning of the mappings as moving of lines within the quadrangle, the 24 equations follow from each other by a sequence of simple substitutions. Just inserting from Table 1 and (37), the first three eqs. of (19) (recall that these are eqs.(12) defining the functional mapping $\mathcal{R}^{(f)}_{123}$) become

$$\begin{aligned}
\mathbf{F}\,(\mathbf{v};\, X, Y', Y, Z', Z)\, \frac{\left(\{\overline{U,X}\}\{\overline{U,Y}\}\{\overline{U,Z}\}\{\overline{X,Y}\}\{\overline{X,Z}\}\right)^N}{E(X,Y)\,E(X,Z)\,E(Y,Z)} &= 0 \\
\mathbf{F}\,(\mathbf{v} + \mathbf{I}_Y^{Y'};\, Y', X', X, Z', Z)\, \frac{\left(\{\overline{U,X}\}\{\overline{U,Y}\}\{\overline{U,Z}\}\right)^N}{E(X,Z)\,E(X,Y')\,E(Y',Z)} &= 0 \\
\mathbf{F}\,(\mathbf{v};\, Z, Y', Y, X', X)\, \frac{\left(\{\overline{U,X}\}\{\overline{U,Y}\}\{\overline{U,Z}\}\{X,Z\}\{Y,Z\}\right)^N}{E(X,Y)\,E(X,Z)\,E(Y,Z)} &= 0. \qquad (38)
\end{aligned}$$

The dependence on U', U appears only in the factors on the right, not in the $\mathbf{F}$. Assuming generic points $U', U, \ldots$ we conclude that the $\mathbf{F}$ must vanish and we combine the essential terms of (38) into

$$\mathfrak{F}\,(\mathbf{v};\, X', X, Y', Y, Z', Z) \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{F}\,(\mathbf{v};\, X, Y', Y, Z', Z) \\ \mathbf{F}\,(\mathbf{v} + \mathbf{I}_Y^{Y'};\, Y', X', X, Z', Z) \\ \mathbf{F}\,(\mathbf{v};\, Z, Y', Y, X', X) \end{pmatrix}. \qquad (39)$$

Then the 24 Hirota equations (19) which describe the functional mappings take the form

$$\begin{aligned}
&\mathfrak{F}\,(\mathbf{v};\ X',X,Y',Y,Z',Z)\ =\ 0\,, &&\mathfrak{F}\,(\mathbf{v}+\mathbf{I}_U^{U'};\ X',X,Y',Y,Z',Z)\ =\ 0\,,\\
&\mathfrak{F}\,(\mathbf{v}+\mathbf{I}_X^{X'};\ U',U,Y',Y,Z',Z)\ =\ 0\,, &&\mathfrak{F}\,(\mathbf{v};\ U',U,Y',Y,Z',Z)\ =\ 0\,,\\
&\mathfrak{F}\,(\mathbf{v};\ U',U,X',X,Z',Z)\ =\ 0\,, &&\mathfrak{F}\,(\mathbf{v}+\mathbf{I}_Y^{Y'};\ U',U,X',X,Z',Z)\ =\ 0\,,\\
&\mathfrak{F}\,(\mathbf{v}+\mathbf{I}_Z^{Z'};\ U',U,X',X,Y',Y)\ =\ 0\,, &&\mathfrak{F}\,(\mathbf{v};\ U',U,X',X,Y',Y)\ =\ 0\,.
\end{aligned} \tag{40}$$

To get the order in which the Hirota equations are written in eq.(68) of [9] and in (19), read these equations on the left hand side up to down and then continue on the right hand side from the bottom up. So e.g. the last line of (40) collects the $10th$ to $15th$ eqs. of (68) [9] or (19). As already mentioned with (20), eqs. (40) incorporate the functional TE.

2.4 Theta-parameterization of the simple modified tetrahedron equation

Using the parameterization (31) for the MTE (17) with (18) we find that the functional mapping just produces a permutation of the four pairs of points $X, X', \ldots, U, U' \in \Gamma_g$, together with shifts in the vector $\mathbf{v}$. Of course, the result corresponds to (40) and is explicitly:

Theorem 1 *The simple modified tetrahedron equation may be parameterized in the terms of* Γ_g, $\mathbf{v} \in \mathrm{Jac}(\Gamma_g)$ *and four pairs* $X', X,\ Y', Y,\ Z', Z,\ U', U \in \Gamma_g$ *by the definition (30),(31),(18) as follows:*

$$\begin{aligned}
\mathbf{R}^{(1)} &= \mathbf{R}(\mathbf{v};\ X',X;Y',Y;Z',Z)\,, & \mathbf{R}^{(5)} &= \mathbf{R}(\mathbf{v}+\mathbf{I}_U^{U'};\ X',X;Y',Y;Z',Z)\,,\\
\mathbf{R}^{(2)} &= \mathbf{R}(\mathbf{v}+\mathbf{I}_X^{X'};\ U',U;Y',Y;Z',Z)\,, & \mathbf{R}^{(6)} &= \mathbf{R}(\mathbf{v};\ U',U;Y',Y;Z',Z)\,,\\
\mathbf{R}^{(3)} &= \mathbf{R}(\mathbf{v};\ U',U;X',X;Z',Z)\,, & \mathbf{R}^{(7)} &= \mathbf{R}(\mathbf{v}+\mathbf{I}_Y^{Y'};\ U',U;X',X;Z',Z)\,,\\
\mathbf{R}^{(4)} &= \mathbf{R}(\mathbf{v}+\mathbf{I}_Z^{Z'};\ U',U;X',X;Y',Y)\,, & \mathbf{R}^{(8)} &= \mathbf{R}(\mathbf{v};\ U',U;X',X;Y',Y)\,.
\end{aligned} \tag{41}$$

Proof: Each $\mathbf{R}^{(j)}$ is determined by its three Fermat points $x_1^{(j)}$, $x_2^{(j)}$, $x_3^{(j)}$. From [9] these points are known in terms of the line-section parameters, see (18). Inserting the theta-function expressions for the line-sections from Table 1 into (18) one finds that the $x_i^{(j)}$ for $j = 2, \ldots, 8$ are obtained from those for $j = 1$, eqs.(31), by the substitutions seen in (41). □

Using the correspondence between the labels $1, \ldots, 6$ and the line labels U, X, Y, Z, see (36), $\mathbf{R}^{(1)} = \mathbf{R}_{123}$ may also labeled as $\mathbf{R}^{\mathsf{XYZ}}$ etc., and we write the MTE as

$$\begin{aligned}
&\mathbf{R}^{\mathsf{XYZ}}(\mathbf{v})\,\mathbf{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_X^{X'})\,\mathbf{R}^{\mathsf{UXZ}}(\mathbf{v})\,\mathbf{R}^{\mathsf{UXY}}(\mathbf{v}+\mathbf{I}_Z^{Z'})\\
&\qquad = \rho\,\mathbf{R}^{\mathsf{UXY}}(\mathbf{v})\,\mathbf{R}^{\mathsf{UXZ}}(\mathbf{v}+\mathbf{I}_Y^{Y'})\,\mathbf{R}^{\mathsf{UYZ}}(\mathbf{v})\,\mathbf{R}^{\mathsf{XYZ}}(\mathbf{v}+\mathbf{I}_U^{U'}).
\end{aligned} \tag{42}$$

This notation also indicates directly the three pairs of points on the algebraic curve which parameterize the matrices $\mathbf{R}^{(j)}$ in (41).

In [9] we had shown that using simple re-scalings, out of the 24 line-section parameters listed in Table 1 and the 6 parameters $\kappa_1, \ldots, \kappa_6$, only eight parameters are independent. Here we have eight points on Γ_g which can be chosen freely. In addition, 16 phases from taking the N-*th* roots can be chosen freely. In terms of the line-section parameters, the choice of the independent phases is the same as the one explained in [9].

3 The fused vertex weight $\mathfrak{R}$

3.1 Open $N_1 \times N_2 \times N_3$ box

The natural graphical interpretation of the **R**-matrix is a three dimensional vertex, i.e. the intersection of three planes in 3-D space. The six indices σ_j, σ'_j are associated to the edges of the vertex, recall Fig. 1.

The next step is the consideration of intersection of three *sets* of N_1, N_2 and N_3 parallel planes. This produces a finite open cubic lattice of the size $N_1 \times N_2 \times N_3$. We call the corresponding vertex object $\mathfrak{R}$, which is the result of the fusion of elementary **R**-matrices.

The lattice is defined as in (1), but now $n_j = 0, \ldots N_j - 1$. Let $\mathfrak{R}_{123}$ be the matrix associated with the open cube (more precisely, the open parallelepiped):

$$\mathfrak{R}_{123} \equiv \langle \vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3 | \, \mathfrak{R} \, | \vec{\sigma}'_1, \vec{\sigma}'_2, \vec{\sigma}'_3 \rangle = \sum_{\{\sigma\}} \prod_{\mathbf{n}} \langle \sigma_{1,\mathbf{n}}, \sigma_{2,\mathbf{n}+\mathbf{e}_2}, \sigma_{3,\mathbf{n}} | \mathbf{R}_{\mathbf{n}} | \sigma_{1,\mathbf{n}+\mathbf{e}_1}, \sigma_{2,\mathbf{n}}, \sigma_{3,\mathbf{n}+\mathbf{e}_3} \rangle \, . \tag{43}$$

Here the six external multi-spin variables (i.e. the indices of the matrix $\mathfrak{R}_{123}$) are associated with the six surfaces of the cube:

$$\vec{\sigma}_1 = \{\sigma_{1:0,n_2,n_3}\}, \quad \vec{\sigma}_2 = \{\sigma_{2:n_1,N_2,n_3}\}, \quad \vec{\sigma}_3 = \{\sigma_{3:n_1,n_2,0}\}, \quad n_j = 0, ..., N_j - 1 \tag{44}$$

and

$$\vec{\sigma}'_1 = \{\sigma_{1:N_1,n_2,n_3}\}, \quad \vec{\sigma}'_2 = \{\sigma_{2:n_1,0,n_3}\}, \quad \vec{\sigma}'_3 = \{\sigma_{3:n_1,n_2,N_3}\}, \quad n_j = 0, ..., N_j - 1 \, , \tag{45}$$

and the summation in (43) is taken with respect to all internal indices $\sigma_{j,\mathbf{n}}$. Anticipating what will be needed in (62) in order to prove that the fused weights satisfy a MTE of the same form as we had in (42), we introduce a reversed numbering for σ_2 and σ'_2, so that the "initial" external indices are $\sigma_1 = 0, \ \sigma_2 = N_2, \ \sigma_3 = 0$. This reversed numbering in the second space is dictated also by our choice of the line orientations in the lattice and, as consequence of this, in the auxiliary plane (see Fig. 4).

In our next step we want to parameterize all $\mathbf{R}_{\mathbf{n}}$ in (43) such that the fused weight $\mathfrak{R}$ again satisfies a Modified Tetrahedron Equation. We shall show that using a theta-function parameterization this is possible and the $\mathfrak{R}_{ijk}$ obtained will depend on $6(N_1 + N_2 + N_3)$ free parameters.

We use again the generic algebraic curve Γ_g, and one vector $\mathbf{v} \in \mathbb{C}^g$. As in (30) each $\mathbf{R}_{\mathbf{n}}$ will depend on three pairs of points on Γ_g, and to each $\mathbf{R}_{\mathbf{n}}$ we assign different three pairs:

$$X'_{n_1}, X_{n_1}, Y'_{n_2}, Y_{n_2}, Z'_{n_3}, Z_{n_3}, \qquad n_j = 0, ..., N_j - 1. \tag{46}$$

However, the argument $\mathbf{v}$ will be shifted for each $\mathbf{R}_{\mathbf{n}}$ by an amount $\mathbf{I}_{\mathbf{n}}$ which depends on the points assigned to "previous" neighbors: We define

$$\mathbf{R}^{(123)}_{\mathbf{n}} = \mathbf{R}(\mathbf{v} + \mathbf{I}_{\mathbf{n}}; X'_{n_1}, X_{n_1}; Y'_{n_2}, Y_{n_2}; Z'_{n_3}, Z_{n_3}) \, , \qquad n_j = 0, ..., N_j - 1, \tag{47}$$

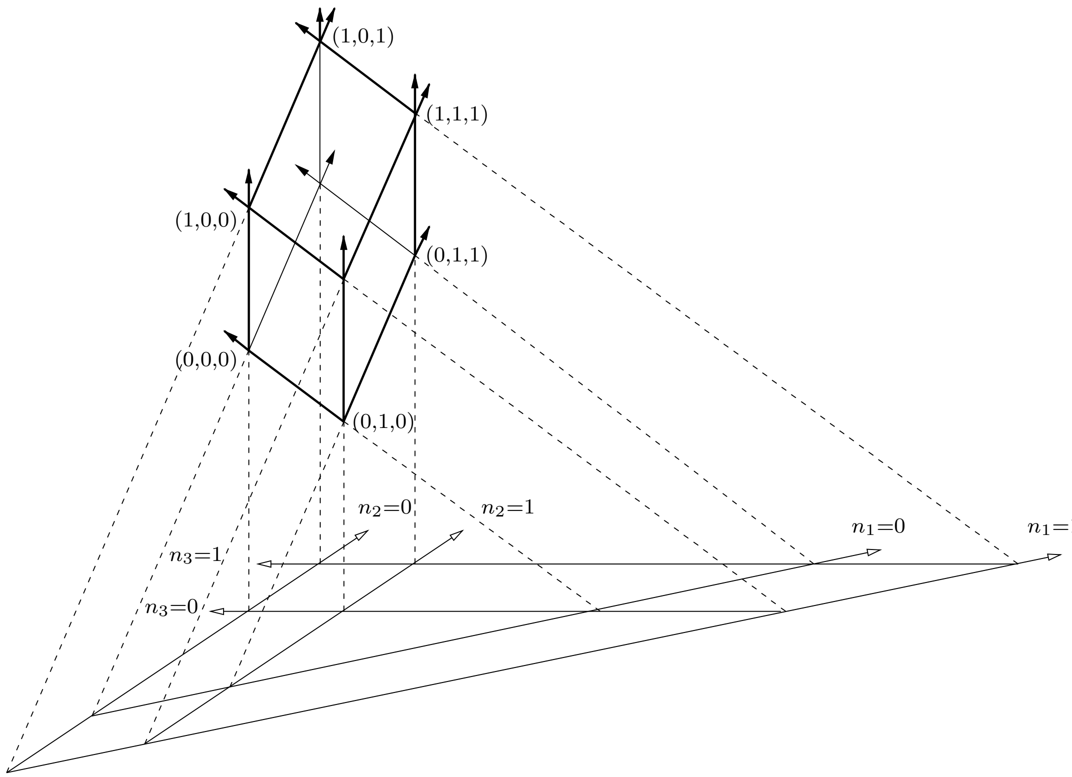

Figure 4: Top: 3-dimensional view of the oriented cube $N_1 = N_2 = N_3 = 2$ (heavy lines) which is formed by six planes (indicated by dashed lines). Bottom: the horizontal auxiliary plane with the three pairs of lines arising from the intersection of the three pairs of planes with the auxiliary plane. This is a generalization of Figs. 1 and 2: If we consider e.g. the point $(0,1,0)$ to correspond to the point A of Fig. 1, then the inner triangle in the auxiliary plane corresponds to the left shaded triangle of Fig. 1 and to the left part of Fig. 2. So the initial external spin (Weyl) variables (44) can be considered to sit at the three times four intersection points of the auxiliary plane. In order to get a similar picture for the final external variables we have to place the auxiliary plane above the cube.

where

$$\mathbf{I_n} \;=\; \sum_{m_1=0}^{n_1-1} \mathbf{I}_{X_{m_1}}^{X'_{m_1}} + \sum_{m_2=0}^{n_2-1} \mathbf{I}_{Y_{m_2}}^{Y'_{m_2}} + \sum_{m_3=0}^{n_3-1} \mathbf{I}_{Z_{m_3}}^{Z'_{m_3}} \,. \tag{48}$$

Now eqs. (43) and (47) define the matrix function

$$\mathfrak{R}_{123}(\mathbf{v}) \;=\; \mathfrak{R}(\mathbf{v}; X', X; Y', Y; Z', Z) \tag{49}$$

where X', X, Y', Y, Z', Z stand for the ordered lists of divisors,

$$X = (X_0, X_1, ..., X_{N_1-1}), \;\; X' = (X'_0, X'_1, ..., X'_{N_1-1}), \;\; Y = (Y_0, Y_1, \ldots), \;\; \text{etc.} \tag{50}$$

As to the index structure, recall (9),

$$\mathfrak{R}_{123} \;\in\; \mathrm{End}\left(\mathcal{V}^{N_2N_3} \otimes \mathcal{V}^{N_1N_3} \otimes \mathcal{V}^{N_1N_2}\right) \;, \tag{51}$$

where in the same way as before we enumerate the number of $\mathcal{V}^{N_jN_k}$ in the tensor product (e.g (43) are the matrix elements of $\mathfrak{R}_{123}$). In Fig. 5 we show the intersection lines of

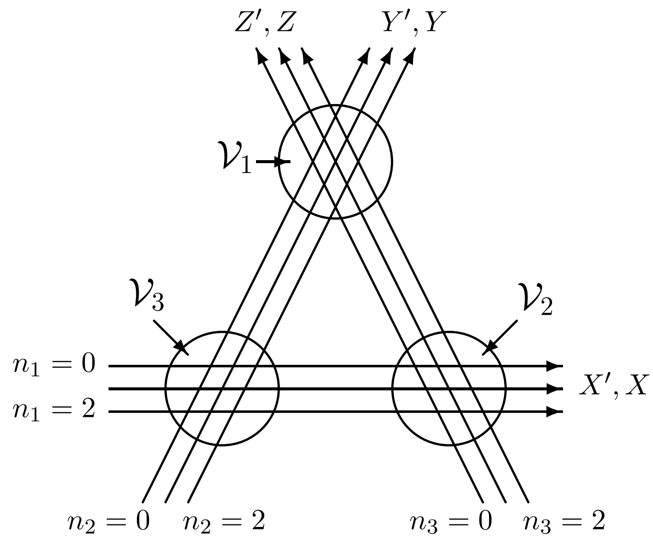

Figure 5: Ordering of the indices of the matrix $\mathfrak{R}_{123}$, shown for the case $N_1 = N_2 = N_3 = 3$ by drawing the auxiliary triangle in the auxiliary plane, as in the bottom part of Fig. 4.

the planes of a $N_1 \times N_2 \times N_3$ cube which appear in an auxiliary plane (as in the bottom part of Fig. 4), which intersects the "initial" edges corresponding to (44). On the section the $N_1 + N_2 + N_3$ planes become lines, and the edges of the cubic lattice become the intersection points of $N_1 N_2 + N_2 N_3 + N_1 N_3$ lines in the auxiliary plane. The intersection points are gathered into three sets $\mathcal{V}_1 = \mathcal{V}^{N_2 N_3}$ etc., and the index of $\mathcal{V}_j$ is the number of the corresponding $\mathcal{V}^{N_k N_l}$ in the tensor product in (51). Fig. 5 is helpful to arrange the numbering in (48) and the correct assignment of X'_{n_1}, X_{n_1}, etc.

3.2 The Modified Tetrahedron equation for the fused weights

For writing the MTE, apart from the three pairs of sets $X', X;\ Y', Y;\ Z', Z$ of (49), (50) we need a fourth pair

$$U \;=\; (U_0,\, U_1,\, ...,U_{N_0-1}\,); \qquad U' \;=\; (U'_0,\, U'_1,\, ...,U'_{N_0-1}\,).$$

Applying the definition (43), in addition to (49) we construct the matrices

$$\mathfrak{R}_{145}(\mathbf{v}) = \mathfrak{R}(\mathbf{v}; U', U; Y', Y; Z', Z)\,; \qquad \mathfrak{R}_{246}(\mathbf{v}) = \mathfrak{R}(\mathbf{v}; U', U; X', X; Z', Z);$$

$$\mathfrak{R}_{356}(\mathbf{v}) = \mathfrak{R}(\mathbf{v}; U', U; X', X; Y', Y).$$

Their index structure is defined by

$$\begin{array}{lll} \mathcal{V}_1 = \mathcal{V}^{N_2 N_3}\,, & \mathcal{V}_2 = \mathcal{V}^{N_3 N_1}\,, & \mathcal{V}_3 = \mathcal{V}^{N_1 N_2}\,, \\ \mathcal{V}_4 = \mathcal{V}^{N_0 N_3}\,, & \mathcal{V}_5 = \mathcal{V}^{N_0 N_2}\,, & \mathcal{V}_6 = \mathcal{V}^{N_0 N_1}\,. \end{array} \tag{52}$$

so that e.g. $\mathfrak{R}_{145}$ is acting in a space of dimension $N^{N_2 N_3 + N_0 N_3 + N_0 N_2}$. For $\mathfrak{R}_{145}$ in analogy to the definitions (47),(48) one uses

$$\mathbf{R}^{(145)}_{\mathbf{n}} \;=\; \mathbf{R}(\mathbf{v} + \mathbf{I}_{\mathbf{n}}; U'_{n_0}, U_{n_0}; Y'_{n_2}, Y_{n_2}; Z'_{n_3}, Z_{n_3}) \tag{53}$$

with

$$\mathbf{I_n} \ = \ \sum_{m_0=0}^{n_0-1} \mathbf{I}_{U_{m_0}}^{U'_{m_0}} + \sum_{m_2=0}^{n_2-1} \mathbf{I}_{Y_{m_2}}^{Y'_{m_2}} + \sum_{m_3=0}^{n_3-1} \mathbf{I}_{Z_{m_3}}^{Z'_{m_3}} , \tag{54}$$

similarly for $\mathfrak{R}_{246}$ and $\mathfrak{R}_{356}$.

Theorem 2 *The matrices $\mathfrak{R}$ defined in (43) obey the modified tetrahedron equation*

$$\begin{aligned} &\mathfrak{R}_{123}(\mathbf{v})\, \mathfrak{R}_{145}(\mathbf{v}+\mathbf{I}_X)\, \mathfrak{R}_{246}(\mathbf{v})\, \mathfrak{R}_{356}(\mathbf{v}+\mathbf{I}_Z) \ = \\ &\qquad \rho\, \mathfrak{R}_{356}(\mathbf{v})\, \mathfrak{R}_{246}(\mathbf{v}+\mathbf{I}_Y)\, \mathfrak{R}_{145}(\mathbf{v})\, \mathfrak{R}_{123}(\mathbf{v}+\mathbf{I}_U) \end{aligned} \tag{55}$$

where

$$\begin{aligned} \mathbf{I}_U \ &= \ \textstyle\sum_{n_0=0}^{N_0-1} \mathbf{I}_{U_{n_0}}^{U'_{n_0}} \ ; \qquad & \mathbf{I}_X \ &= \ \textstyle\sum_{n_1=0}^{N_1-1} \mathbf{I}_{X_{n_1}}^{X'_{n_1}} \ ; \\ \mathbf{I}_Y \ &= \ \textstyle\sum_{n_2=0}^{N_2-1} \mathbf{I}_{Y_{n_2}}^{Y'_{n_2}} \ ; \qquad & \mathbf{I}_Z \ &= \ \textstyle\sum_{n_3=0}^{N_3-1} \mathbf{I}_{Z_{n_3}}^{Z'_{n_3}} \ . \end{aligned} \tag{56}$$

Proof: The main content of this theorem is the appearance of the specific set of shifts (48), (56). For the proof it is convenient to introduce some compact notations. Instead of using the number labels for the $\mathfrak{R}$ we shall the labels $\mathsf{U}, \mathsf{X}, \mathsf{Y}, \mathsf{Z}$ just as these were introduced in (36) and (42) for the single vertex matrices $\mathbf{R}$. So, for the box $\mathfrak{R}$-matrix and its sets of divisors we write,

$$\mathfrak{R}_{123}(\mathbf{v}) \implies \mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v}); \qquad \mathfrak{R}_{145}(\mathbf{v}) \implies \mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}); \qquad \textit{etc.} \tag{57}$$

In this short notation , formulas (43), (47) imply the definition

$$\mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v}) \ = \prod_{n_1=0\uparrow N_1-1} \ \prod_{n_2=N_2-1\downarrow 0} \ \prod_{n_3=0\uparrow N_3-1} \mathbf{R}^{X_{n_1} Y_{n_2} Z_{n_3}}(\mathbf{v}+\mathbf{I_n}) \tag{58}$$

where we use ordered products

$$\prod_{n_1=0\uparrow N_1-1} \mathfrak{f}_{n_1} \ = \ \mathfrak{f}_0\mathfrak{f}_1\cdots\mathfrak{f}_{N_1-1} \ , \qquad \prod_{n_2=N_2-1\downarrow 0} \mathfrak{f}_{n_2} \ = \ \mathfrak{f}_{N_2-1}\cdots\mathfrak{f}_1\mathfrak{f}_0 \ . \tag{59}$$

$\mathbf{I_n}$ for the triple (X,Y,Z) is given by (48). Each $\mathbf{R}^{\mathsf{X}_{n_1}\mathsf{Y}_{n_2}\mathsf{Z}_{n_3}}$ acts non-trivially just in three of all the spaces (52) and the order is relevant just for neighboring indices X_{n_1} or Y_{n_2} or Z_{n_3}. The analogous modifications required to get the other matrices $\mathfrak{R}_{145}$, etc. should be evident.

Now we turn to the proof of the theorem which will be by mathematical induction. The theorem claims the validity of the MTE for arbitrary N_0, N_1, N_2, N_3. For the initial point $N_0 = N_1 = N_2 = N_3 = 1$ the MTE holds because it is just (42). Then to prove the theorem, we reduce the MTE (55) for some N_j to MTEs with $N'_j \leq N_j$. Thus one has four similar steps of the induction. Here we consider the induction step for the X-direction, the other steps follow analogously.

We split the list X into two sublists of length $N_1^{(1)}$ and $N_1^{(2)} = N_1 - N_1^{(1)}$:

$$X^{(1)} = (X_0, X_1, ..., X_{N_1^{(1)}-1}) \ , \quad X^{(2)} = (X_{N_1^{(1)}}, ..., X_{N_1-1}) \ , \tag{60}$$

so that $\mathbf{I}_{X^{(1)}} = \sum_{n_1=0}^{N_1^{(1)}-1} \mathbf{I}_{X_{n_1}}$ and $\mathbf{I}_{X^{(2)}} = \mathbf{I}_X - \mathbf{I}_{X^{(1)}}$. According to this splitting and due to the definition (58)

$$
\begin{aligned}
\mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v}) &= \mathfrak{R}^{\mathsf{X^{(1)}YZ}}(\mathbf{v})\ \mathfrak{R}^{\mathsf{X^{(2)}YZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\,, \\
\mathfrak{R}^{\mathsf{UXZ}}(\mathbf{v}) &= \mathfrak{R}^{\mathsf{UX^{(2)}Z}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\,\mathfrak{R}^{\mathsf{UX^{(1)}Z}}(\mathbf{v})\,, \\
\mathfrak{R}^{\mathsf{UXY}}(\mathbf{v}) &= \mathfrak{R}^{\mathsf{UX^{(2)}Y}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\ \mathfrak{R}^{\mathsf{UX^{(1)}Y}}(\mathbf{v})\,.
\end{aligned}
\tag{61}
$$

The meaning of notations $\mathsf{X}^{(1)}$ and $\mathsf{X}^{(2)}$ used in (61) should be evident from (58). Observe also that because of the reverse numbering with respect to the middle superscript of $\mathfrak{R}$ in (58), the factors in the latter two equations appear in reverse order. Since $\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v})$ contains no X, neither as subscript nor in the argument, it will not be split. However, we have to put $\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_X) = \mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_{X^{(2)}})$.

Now substituting (61) to the LHS of (55) written in our new superscript notation, and abbreviating $\mathbf{v}_1 \equiv \mathbf{v}+I_{X^{(1)}}$ we get

$$
\begin{aligned}
&\mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_X)\,\mathfrak{R}^{\mathsf{UXZ}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{UXY}}(\mathbf{v}+\mathbf{I}_Z) \\
&= \mathfrak{R}^{\mathsf{X^{(1)}YZ}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{X^{(2)}YZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\,\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_{X^{(2)}})\,\mathfrak{R}^{\mathsf{UX^{(2)}Z}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\times \\
&\qquad\times\mathfrak{R}^{\mathsf{UX^{(1)}Z}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{UX^{(2)}Y}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_Z)\,\mathfrak{R}^{\mathsf{UX^{(1)}Y}}(\mathbf{v}+\mathbf{I}_Z) \\
&= \mathfrak{R}^{\mathsf{X^{(1)}YZ}}(\mathbf{v})\ \Big[\mathfrak{R}^{\mathsf{X^{(2)}YZ}}(\mathbf{v}_1)\,\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}_1+\mathbf{I}_{X^{(2)}})\,\mathfrak{R}^{\mathsf{UX^{(2)}Z}}(\mathbf{v}_1)\,\mathfrak{R}^{\mathsf{UX^{(2)}Y}}(\mathbf{v}_1+\mathbf{I}_Z)\Big]\times \\
&\qquad\times\mathfrak{R}^{\mathsf{UX^{(1)}Z}}(\mathbf{v})\mathfrak{R}^{\mathsf{UX^{(1)}Y}}(\mathbf{v}+\mathbf{I}_Z) \\
&= \mathfrak{R}^{\mathsf{X^{(1)}YZ}}(\mathbf{v})\ \Big[\mathfrak{R}^{\mathsf{UX^{(2)}Y}}(\mathbf{v}_1)\ \mathfrak{R}^{\mathsf{UX^{(2)}Z}}(\mathbf{v}_1+\mathbf{I}_Y)\ \mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}_1)\ \mathfrak{R}^{\mathsf{X^{(2)}YZ}}(\mathbf{v}_1+\mathbf{I}_U)\Big]\times \\
&\qquad\times\mathfrak{R}^{\mathsf{UX^{(1)}Z}}(\mathbf{v})\mathfrak{R}^{\mathsf{UX^{(1)}Y}}(\mathbf{v}+\mathbf{I}_Z) \\
&= \mathfrak{R}^{\mathsf{UX^{(2)}Y}}(\mathbf{v}_1)\,\mathfrak{R}^{\mathsf{UX^{(2)}Z}}(\mathbf{v}_1+\mathbf{I}_Y)\times \\
&\qquad\times\Big[\mathfrak{R}^{\mathsf{X^{(1)}YZ}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v}+X^{(1)})\,\mathfrak{R}^{\mathsf{UX^{(1)}Z}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{UX^{(1)}Y}}(\mathbf{v}+\mathbf{I}_Z)\Big]\,\mathfrak{R}^{\mathsf{X^{(2)}YZ}}(\mathbf{v}_1+\mathbf{I}_U) \\
&= \mathfrak{R}^{\mathsf{UX^{(2)}Y}}(\mathbf{v}_1)\,\mathfrak{R}^{\mathsf{UX^{(2)}Z}}(\mathbf{v}_1+\mathbf{I}_Y)\times \\
&\qquad\times\Big[\mathfrak{R}^{\mathsf{UX^{(1)}Y}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{UX^{(1)}Z}}(\mathbf{v}+\mathbf{I}_Y)\,\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{X^{(1)}YZ}}(\mathbf{v}+\mathbf{I}_U)\Big]\ \mathfrak{R}^{\mathsf{X^{(2)}YZ}}(\mathbf{v}_1+\mathbf{I}_U) \\
&= \mathfrak{R}^{\mathsf{UX^{(2)}Y}}(\mathbf{v}+\mathbf{I}_{X^{(1)}})\,\mathfrak{R}^{\mathsf{UX^{(1)}Y}}(\mathbf{v})\ \mathfrak{R}^{\mathsf{UX^{(2)}Z}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_Y)\,\mathfrak{R}^{\mathsf{UX^{(1)}Z}}(\mathbf{v}+\mathbf{I}_Y)\times \\
&\qquad\times\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{X^{(1)}YZ}}(\mathbf{v}+\mathbf{I}_U)\,\mathfrak{R}^{\mathsf{X^{(2)}YZ}}(\mathbf{v}+\mathbf{I}_{X^{(1)}}+\mathbf{I}_U) \\
&= \mathfrak{R}^{\mathsf{UXY}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{UXZ}}(\mathbf{v}+\mathbf{I}_Y)\,\mathfrak{R}^{\mathsf{UYZ}}(\mathbf{v})\,\mathfrak{R}^{\mathsf{XYZ}}(\mathbf{v}+\mathbf{I}_U)\,.
\end{aligned}
\tag{62}
$$

From the third to the fourth line of (62) within the inserted brackets we used the MTE for the smaller set $(U, X^{(2)}, Y, Z)$. In order to be able to isolate the terms containing $X^{(2)}$ the order of factors in the second line of (61), which was used in the first step, is crucial. Going from the fifth to the sixth line in the brackets we used the MTE for $(U, X^{(1)}, Y, Z)$. In the other steps of (62) we just commuted or combined various terms. The last step is made possible by the "reverse" order of factors in the last line of (61). □

4 Special cases: Solving the tetrahedron equation

4.1 Compact algebraic curve

Let now $N_0 = N_1 = N_2 = N_3 = M$. Starting from the generic parameterization of the MTE (55), the usual tetrahedron equation is obtained if

$$\mathfrak{R}(\mathbf{v}) \equiv \mathfrak{R}(\mathbf{v}+\mathbf{I}) \qquad \text{and} \qquad \rho = 1 . \tag{63}$$

This is the case when

$$\mathbf{I}_U = \mathbf{I}_X = \mathbf{I}_Y = \mathbf{I}_Z = 0 \qquad \text{on} \qquad \mathrm{Jac}(\Gamma_g) , \tag{64}$$

and the ratios of θ-functions (25) are periodical.

According to Abel's theorem, equation (64) means that there are four meromorphic functions u, x, y, z on Γ_g with the divisors

$$(u) = \sum_{n_0=0}^{M-1} U'_{n_0} - U_{n_0}, \quad (x) = \sum_{n_1=0}^{M-1} X'_{n_1} - X_{n_1}, \quad (y) = \sum_{n_2=0}^{M-1} Y'_{n_2} - Y_{n_2}, \quad (z) = \sum_{n_3=0}^{M-1} Z'_{n_3} - Z_{n_3}. \tag{65}$$

As it is well known (see e.g. Theorem 10-23 of [20]), conditions (64) are a strong restriction for the type of Γ_g: Γ_g is the algebraic curve given by the polynomial equation

$$P(x,y) \stackrel{\text{def}}{=} \sum_{a,b=0}^{M} x^a \, y^b \, p_{a,b} = 0 . \tag{66}$$

The choice of any pair of x, y, z, u produces an equivalent polynomial equation. The form of the polynomial $P(x,y)$ provides the restriction for the genus,

$$g \le (M-1)^2 . \tag{67}$$

Thus we come to

Theorem 3 *Let Γ_g be the compact algebraic curve defined by polynomial equation (66). Let four sets of U'_{n_0}, U_{n_0}, X'_{n_1}, X_{n_1}, Y'_{n_2}, Y_{n_2} and Z'_{n_3}, Z_{n_3}, $n_k = 0, ..., M-1$, are the divisors of four meromorphic functions u, x, y, z on Γ_g. Then the tetrahedron equation is satisfied*

$$\begin{aligned} &\mathfrak{R}_{123}(x,y,z)\, \mathfrak{R}_{145}(u,y,z)\, \mathfrak{R}_{246}(u,x,z)\, \mathfrak{R}_{356}(u,x,y) = \\ &\qquad \mathfrak{R}_{356}(u,x,y)\, \mathfrak{R}_{246}(u,x,z)\, \mathfrak{R}_{145}(u,y,z)\, \mathfrak{R}_{123}(x,y,z) , \end{aligned} \tag{68}$$

where four matrices are the same matrix function of different arguments,

$$\mathfrak{R}(x,y,z) = \mathfrak{R}(\mathbf{v}; X', X; Y', Y; Z', Z) \quad \textit{etc.} \tag{69}$$

defined via (43),(47),(49) and (65).

According to the conventional terminology, one may say that u, x, y, z are the spectral parameters, the moduli of Γ_g are the moduli of the tetrahedron equation, and vector $\mathbf{v}$ is a kind of deformation parameter.

Theorem 3 may be also be proved differently, without mentioning the simple MTE. In this alternative approach one considers the auxiliary linear problem for the whole box and the corresponding mappings. See e.g. [21, 22] for the description of this method and [23] for the parameterization of the classical equations of motion. Remarkably, in this alternative way the spectral curve (66) appears naturally from the linear problem.

4.2 Simple tetrahedron equation for ZBB-case recovered

As it was mentioned, (7) is the **R**-matrix for the Zamolodchikov-Bazhanov-Baxter model. Our scheme contains this case. Formally ZBB's tetrahedron equation corresponds to (68) with $M = 1$. It gives $g = 0$, i.e. the spectral curve is the sphere with

$$E(X,Y) \;=\; \frac{X-Y}{\sqrt{dX\,dY}}\;, \quad \Theta() \equiv 1\;. \tag{70}$$

Here the formal theta-function has no argument since the Jacobian is 0-dimensional. Conditions (64) therefore are out of use, and the parameterization (30)(31) contains the N-th roots of the cross-ratios like $\dfrac{(X-Z)(X'-Z')}{(X'-Z)(X-Z')}$. One may show, the number of independent cross-ratios is the number of variables $X, X', \ldots$ *minus* three. Therefore the single **R**-matrix contains $6-3=3$ independent complex parameters (as it should be), and the simple tetrahedron equation contains $8-3=5$ independent complex parameters (again as it should be). It means that the tetrahedral condition, which appears in Zamolodchikov's parameterization of **R** in the terms of spherical triangles, is taken into account automatically. Moreover, the parameterization with the help of the cross-ratios takes automatically into account the geometric structure of any set of the planes in three dimensional Euclidean space. The parameterization of the inhomogeneous Zamolodchikov-Bazhanov-Baxter model in the terms of cross-ratios corresponding to the $g=0$ limit of (30),(31)(47) was already used in [24].

4.3 Chessboard model

Previously derived "chessboard models" of the lattice statistical mechanics based on the modified tetrahedron equation [10, 11] are of course related to our considerations. The term "chessboard" appeared as the visual interpretation of the cubic lattice with $M = 2$ to be homogeneous. It means that the cubic lattice consists of eight different types of vertices (i.e. eight different types of the Boltzmann weights) – a kind of three dimensional analogue of the chess board with eight different colors of the cells.

The models described in [10, 11] are at the first the so-called IRC-type models, but with the help of vertex-IRC correspondence [8] one may construct their vertex reformulation. Thus the model implicitly described in [11] is equivalent to $M = 2$, $g = 1$ of our scheme. For $g = 1$ the curve and its Jacobian are isomorphic, so that without loss of generality one may chose

$$\mathbf{I}_Y^X \;= \mathtt{X} - \mathtt{Y} \,\in\, \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau\;. \tag{71}$$

Further, one may use θ_1

$$\Theta(v) \;=\; \theta_1(v,\tau) \;\equiv\; \sum_{n=-\infty}^{\infty} e^{i\pi\tau(n+1/2)^2+2i\pi(v+1/2)(n+1/2)} \tag{72}$$

as basic theta-function, and $E(X,Y) \sim \theta_1(\mathtt{X}-\mathtt{Y})$ as the prime form. This formulas simplify the definitions (25). Periodicity conditions (65) may be chosen

$$\mathtt{X}_0' - \mathtt{X}_0 + \mathtt{X}_1' - \mathtt{X}_1 = \mathtt{Y}_0' - \mathtt{Y}_0 + \mathtt{Y}_1' - \mathtt{Y}_1 = \mathtt{Z}_0' - \mathtt{Z}_0 + \mathtt{Z}_1' - \mathtt{Z}_1 = \mathtt{U}_0' - \mathtt{U}_0 + \mathtt{U}_1' - \mathtt{U}_1 = 1. \tag{73}$$

Note, 1 in the right hand side of (73) is equivalent to τ (due to the Jacobi transform), while 0 instead of 1 in the right hand side of (73) gives a trivial model.

The model explicitly described before in [10] corresponds to $M=2$, $g=1$, $\mathtt{X}_0=\mathtt{X}_1$, $\mathtt{X}_0'=\mathtt{X}_1'$, $\mathtt{Y}_0=\mathtt{Y}_1$, $\mathtt{Y}_0'=\mathtt{Y}_1'$, $\mathtt{Z}_0=\mathtt{Z}_1$, $\mathtt{Z}_0'=\mathtt{Z}_1'$ etc. with the condition (73). This choice makes the identification of the parameters in (43)

$$\mathbf{R_n}=\mathbf{R_{n+e_1+e_2}}=\mathbf{R_{n+e_1+e_3}}=\mathbf{R_{n+e_2+e_3}}\;, \tag{74}$$

so the cells of this three dimensional chess board have only two "colors".

Note that the vertex-IRC duality is not exact because it changes the boundary conditions.

5 Conclusions

We considered a large class of integrable 3-D lattice models which have Weyl variables at *N-th* root of unity as dynamic variables. We have shown how the Boltzmann weights can be conveniently parameterized in terms of *N-th* roots of theta functions on a Jacobian of a compact Riemann surface. The Fermat relations of the points determining the Boltzmann weights are simple Fay identities and the classical equations determining the scalar parameters are a consequence of a double Fay identity. In the modified tetrahedron equation we have four pairs of arbitrary points on the Riemann surface in simple permuted combinations.

This parameterization allows a compact formulation of the rules to form fused Boltzmann weights $\mathfrak{R}\in\mathrm{End}\,\mathbb{C}^{3NM^2}$ which are the partition functions of an open boxes of arbitrary size. The $\mathfrak{R}$ obey the modified tetrahedron equation and are again parameterized terms on N-th roots of theta-functions on the Jacobian of a genus $g=(M-1)^2$ compact Riemann surface Γ_g. The spectral parameters of the vertex weight $\mathfrak{R}$ are three meromorphic functions on the spectral curve Γ_g. For the case that the Jacobi transforms become trivial the $\mathfrak{R}$ obey the simple tetrahedron equation. The Zamolodchikov-Baxter-Bazhanov model and the Chessboard model are obtained as special cases.

So, the scheme discussed here contains and generalizes many known 3-D integrable models. The hope is that the framework is now sufficiently general to contain physically interesting models with a non-trivial phase structure. However, to get information on partition functions, either analytically or approximately, is still a very difficult open problem. There is no way known to generalize Baxter's special method [26] by which he obtained the partition function of the ZBB-model.

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References

[1] A. B. Zamolodchikov, Commun. Math. Phys. **79** (1981) 489-505

[2] R. J. Baxter, Commun. Math. Phys. **88** (1983) 185-205

[3] V. V. Bazhanov and R. J. Baxter, J. Stat. Phys. **69** (1992) 453; **71** (1993) 839

[4] V. Mangazeev, R. Kashaev and Yu. Stroganov, Int. J. Mod. Phys. **A8** (1993) 1399-1409; Int. J. Mod. Phys. **A8** (1993) 587-601

[5] S. M. Sergeev, V. V. Mangazeev and Yu. G. Stroganov, J. Stat. Phys. **82** (1996) 31

[6] I. G. Korepanov, Comm. Math. Phys. **154** (1993) 85–97.

[7] J. Hietarinta, J. Phys. A: Math. Gen. **27** (1994) 5727–5748

[8] H. E. Boos, V. V. Mangazeev, S. M. Sergeev and Yu. G. Stroganov, Mod. Phys. Let. **A11** (1996) 491-498

[9] G. von Gehlen, S. Pakuliak and S. Sergeev, J. Phys. A: Math. Gen. **36** (2003) 975

[10] V. V. Mangazeev and Yu. G. Stroganov, Mod. Phys. Lett. **A8** (1993) 3475-3482; V. V. Mangazeev, S. M. Sergeev and Yu. G. Stroganov, Int. J. Mod. Phys. **A9** (1994) 5517-5530.

[11] H. E. Boos, V. V. Mangazeev, S. M. Sergeev, Int. J. Mod. Phys. A **10** (1995) 4041

[12] J.-M. Maillard and S. M. Sergeev, Phys. Lett. **B405** (1997) 55

[13] D. Mumford, "Tata lectures of Theta I,II", Boston-Basel-Stuttgart, Birkhäuser, 1983, 1985

[14] I. Krichever, Russian Math. Survey **33** (1978) 215

[15] T. Shitota, Inventione mathematicae **83** (1986) 333

[16] M. Mulase, J. Diff. Geom. **19** (1984) 403

[17] I. Krichever, P. Wiegmann and A. Zabrodin, Commun. Math. Phys. **193** (1998) 373

[18] S. M. Sergeev, J. Nonlin. Math. Phys. **7** (2000) 57

[19] R. J. Baxter and P. J. Forrester, J. Phys. A: Math. Gen. **18** (1985) 1483

[20] G. Springer, "Introduction to Riemann Surfaces", Chelsea Pub. Co. New-York, sc. ed., 1981

[21] S. M. Sergeev, J. Phys. A: Math. Gen. **32** (1999) 5639

[22] S. M. Sergeev, "Integrable three dimensional models in wholly discrete space-time", in *"Integrable Structures of Exactly Solvable Two-Dimensional Models etc."*, eds. S. Pakuliak and G. von Gehlen, NATO Science Series II **35**, Kluwer, Dordrecht (2001), p. 293-304

[23] S. Pakuliak and S. Sergeev, “Spectral curves and parameterizations of a discrete integrable 3-dimensional model”, Preprint MPI 02-85 (2002), nlin.SI/0209019

[24] S. Sergeev, “Functional equations and quantum separation of variables for 3D spin models”, Preprint MPI 02-46 (2002)

[25] G. von Gehlen, S. Pakuliak and S. Sergeev, “The modified tetrahedron equation and its solutions“, preprint nlin.SI/0303043

[26] R. J. Baxter, Physica **18D** (1986) 321