A Frequency-domain Robust Instability Criterion for Time-varying and Non-linear Systems*

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Abstract—This paper presents an instability version of an earlier result on robust stability analysis using a modified Nyquist plot. The result is shown to be useful for presenting robust stability and instability margins for feedback systems.

1. Introduction

Usable tools for stability analysis of nonlinear systems give sufficient conditions for certain stability properties. While generally not a desirable situation, it is often useful to determine conditions under which a system will be unstable. Therefore specific attention has been given to developing instability counterparts to the main stability results. In recent years, researchers in control system stability have been concerned with analysis of robust stability, that is, conditions when a feedback system is stable for limited uncertainty in the subsystems. In this paper, a robust instability result is given which relates to an earlier instability result (Tsypkin and Polyak, 1992).

Absolute stability criteria guarantee stability of the family of systems obtained by choosing the allowed feedback characteristics. These results provide tests for robust stability where the uncertainty lies in the controller gain element. One of the major classes of stability criteria is that referred to as Circle Criteria where the Nyquist locus $G(j\omega)$ should avoid and appropriately encircle a disc defined by the gain bound.

Recent results have considered uncertainty in the linear part. Replacing the usual Nyquist locus by bands whose width reflect the uncertainty, Nyquist and Circle type criteria can be stated. The results by Tsypkin and Polyak (1992) are significant in that they use scaling to reformulate the stability criteria as requiring a modified Nyquist locus to encircle a disk defined by the uncertainty bound.

The first frequency domain instability condition for the absolute stability problem is due to Brockett and Lee (1967); they use Lyapunov methods to obtain Circle type criteria. Hill and Moylan (1978, 1983) developed a general theory for deriving input–output and corresponding Lyapunov instability criteria; this allows inter alia the flexible design of refined criteria to suit particular situations. Other references on instability are given in Hill and Moylan (1983).

The result presented here considers the situation where there is unstructured uncertainty in the linear part $G(s)$. An instability counterpart of the robust absolute stability result by Polyak and Tsypkin (1992) is derived.

2. A circle instability criterion

We consider the system shown in Fig. 1 and present an input–output version of the Brockett and Lee (1967) Circle Instability Criterion.

Firstly note that the feedback system is unstable if there exists some input $r \in \mathbb{L}_2$ which produces an output $y \notin \mathbb{L}_2$. Assume that $G(s)$ is a strictly proper rational function which has $p$ poles in the open half-plane $\text{Re} \{s\} > 0$. Also assume the feedback characteristic satisfies

$$0 < \kappa \leq \frac{f(\omega)}{\sigma} \leq \bar{k} \quad \forall \omega \in \mathbb{R}_+; \quad \forall \sigma \in \mathbb{R}.$$  

(1)

It is convenient to define $D(a,b)$ as the open disk in $\mathbb{R}^2$ whose diameter is the line segment joining the points $(a,0)$ and $(b,0)$. The closure of this disk is denoted $\overline{D}(a,b)$.

Theorem 1. Suppose that the Nyquist locus $G(j\omega)$ does not intersect disk $D(-\kappa^{-1}, -\kappa^{-1})$ and encircles it fewer than $p$ times in the counter-clockwise sense. Then the feedback system is input–output unstable.

This result can be obtained directly from a more general result in Hill and Moylan (1983); it is a refinement of a similar result presented in Desoer and Vidyasagar (1975). By adding some minor technical assumptions and using a Kalman–Yakubovich type lemma, Theorem 1 can be translated into statements about Lyapunov instability (Hill and Moylan, 1983). These are the criteria presented by Brockett and Lee (1967).

The encirclement condition is stated in terms of the closed Nyquist locus as $\omega$ goes from $-\infty$ to $\infty$. It can be restated for $p/2$ encirclements as $\omega$ goes from 0 to $\infty$.

The stability counterpart of Theorem 1 requires that the Nyquist locus does not intersect the closed disk $D[-\kappa^{-1}, -\kappa^{-1}]$ and encircles it exactly $p$ times—see Desoer and Vidyasagar (1975).

3. The robust absolute instability problem

Assume now that the rational transfer function $G(s)$ is not completely known. Rather a nominal model $G_1(s)$ is known. The true plant is $G(s) = G_1(s) + \Delta G(s)$ where the uncertainty $\Delta G(s)$ satisfies

$$|\Delta G(j\omega)| \leq \gamma \sigma(\omega), \quad \forall \omega \in \mathbb{R}_+.$$  

(2)

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\[ y > 0 \] represents an uncertainty margin and \( r(\omega) \geq 0 \) is a given bounded scalar function. We assume that \( G^0(s) \) and \( G(s) \) have the same number of unstable poles.

The nonlinear feedback function is still described by the sector condition (1). However, now it is convenient to parameterize the bounds in terms of \( \gamma \) according to

\[ \gamma = k(1 + c\gamma)^{-1}; \quad \tilde{\gamma} = k(1 - c\gamma)^{-1}, \]

where \( c, k \) are positive constants. For fixed \( c, \gamma \) increasing \( \gamma \) expands the disk \( D(-k^{-1}, k^{-1}) \) while keeping its centre fixed at \(-1/k\). The radius is \( c/\gamma k \). This expansion corresponds to a larger sector uncertainty in (1).

The problem is to construct a criterion which guarantees instability of the family of systems defined by Fig. 1 and the uncertainty bounds (1) and (2) for a fixed \( \gamma \). Further, it is of interest to find the largest \( \gamma \), i.e. \( \gamma_{\text{max}} \), which preserves instability.

Note that if \( r(\omega) = 0 \) and \( \gamma \) is fixed, we arrive at (an input–output version of) the absolute instability problem considered by Brockett and Lee (1967). If \( c = 0 \), then \( k = k' \) and function \( f(\sigma, t) \) is linear time-invariant; the problem reduces to that of robust instability of linear systems.

The above problem formulation allows simultaneous consideration of linear plant uncertainty and nonlinear controller uncertainty via a parameter \( \gamma \). There are of course different versions of this problem such as when the nonlinear sector bounds \( k, k' \) are fixed and the largest \( \gamma \) in (2) must be found. These will be discussed later.

Finally, we make a comment on the practicality of the problem. In some applications, like oscillator design, instability is the desired property. It is required that this property be preserved in the face of linear dynamical and nonlinear uncertainty. Generally, of course, instability is undesirable; nevertheless, it is useful in design to know how fragile is any assessment of instability or stability when uncertainty is allowed for.

4. Robust circle instability criterion

The disk in Theorem 1 represents the nonlinear uncertainty and must be avoided by the Nyquist locus. With \( \rho \) (less than \( \rho_0 \)) counter-clockwise encirclements of the closed (open) disk, the system is stable (unstable). For \( r(\omega) = \gamma \) constant, it is possible to incorporate the linear uncertainty by appropriately enlarging the disk. In general, the circle becomes frequency dependent. In order to retrieve a fixed circle, Tsypkin and Polyak (1992) introduce a modified frequency response

\[ G(\omega) = \frac{G^0(\omega) + k^{-1}}{r(\omega) + ck^{-1}} - k^{-1} \]

for \( 0 \leq \omega \leq \infty \). Assume that the nominal linear system \( G^0(s) \) has \( \rho_0 \) poles in the open right-half plane.

Theorem 2. (Tsypkin and Polyak, 1992) If the modified frequency response \( G(\omega) \) does not intersect the closed (open) disk centred at \(-k^{-1} \), of radius \( \gamma \), and encircles it \( \rho_0/2 \) times in the counter-clockwise direction, then the class of systems (1), (2) with fixed \( \gamma \) is robust absolute stable.

For the standard absolute stability problem, \( r(\omega) = 0 \), this criterion is a constant scaled version of the standard circle criterion.

We proceed to prove an instability version of Theorem 2 by invoking Theorem 1.

Theorem 3. (Robust Circle Instability Criterion.) If the modified frequency response \( G(\omega) \) does not intersect the open disk \( D(-k^{-1} - \gamma, -k^{-1} + \gamma) \) and encircles it fewer than \( \rho_0/2 \) times in the counter-clockwise direction, then the class of systems (1), (2) with fixed \( \gamma \) is robust absolute unstable.

Proof. It is sufficient to prove that the given conditions imply that any system in the class defined by (1), (2) satisfies the Circle Instability Criterion of Theorem 1.

The non-intersection property for \( G(\omega) \) corresponds to

\[ \left| \frac{G^0(\omega) + k^{-1}}{r(\omega) + ck^{-1}} - k^{-1} \right| \geq \gamma \]

for \( 0 \leq \omega \leq \infty \), or

\[ |G^0(\omega) + k^{-1}| \geq \gamma (r(\omega) + ck^{-1}), \]

Hence for all admissible \( G(\omega) \) satisfying (1)

\[ |G(\omega) + k^{-1}| \geq |G^0(\omega) + k^{-1}| - |\Delta G(\omega)| \geq \gamma c k^{-1}, \]

that is, the frequency response \( G(\omega) \) does not intersect the open disk \( D(-k^{-1}, -k^{-1}) \).

It remains to check the encirclements of this disk. From (3), we see that \( G(\omega) \) and \( G^0(\omega) \) encircle the disks \( D(-k^{-1} - \gamma, -k^{-1} + \gamma) \) and \( D(-k^{-1}, -k^{-1}) \) the same number of times, respectively. Further, from (2) and (6), it is clear that \( G(\omega) \) and \( G^0(\omega) \) encircle the disk \( D(-k^{-1}, -k^{-1}) \) the same number of times. Thus the given conditions imply that the frequency response \( G(\omega) \), \( 0 \leq \omega \leq \infty \), does not intersect and encircles the disk \( D(-k^{-1}, -k^{-1}) \) fewer than \( \rho_0/2 \) times in the counter-clockwise direction.

Now, by assumption, \( G^0(\omega) \) and \( G(s) \) have the same number of right half-plane poles, i.e. \( \rho_0 \).

The result then follows from Theorem 1.

Remarks.

(1) Theorems 2 and 3 are clearly complementary: if \( \tilde{G}(\omega) \) does not intersect the closed (open) disk centred at \(-k^{-1} \), of radius \( \gamma \), and encircles it \( \rho_0/2 \) (fewer than \( \rho_0/2 \)) times, then the class of systems with fixed \( \gamma \) is input–output robust absolute stable (unstable).

(2) If \( \rho_0 = 0 \), the class of systems is robust unstable if the encirclements by \( G^0(\omega) \) are one or more in the clockwise direction. This is illustrated in Fig. 2.
5. Special situations

In this Section, we briefly consider some particular situations as special cases and extensions of Theorem 3.

5.1. Linear feedback. If \( c = 0 \), i.e. \( \hat{k} = k = k \), then \( f(\sigma, t) \) represents a constant linear gain: \( f(\sigma, t) = k\sigma \). In this case, we have

\[
\hat{G}(j\omega) = \frac{G^0(j\omega) + k^{-1}}{r(\omega)} - k^{-1}. \tag{7}
\]

The stability analysis reduces to that considered elsewhere for robust stability of linear feedback systems. Chen and Desoer (1982) effectively show that the criterion

\[
\left| \frac{G^0(j\omega) + k^{-1}}{r(\omega)} \right| > \gamma \tag{8}
\]

is necessary and sufficient for robust stability if the nominal feedback system is input-output stable. An instability counterpart of this result is not available.

Theorem 3 gives that if \( G^0(j\omega), 0 \leq \omega \leq \infty \), satisfies:

(i) \[
\left| \frac{G^0(j\omega) + k^{-1}}{r(\omega)} \right| \geq \gamma \tag{9}
\]

(ii) it encircles the disk \( D(-k^{-1} - \gamma, -k^{-1} + \gamma) \) fewer than \( \pi\omega/2 \) times; then the linear feedback system is robust unstable.

5.2. Fixed uncertainty bounds. By fixing \( \gamma \), the nonlinear uncertainty bounds \( \hat{k}, \hat{k} \) become fixed. In this case, it is interesting to assume \( r(\omega) = r \), constant and let \( \beta = \gamma r \). Then (2) can be written as

\[
\|\Delta G\|_{\infty} \leq \beta \tag{10}
\]

where \( \|\cdot\|_{\infty} \) denotes the \( L^\infty \)-norm. The modified frequency response becomes

\[
\hat{G}(j\omega) = \frac{G^0(j\omega) + k^{-1}}{\beta/\gamma + ck^{-1}} - k^{-1}.
\]

Let \( R = (k^{-1} - \bar{k}^{-1})/2 \) and select

\[
\gamma = R + \beta, \quad c = kR/\gamma.
\]

Then, \( \hat{G}(j\omega) \) coincides with \( G^0(j\omega) \). The robust absolute stability (instability) criterion states that \( G^0(j\omega) \) should not intersect the closed (open) disk centred at \(-k^{-1} + j0\) with radius \( R + \beta \) and encircle it \( \rho_0/2 \) (fewer than \( \rho_0/2 \)) times in the counterclockwise direction.

This criterion corresponds to merely enlarging the usual Circle Criteria disks by a margin \( \beta \) for the linear uncertainty.

November's illustration a robust unstable situation when \( \rho_0 = 0 \). While appealing for its simplicity, the conservative bound in (10) allows a less refined analysis that can be obtained from the general result.

For \( \beta = 0 \), the robust instability criterion of course reduces to the Circle Instability Criterion of Theorem 1.

5.3. Fixed nonlinearity bounds. The criteria in Theorems 2 and 3 enable computation of a maximal \( \gamma - \gamma_{\text{max}} \) for stability and instability. This robustness margin measures both the largest nonlinearity and linear uncertainty simultaneously which preserves the stability or instability property. This can be convenient when considering general robustness of a feedback loop.

In some cases, the particular nonlinearity of concern may be rather well-defined; thus the gain bounds \( \hat{k}, \hat{k} \) can be fixed. It remains to determine the largest linear uncertainty bound. This is easily found as \( \beta_{\text{max}} \) in the \( H^\infty \) bound case above, but is more difficult to find when using the frequency dependent bound. The modified frequency response can be rewritten

\[
\hat{G}(j\omega) = \frac{G^0(j\omega) + k^{-1}}{r(\omega) + R/\gamma} - k^{-1},
\]

where \( k, R \) are fixed. The disk and modified locus both depend on \( \gamma \). Determination of \( \gamma_{\text{max}} \) requires an iterative procedure.

6. Examples

Consider the class of systems defined by uncertainty bounds (1), (2). For a certain nominal linear system \( G^0(s) \) and feedback gain \( k \), the results above provide a means to calculate an overall margin of stability or instability as the case may be in the form of \( \gamma_{\text{max}} \). After choosing the uncertainty profile and sector parameter \( c \), \( \gamma \) represents the total uncertainty. In this, the full range of feedback gains can be classified as robust stable or unstable with a certain margin. Gains with small margins must be treated carefully since system behaviour is very sensitive to uncertainty.

Example. To illustrate the idea of the stability margin \( \gamma \) we will use as an example the system in Example 1 of Brockett and Lee (1967), i.e.

\[
G^0(s) = \frac{-s + 1}{(s + 1)^2}.
\]

Two different uncertainties are tried.
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8

FIG. 4. Maximum $\gamma$ for different values of $k$ for Example 1 ($c = 1/3$).

(A) For the first uncertainty, we assume that the actual plant has an unmodelled time constant. For example

$$G(s) = G^0(s) \frac{s}{s + a},$$

which means that

$$\Delta G(s) = -G^0(s) \frac{s}{s + a}.$$

Assume that $a = 10$ which yields

$$|\Delta G(j\omega)| \approx |G^0(j\omega)| \cdot \left|\frac{j\omega}{j\omega + 10}\right|.$$

(B) For the second uncertainty, we assume that the unmodelled dynamics corresponds to a resonant mode instead. Then

$$G(s) = G^0(s) \frac{\omega_0^2}{s^2 + 2\xi\omega_0s + \omega_0^2},$$

which in this case leads to

$$\Delta G(s) = -G^0(s) \frac{s^2 + 2\xi\omega_0s + \omega_0^2}{s^2 + 2\xi\omega_0s + \omega_0^2}.$$

In the plots we have used $\xi = 0.7$ and $\omega_0 = 10$, i.e.

$$|\Delta G(j\omega)| \approx |G^0(j\omega)| \cdot \left|\frac{-\omega_0^2 + 14j\omega}{-\omega_0^2 + 14j\omega + 100}\right|. $$

The maximum $\gamma$ guaranteeing stability or instability has been obtained for different values of $k$, using a fixed size of the nonlinear sector $c = 1/3$. A plot of $\gamma$ versus $k$ is given in Fig. 4 for the system without uncertainty and for the two different uncertainty bounds. If $\gamma < 1$ no statement can be made, and for those cases $\gamma$ in the plot has been rounded to 1. This ‘grey zone’ is obviously relatively insensitive to the uncertainty on the stability side. For uncertainty (A) the ‘grey zone’ increases significantly on the instability side, and for uncertainty (B) there are no values of $k$ guaranteeing an unstable system.

In Fig. 5 the modified curves for the three different cases are shown for $k = 5$.

7. Conclusions

This paper has considered instability analysis of feedback loops in the presence of nonlinear and linear dynamic uncertainty, through combining elements of absolute stability and robustness. The main result gives a criterion for robust absolute instability which is a generalization of the Circle Instability Criterion. The result is the counterpart of earlier stability results by Tsypkin and Polyak (1992).

The formulation provides a convenient overall robustness measure $\gamma_{\text{max}}$ by just plotting one frequency response. More refined analysis based on independent bounds can be done by iterative methods.

The main result can be generalized in several directions in parallel with stability results which follow Tsypkin and Polyak (1992). The use of Nyquist arrays for multi-input/multi-output systems has been studied by Tsypkin and
Isaksson (1991). Discrete-time results could also be derived. This paper has used an input–output framework. The results are easily translated into Lyapunov versions via Kalman–Yakubovich type lemmas (Hill and Moylan, 1983).

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References