Classes of graphs for which upper fractional domination equals independence, upper domination, and upper irredundance

Grant A. Cheston\textsuperscript{a,*}, Gerd Fricke\textsuperscript{b}

\textsuperscript{a}Department of Computational Science, University of Saskatchewan, Saskatoon, Sask., Canada S7N 0W0
\textsuperscript{b}Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, USA

Received 4 December 1990; revised 8 April 1993

Abstract

This paper investigates cases where one graph parameter, upper fractional domination, is equal to three others: independence, upper domination and upper irredundance. We show that they are all equal for a large subclass, known as strongly perfect graphs, of the class of perfect graphs. They are also equal for odd cycles and upper bound graphs. However for simplicial graphs, upper irredundance might not equal the others, which are all equal. Also for many subclasses of perfect graphs other than the strongly perfect class, independence, upper domination and upper irredundance are not necessarily equal. We also show that if the graph join operation is used to combine two graphs which have some of the parameters equal, the resulting graph will have the same parameters equal.

1. Introduction

There are a large number of graph parameters that are used to indicate various features of graph. In this paper, we investigate when four parameters that involve maximization are equal.

We will consider undirected graphs, $G = (V, E)$, with no multiple edges or self-loops. For $U \subseteq V$, we will denote by $\langle U \rangle$, the subgraph of $G$ induced by $U$. The open neighbourhood of a vertex $v$, $N(v)$, is the set of vertices adjacent to $v$. The closed neighbourhood of $v$ is given by $N[v] = N(v) \cup \{v\}$. These definitions can be expanded to sets as follows: for $U \subseteq V$

\[
N(U) = \bigcup_{v \in U} N(v),
\]
\[
N[U] = \bigcup_{v \in U} N[v].
\]

*Corresponding author.
For \( S \subseteq V \), \( S \) is an independent (also called stable) set if no two vertices in \( S \) are adjacent. The maximum cardinality of an independent set of a graph is called the independence number, \( \beta(G) \), of the graph. Also for \( S \subseteq V \) and a vertex \( u \in S \), \( u \) is said to have private neighbour \( w \) if (i) \( w \) is in the closed neighbourhood of \( u \), i.e. \( w \in N[u] \), and (ii) \( w \) is not in the neighbourhood of any other vertex in \( S \), i.e. \( w \notin N[S - \{ u \}] \). An irredundant set of vertices is a set \( S \) where every vertex \( v \) in \( S \) has a private neighbour. The upper irredundance number, \( IR(G) \), is the maximum cardinality of an irredundant set for graph \( G \).

A set \( S \subseteq V \) is a dominating set if \( N[S] = V \). A dominating set is minimal if no proper subset is dominating. The domination number, \( \gamma(G) \), is the minimum cardinality over all (minimal) dominating sets. The upper domination number, \( \Gamma(G) \), is the maximum cardinality over all minimal dominating sets.

In recent years, several authors have started investigations of fractional variations of graph parameters (e.g. \[1, 2, 8, 12, 15, 17, 20, 33\]). The fractional variation corresponding to domination has been defined \([6]\) using a function \( f \) from \( V \) to the closed real interval \([0, 1]\). We say \( f \) is a (fractional) dominating function if for each \( v \in V \), it is true that \( f(N[u]) = 1 \), where for a set \( S \),

\[
f(S) = \sum_{v \in S} f(v).
\]

Given a dominating function \( f \), we say it is minimal dominating if it is minimal among dominating functions under the usual partial ordering for real valued functions (i.e. \( f \leq g \) if \( f(v) \leq g(v) \) for all \( v \)). In discussing minimal dominating functions, the following observation is very useful \([6]\).

**Lemma 1.** Let \( f \) be a dominating function for a graph \( G \). Then \( f \) is a minimal dominating function if and only if whenever \( f(y) > 0 \) there exists some \( z \in N[y] \) such that \( f(N[z]) = 1 \).

The intuitive explanation of this result is that if \( f(y) > 0 \) and \( f \) is minimal dominating, then all the contribution of \( f \) at \( y \) is needed to make some neighbourhood sum equal to 1 (or else \( f(y) \) could be reduced and \( f \) still dominate). Now the upper fractional domination number, \( \Gamma_f(G) \), is the maximum of \( f(V) \) over all minimal dominating functions \( f \). It follows from the definition of \( \Gamma_f(G) \) that if \( f \) is restricted to mapping to \([0, 1]\), then this definition reduces to the definition for \( \Gamma(G) \). It is known \([6]\) that \( \Gamma_f(G) \) is computable and rational, and like \( \Gamma(G) \), is NP-hard \([16]\) to compute.

Since it seems \([6]\) that \( \Gamma_f(G) \) may be even more difficult to compute than \( \beta(G) \), \( \Gamma(G) \) or \( IR(G) \), we have decided to take the approach of trying to discover when it equals the other parameters. Cockayne et al. \([11]\) have shown that for all graphs \( G \), \( \beta(G) \leq \Gamma(G) \leq IR(G) \). Since the fractional problem is a relaxed version of a 0–1 maximization problem, \( \Gamma(G) \leq \Gamma_f(G) \). We now show that \( \Gamma_f(G) \leq IR(G) \).
Theorem 1. For any graph $G$, $\Gamma_f(G) \leq IR(G)$.

Proof. Let $g : V \rightarrow [0, 1]$ be a minimal dominating function of $G$ where $g(V) = \Gamma_f(G)$. Also let $S = \{v_1, v_2, \ldots, v_m\}$ be the set of vertices with $g(N[v_i]) = 1$. Note that since $g$ is minimal, every vertex $u \in V$ with $g(u) > 0$ is adjacent to at least one vertex in $S$. Hence $S$ dominates the set $P$ of vertices with positive function values. Let $D \subseteq S$ be a minimal subset of $S$ which dominates $P \cup S$. Since $D$ is minimal, $D$ is an irredundant set of $\langle P \cup S \rangle$, and therefore of $G$. Thus $IR(G) \geq |D|$. But

$$|D| = \sum_{v_i \in D} 1 = \sum_{v_i \in D} g(N[v_i]) \quad \text{since } D \subseteq S \text{ and for all } v \in S, g(N[v]) = 1 \geq \sum_{v \in V} g(v) \geq g(v) > 0 \text{ implies } v \in P, \text{ and } D \text{ dominates } P$$

$$= g(V) = \Gamma_f(G).$$

Therefore $\Gamma_f(G) \leq |D| \leq IR(G)$.  $\Box$

As a result of the theorem, we have $\beta(G) \leq \Gamma_f(G) \leq \Gamma_f(G) \leq IR(G)$. In the remaining sections, we investigate cases where these parameters are equal, especially when $\Gamma_f(G)$ is involved.

2. Classes of graphs with $\beta(G) = \Gamma_f(G) = \Gamma_f(G) = IR(G)$

Equality of the three parameters $\beta(G)$, $\Gamma_f(G)$, and $IR(G)$ has already been investigated by a number of authors. None of them considered $\Gamma_f(G)$, but when $\Gamma_f(G) = IR(G)$ its equality relationship with the others follows from the above result. Cockayne et al. [10] have shown that if a graph does not contain (as an induced subgraph) any of the four forbidden subgraphs indicated in Fig. 1 (each dashed line can be either present or absent), then $\Gamma_f(G) = IR(G)$. Favaron [13] obtained this same result if a graph does not contain $K_{1,3}$, $G_2$, or $G_3$ (see Fig. 2).
Forbidden subgraphs have also been used to obtain sufficient conditions for the stronger condition $\beta(G) = \Gamma(G) = IR(G)$. In particular, Jacobson and Peters [26] have shown that if a graph does not contain $K_{1,3}$, $C_4$ (the 4 cycle), or $G_4$ (see Fig. 3) then the three parameters are equal. They are also equal if $G$ does not contain either $C_4$ or $\bar{C}_4$ (the complement of $C_4$).

Properties of graphs have also been used to establish equality of the three parameters. Cockayne et al. [10] showed that bipartite graphs have $\beta(G) = \Gamma(G) = IR(G)$. They also showed that the parameters are equal if $G$ has no vertices of degree 0 and $\gamma(G) + IR(G) = |E|$. Jacobson and Peters established equality of the parameters for chordal graphs [26], peripheral graphs, and for any graph where the maximum degree of any vertex is two [27].

Cockayne et al. [11] showed that the representative graph of any hereditary hypergraph with no degree 0 vertices has the parameters equal (see [11] for the related definitions). It is easy to see that all such representative graphs, including middle graphs and independence graphs, are upper bound graphs. A graph $G = (V, E)$ is
called an upper bound graph \cite{29} if there exists a partially ordered set \((P, \leq)\) such that \(V = P\) and \((x, y) \in E\) if \(x \neq y\) and there exists a \(z \in P\) with \(x \leq z\) and \(y \leq z\). It was shown by Cheston et al. \cite{7} that the three parameters are equal for upper bound graphs.

Given an arbitrary graph \(G\), the trestled graph of index \(k\), \(T_k(G)\), is the graph obtained from \(G\) by adding \(k\) copies of \(K_2\) for each edge \((u, v)\) of \(G\) and joining \(u\) and \(v\) to the respective endvertices of each \(K_2\). See Fig. 4 for an example of the structure when \(k = 2\). Fellows et al. \cite{14} show that the three parameters are equal for all trestled graphs.

Finally Golumbic and Laskar \cite{19} showed that \(\beta(G) = \Gamma(G) = IR(G)\) for the class of circular arc graphs. A graph \(G = (V, E)\) is a circular arc graph \cite{28, 18} if there exists a set of arcs of a circle with each arc corresponding to a vertex of \(G\), and two vertices are adjacent iff the corresponding arcs have a nonempty intersection on the circle.

The main class to be considered here is a subclass called strongly perfect graphs \cite{4} of the class of perfect graphs. A graph \(G\) is called perfect if for each induced subgraph \(H\) of \(G\), the size of the largest clique (maximal complete subgraph) in \(H\) equals the chromatic number of \(H\) (the fewest number of colours needed to colour the vertices of \(H\) in such a way that no 2 adjacent vertices have the same colour). An intuition for this class can be obtained from the strong perfect graph conjecture. A chordless cycle of length at least four is called a hole, and the complement of such a cycle is called an antihole. A graph is called Berge if it contains as an induced subgraph neither an odd hole nor an odd antihole. If a graph is perfect then it is Berge, and the strong perfect graph conjecture asserts that a graph is perfect if and only if it is Berge. In recent years, many results have been developed while trying to prove or disprove this conjecture \cite{3}.

A set \(S\) is called a stable transversal if \(|S \cap C| = 1\) for all \(C \in C\) where \(C\) is the set of all (maximal) cliques. A graph \(G\) is called strongly perfect if \(G\) and each of its induced subgraphs has a stable transversal. It is known that every strongly perfect graph is perfect \cite{4}, and that the class of strongly perfect graphs includes perfectly ordered
graphs [9], Meyniel graphs [30, 34], and many other classes which are subclasses of these two classes.

**Theorem 2.** If \( G \) is a strongly perfect graph, then \( \beta(G) = \Gamma(G) = \Gamma_f(G) = \text{IR}(G) \).

**Proof.** (The general approach of the proof follows that of Jacobson and Peters [26].)

Let \( G \) be a strongly perfect graph, and \( I \) be an arbitrary IR-set (i.e. a maximum irredundant subset of vertices with \( |I| = \text{IR}(G) \)).

If \( I \) is an independent set, then \( \beta(G) \geq |I| = \text{IR}(G) \) and the proof is complete.

Assume \( I \) is not independent. Let \( R = \{ x \mid x \in I, \exists y \in I, (x, y) \in E \} \), i.e. \( R \) is the subset of \( I \) that consists of the nonisolated (degree greater than zero) vertices of \( \langle I \rangle \).

For each \( x \in R \), let \( x' \) be a (particular) private neighbour of \( x \) not in \( I \) and \( R' = \{ x' \mid x \in R \} \). Consider the graph \( H = \langle I \cup R \rangle \). Note that since \( R' \) is a set of private neighbours for vertices in \( R \), the only edges in \( H \) from \( I \) to \( R' \) are \( (x, x') \) for some \( x \in R \), and the vertices of \( I - R \) are isolated in \( H \) (see Fig. 5). Since \( G \) is strongly perfect, \( H \) must have a stable transversal. Let \( S \) be a stable transversal of \( H \). Consider each \( (x, x') \) edge for \( x \in R \). It forms a maximal clique since no vertex of \( R' - \{ x' \} \) is adjacent to \( x \), and no vertex of \( I - \{ x \} \) is adjacent to \( x' \). Thus exactly one of \( x \) or \( x' \) must be in \( S \). Also \( S \) must contain \( I - R \).

\[ \therefore |S| = |I| = \text{IR}(G). \]

Since stable transversals are independent sets, \( S \) is an independent set of size \( \text{IR}(G) \).

\[ \therefore \beta(G) \geq |S| = \text{IR}(G), \]

\[ \therefore \beta(G) = \Gamma(G) = \Gamma_f(G) = \text{IR}(G). \]

**Corollary 1.** If \( G \) is an even cycle, tree, bipartite, cograph, permutation, comparability, chordal, co-chordal, peripheral, purity, Gallai, perfectly orderable, or Meyniel graph then

\[ \beta(G) = \Gamma(G) = \Gamma_f(G) = \text{IR}(G). \]
Proof. These and others are all strongly perfect graphs [5, 18]. □

There are another couple classes of graphs that have not been publicized as subclasses of the strongly perfect class. For a class of graphs, the class of \( \mathcal{J} \)-cographs is defined recursively as follows:

(i) A graph in \( \mathcal{J} \) is a \( \mathcal{J} \)-cograph.
(ii) If \( G_1, G_2, \ldots, G_k \) are \( \mathcal{J} \)-cographs, then so is their disjoint union

\[
G_1 \cup G_2 \cup \cdots \cup G_k.
\]

(iii) If \( G \) is a \( \mathcal{J} \)-cograph, then so is its complement \( \bar{G} \).

If \( \mathcal{J} \) is the trivial class consisting of only a single vertex graph, then the \( \mathcal{J} \)-cograph class is the standard cograph class. If \( \mathcal{J} \) is the class of trees, then the tree-cograph class [36] is obtained. Other classes can also be defined, for example chordal-cographs. Also, if \( \mathcal{J} \) is a class of perfect graphs, the \( \mathcal{J} \)-cographs are perfect since the class of perfect graphs is closed under union and complementation.

One of the perfect classes referred to in Corollary 1 is the perfectly orderable class. This class of graphs is characterized by the existence of a linear order \( < \) on the set of vertices such that no induced chordless \( P_4 \) path with vertices \( a, b, c, d \) and edges \( (a, b), (b, c), (c, d) \) has \( a < b \) and \( d < c \) (this is called the forbidden orientation).

Theorem 3. If \( \mathcal{J} \) is a class of graphs such that for all \( H \in \mathcal{J} \) both \( H \) and \( \bar{H} \) are perfectly orderable, then all \( \mathcal{J} \)-cographs are perfectly orderable.

Proof. Let \( G \) be an arbitrary \( \mathcal{J} \)-cograph. Then \( G \) can be built up from graphs in \( \mathcal{J} \) by means of union and complement operations. Let \( H_1, H_2, \ldots, H_r \) be the collection of \( \mathcal{J} \)-graphs used to construct \( G \), and let \( A_i \) be the set of vertices from \( G \) that correspond to the graph \( H_i \). In \( G \), the subgraph induced by \( A_i \) is either \( H_i \) or \( \bar{H}_i \), dependent on whether there was an even or odd number of complement operations after \( H_i \) joined the construction. Order the vertices of \( G \) as follows:

Consider each of the \( r \) sets \( A_i \) in some arbitrary order:
- if \( \langle A_i \rangle = H_i \), then order the vertices of \( A_i \) according to the perfect ordering of \( H_i \) placing them after the vertices of preceding \( A_i \)'s;
- if \( \langle A_i \rangle = \bar{H}_i \), then order the vertices of \( A_i \) according to the perfect ordering of \( \bar{H}_i \) placing them after the vertices of preceding \( A_i \)'s.

We now need to show that this ordering does not induce a \( P_4 \) with the forbidden orientation specified above.

Consider an induced \( P_4 \) in \( G \). First we claim that all the vertices of \( P_4 \) must belong to the same set \( A_i \). This follows by induction since a union operation implies \( P_4 \) must belong to one of the graphs unioned, and a union followed by a complement induces a subgraph that contains either a \( K_{1,3} \) or a \( K_{2,2} \) if vertices are not all in the same graph during the union.
Now since all the vertices of $P_4$ belong to the same set $A_i$, by the ordering of the vertices in $\langle A_i \rangle$, $P_4$ can not have the forbidden orientation. Therefore $G$ is perfectly orderable.  \[ \square \]

**Corollary 2.** Tree-cographs and chordal-cographs are perfectly orderable.

**Proof.** Co-chordal graphs are graphs whose complement is chordal. Since both chordal and co-chordal graphs are known to be perfectly orderable, it follows that chordal-cographs are perfectly orderable. Trees are a subclass of chordal graphs, so the result follows for tree-cographs.  \[ \square \]

As we will show in the next section, not all perfect graphs have the property $\beta(G) = \Gamma(G) = \Gamma_f(G) = IR(G)$. In particular, the property does not hold for the (perfectly orderable) cograph class.

Note that a strongly perfect graph $G$ satisfies the stronger property that $\beta(H) = \Gamma(H) = \Gamma_f(H) = IR(H)$ for all induced subgraphs $H$ of $G$. Graphs with this property are characterized by Jacobson and Peters [27]. They also show that circular arc graphs satisfy this characterization and hence have the stronger property. Obviously a tristled graph $T_k(G)$ does not have the stronger property as $G$ is an induced subgraph of $T_k(G)$, and $G$ can be any graph. Also upper bound graphs do not have the stronger property for the parameters. To see this note that for any graph $G$, the graph $\hat{G}$ can be constructed as follows: for every edge $(u, v) \in E(G)$, add a unique vertex $w$ and edges $(u, w)$ and $(v, w)$. Then $\hat{G}$ is an upper bound graph which has $G$ as an induced subgraph. Therefore this shows that every graph is an induced subgraph of some upper bound graph. It would be useful to have a characterization of the weak version of the property (when the four parameters are equal for the graph, but not necessarily for all induced subgraphs).

### 3. Some classes of graphs where the four parameters are not equal

The first class that we will consider is one where $\beta(G) = \Gamma(G) = \Gamma_f(G)$ but $IR(G)$ is not necessarily equal to the three others. A simplicial vertex is a vertex that appears in exactly one clique (i.e. $v$ is a simplicial vertex iff every two neighbours of $v$ are adjacent), and a clique containing one or more simplicial vertices is called a simplex. A graph is called simplicial [7] if every vertex of $G$ is either a simplicial vertex or adjacent to a simplicial vertex. Hence in a simplicial graph, every vertex belongs to a simplex. One can also define edge simplicial graphs to be graphs where every edge belongs to a simplex. It turns out that the class of edge simplicial graphs is the same as the class of upper bound graphs [7]. Therefore the class of simplicial graphs is a natural extension of the class of upper bound graphs.

It is known [7] that for simplicial graphs, $\beta(G) = \Gamma(G)$, and simplicial graphs exist with $\Gamma(G) < IR(G)$. We now prove that for simplicial graphs $\Gamma(G) = \Gamma_f(G)$. To
prove this result, we use a stronger version of Lemma 1 that can be proved for minimal dominating functions in the context of simplicial graphs.

**Lemma 2.** If $G = (V, E)$ is a simplicial graph and $f$ is a minimal dominating function for $G$, then $f(y) > 0$ implies there exists a simplicial vertex $v$ such that $v \in N[y]$ and $f(N[v]) = 1$.

**Proof.** Since $f$ is minimal dominating and $f(y) > 0$, there exists $z \in N[y]$ such that $f(N[z]) = 1$. Also since $G$ is simplicial, there exists a simplicial vertex $v \in N[z]$.

Suppose $v \notin N[y]$. Then

$$1 < f(N[v]) < f(N[v]) + f(y) = f(y) + f([y]) = 1.$$ 

But this is impossible. Therefore $v \notin N[y]$, and $v$ is simplicial. Also $1 < f(N[v]) = 1$, so $f(N[v]) = 1$. □

**Theorem 4.** For $G$ a simplicial graph, $\Gamma(G) = \Gamma_f(G)$.

**Proof.** Suppose $f$ is a minimal dominating function for $G = (V, E)$. We begin by defining a partition on $V$ such that the number of equivalence classes is equal to $s$, where $s$ is the number of simplices in $G$. This partition is defined by the following algorithm:

1. Initialize all vertices to be unmarked.
2. While there exists an unmarked vertex $u$ that is in more than one simplex and has $f(u) > 0$
   - Find such a vertex.
   - Find a simplicial vertex $v \in N[u]$ with $f(N[v]) = 1$.
   - Form a class to consist of all unmarked vertices of $N[v]$.
   - Mark all vertices in the class.
3. While there is an unmarked simplicial vertex
   - Find an unmarked simplicial vertex $v$.
   - Form a class to consist of all unmarked vertices of $N[v]$.
   - Mark all vertices in the class.

Note that if a vertex $u$ is unmarked, then all adjacent simplicial vertices are also unmarked. Thus the vertex $v$ selected in step 2 is unmarked. Also it is clear that the classes form a partition of $V$ since every vertex is adjacent to a simplicial vertex. Also the number of classes is equal to $s$, the number of simplices, since (i) each class is a subset of the vertex set for a distinct simplex, and (ii) there must exist a class for each
simplex. But \( s \leq \beta(G) \) since selecting one simplicial vertex from each simplex forms an independent set. Also \( \beta(G) \leq s \) since no independent set can contain 2 vertices from the same simplex. Therefore the number of classes in the partition is \( \beta(G) \).

We now want to show that for any class \( C \), \( f(C) \leq 1 \). If the class was formed in step 2, then \( f(C) \leq f(N[v]) = 1 \) so the property holds. When step 2 is completed, every unmarked vertex \( u \) that is in more than one simplex has \( f(u) = 0 \). Thus the vertices of a class formed in step 3 either have a function value of 0, or else belong to a single simplex. Let \( C \) be the class formed in step 3 from \( N[v] \). If \( y \in C \) and \( f(y) > 0 \), then \( y \) belongs to only one simplex, \( N[v] \). By Lemma 2, there exists a simplicial vertex \( v' \in N[y] \) with \( f(N[v']) = 1 \). Since \( y \) is in only one simplex, \( N[v'] = N[v] \) and \( f(C) \leq f(N[v]) = 1 \). Therefore \( f(C) \leq 1 \) for all classes \( C \).

Thus for any minimal dominating function \( f \),

\[
\sum_{C} f(C) = 1 = \beta(G).
\]

Therefore \( \Gamma_f(G) \leq \beta(G) \). But \( \beta(G) \leq \Gamma(G) \leq \Gamma_f(G) \), and hence \( \Gamma(G) = \Gamma_f(G) \). \( \Box \)

It is also interesting to note the following.

**Corollary 3.** Suppose \( f \) is a minimal dominating function with weight \( \Gamma_f(G) \) for \( G \) a simplicial graph. Then for each simplicial vertex \( v \), \( f(N[v]) = 1 \), and no vertex with a positive function value belongs to more than one simplex.

**Proof.** Suppose \( v \) is a simplicial vertex with \( f(N[v]) > 1 \). Then the simplex \( N[v] \) will never be selected in step 2, so the class \( C \) for the \( N[v] \) simplex is formed in step 3. If there exists a \( y \in C \) with \( f(y) > 0 \), then by the construction, \( y \) belongs to only one simplex \( (N[v]) \), and by Lemma 2, \( f(N[v]) = 1 \). Since we assumed that \( f(N[v]) > 1 \), this means that \( f(C) = 0 \). But this is impossible since each class must have a sum of 1 in order to obtain \( f(V) = \beta(G) \). Hence for every simplicial vertex \( v \), \( f(N[v]) = 1 \).

Now suppose that \( u \) is a vertex in more than one simplex. In the construction of the classes, \( u \) is only put in one class. Thus there is a simplicial vertex \( v \) and simplex \( N[v] \) with \( u \in N[v] \), and a class \( C \) with \( C \subseteq N[v] - \{u\} \). Then \( f(C) \leq f(N[v] - \{u\}) = f(N[v]) - f(u) \leq f(N[v]) = 1 \), since \( v \) is simplicial. Since \( V \) has weight \( \Gamma_f(G) = \beta(G) \), all classes must have a weight of 1. Thus \( f(u) = 0 \). \( \Box \)

In the previous section, we showed that \( \beta(G) = \Gamma(G) = \Gamma_f(G) = IR(G) \) for the class of strongly perfect graphs. But this class does not include all perfect graphs. We will now give several subclasses of the class of perfect graphs where the four parameters are not necessarily equal.

We begin with the cograph extension of strongly perfect graphs. \( G_3 \) (of Fig. 2) and \( G_5 = \overline{C}_6 \) (in Fig. 6) both are complements of bipartite graphs. Hence the graphs are cocomparability graphs, coperfectly orderable graphs, (perfectly orderable) cographs, and (strongly perfect) cographs. (Unless otherwise indicated, the definitions for and
the containment relationships amongst the classes of graphs mentioned in this paper can be found in [5]. Also the two graphs $G_3$ and $G_4$ are claw-free, dart-free [35], and pan-free [31] Berge graphs. Finally the two graphs are superperfect and murky [22]. Each of these classes is a subclass of perfect graphs. For our purposes here, the key facts are that

$$2 = \beta(G_4) < \Gamma(G_4) = \Gamma_f(G_4) = IR(G_4) = 3,$$

and

$$2 = \beta(G_3) = \Gamma(G_3) = \Gamma_f(G_3) < IR(G_3) = 3.$$ 

Therefore $\beta$, $\Gamma$, $\Gamma_f$, and $IR$ are not necessarily equal for graphs in these classes.

Graphs $G_6$ and $G_7$ in Fig. 7 imply that these four parameters are not equal for a number of other classes of perfect graphs. First, the parameter values for these
Fig. 8. Graphs are as follows:

\[ 3 = \beta(G_6) < \Gamma(G_6) = \Gamma_f(G_6) = IR(G_6) = 4, \]

and

\[ 3 = \beta(G_7) = \Gamma(G_7) = \Gamma_f(G_7) < IR(G_7) = 4. \]

(The initial equalities for \( G_7 \) follow from \( G_7 \) being simplicial.) These two graphs are line graphs of bipartite graphs. This implies the graphs are \((K_4 - e)\)-free Berge [32, 37], unimodular, neighbourhood perfect [21], and alternately colourable graphs [24].

The graphs are also bounded tolerance graphs, which implies they are tolerance graphs, alternately orderable, bip*, weakly chordal, strictly quasi-parity, and quasi-parity. They are also clique separable, locally perfect, and slender [23]. \( G_6 \) is also Dilworth 4 (but \( G_7 \) is not). Thus for these subclasses of perfect graphs, the four parameters are not necessarily equal. Finally we note that \( G_6 \) and \( G_7 \) are circle graphs. While this class is not a perfect class of graphs, it is mentioned because of its similarity to circular arc graphs previously mentioned.

For graphs \( G_8 \) and \( G_9 \) of Fig. 8,

\[ 4 = \beta(G_8) < \Gamma(G_8) = \Gamma_f(G_8) = IR(G_8) = 5, \]

and

\[ 4 = \beta(G_9) = \Gamma(G_9) = \Gamma_f(G_9) < IR(G_9) = 5. \]

As well as belonging to a number of other classes, these graphs are both slender and slim [25]. Thus these are two classes of perfect graphs where the parameters are not necessarily equal.
For all the classes of perfect graphs that we have considered, if it is not contained in the strongly perfect class, then the class contains a graph with the parameters not equal. Nevertheless, it is not surprising that if a graph does not belong to the strongly perfect class, it can still have the parameters equal. In particular, the graph $G_{10}$ in Fig. 9 is an example of a perfect graph that is not strongly perfect but has $\beta = IR$ for all induced subgraphs.

None of the graphs so far have had the property that $\Gamma(G) \not> \Gamma_f(G)$. Very little is known about such graphs. We now look at two examples, one of which is a perfect graph.

Consider the graph $G_{11}$ shown in Fig. 10, where the $K_6$ that appears in eight places indicates that a complete graph on 6 vertices is formed by the 6 vertices surrounding each occurrence of $K_6$. For $G_{11}$, $\beta(G_{11}) = 8$, $\Gamma(G_{11}) = 14$, $\Gamma_f(G_{11}) = 14\frac{2}{3}$ [6], and $IR(G_{11}) = 22$. The $14\frac{2}{3}$ comes from $\frac{2}{3}$ at vertices 2, 6, 14, 18, 31, 35, 43, 47, and $\frac{1}{3}$ at 12, 24, 25 and 37. It is easy to see that the graph is Berge and $(K_4 - e)$-free, and hence a $(K_4 - e)$-free Berge graph. Thus the class of $(K_4 - e)$-free Berge graphs, and classes that contain it like dart-free graphs, unimodular graphs and perfect graphs, do not necessarily have $\Gamma(G) = \Gamma_f(G)$. This is also true for locally perfect and neighbourhood perfect graphs as $G_{11}$ is contained in both these classes.

To our knowledge the graph $G_{12}$ given in Fig. 11 (where vertices 1, 2, 3, ..., 7 form a complete graph) is the graph of the fewest vertices with $\Gamma(G) < \Gamma_f(G)$. This graph has $\beta(G_{12}) = 4$, and $IR(G_{12}) = 6$ with the vertices $\{2, 3, ..., 7\}$ forming an irredundant set of maximum size. $\Gamma_f$ is maximized by making the values at vertices $\{2, 3, ..., 7\}$ as large as possible subject to the minimal dominating constraint. In order to dominate $\{14, 15, 16\}$, the best that can be done is $\Gamma_f(G_{12}) = 4\frac{1}{2}$ (with $\frac{1}{2}$ at 2, 3, ..., 7, and $\frac{1}{4}$ at 8, 9, ..., 13). $\Gamma$ cannot do as well as it must have at least two ones at vertices in $\{8, 9, ..., 13\}$ in order to cover 14, 15, and 16. As a result, the best is $\Gamma(G_{12}) = 4$ with for example ones at 8, 10, 6 and 7.
4. The join operator and the parameters

Many graphs can be built from simpler graphs by means of graph operations. If a parameter can be evaluated on the simpler graphs, and the value of the parameter for a graph that results from the operation can be expressed in terms of the parameter values of the simpler graphs, then the value of the parameter of the resulting graph can be obtained. This can provide an easy way to determine certain parameters for some graphs. In this section, we consider the four parameters $\beta$, $\Gamma$, $\Gamma_f$, and $IR$ and the graph join operator.

Given two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where we assume $V_1 \cap V_2 = \emptyset$, the graph resulting from the join operation, $G_1 + G_2 = (V, E)$, is given by

$$V = V_1 \cup V_2$$

and

$$E = E_1 \cup E_2 \cup \{(x, y) | (x \in V_1 \text{ and } y \in V_2) \text{ or } (x \in V_2 \text{ and } y \in V_1)\}.$$ 

The following theorem expresses the value of a parameter in terms of its value for the graphs joined.
Theorem 5. $\zeta(G_1 + G_2) = \max(\zeta(G_1), \zeta(G_2))$ where $\zeta$ is any of the parameters $\beta$, $\Gamma$, $\Gamma_f$, or $1R$.

Proof. For each of the parameters $\zeta$, it is obvious that

$$\zeta(G_1 + G_2) \geq \max(\zeta(G_1), \zeta(G_2))$$

since any set or function that has the required properties on either $G_1$ or $G_2$ has the same properties on $G_1 + G_2$.

Now consider the three integer parameters $\beta$, $\Gamma$, and $1R$. For $\Gamma$ and $1R$, if the required set for $G_1 + G_2$ has a member from both $V_1$ and $V_2$, then the largest size the set can be is 2. For $\beta$, there cannot be members from both $V_1$ and $V_2$. In both cases $\zeta(G_1 + G_2) = \max(\zeta(G_1), \zeta(G_2))$, so we immediately obtain the result.

Now we want to show $\Gamma_f(G_1 + G_2) = \max(\Gamma_f(G_1), \Gamma_f(G_2))$.

If $G_1 + G_2$ is a complete graph then $\beta(G_1 + G_2) = \Gamma(G_1 + G_2) = \Gamma_f(G_1 + G_2) = 1 = \Gamma_f(G_1) = \Gamma_f(G_2)$. Therefore, assume that $G_1 + G_2$ is not a complete graph, so that $2 \leq \beta(G_1 + G_2) \leq \Gamma(G_1 + G_2) \leq \Gamma_f(G_1 + G_2)$, since $G_1 + G_2$ has at least two nonadjacent vertices.

Now let $g: V \to [0,1]$ be a minimal dominating function on $G_1 + G_2$ with $g(V) = \Gamma_f(G_1 + G_2)$. Without loss of generality, assume that $g(V_1) \geq g(V_2)$. Then
since $\Gamma_f(G_1 + G_2) \geq 2$, it follows that $g(V_1) \geq 1$ since $\Gamma_f(G_1 + G_2) = g(V) = g(V_1) + g(V_2)$.

To see that this implies $g(V_2) \leq 1$, suppose to the contrary that $g(V_2) > 1$. Then reduce any positive value assigned to a vertex in $G_2$ by an $\varepsilon > 0$ such that $g(V_2) - \varepsilon > 1$. This new function, call it $g'$, is still dominating, since every vertex in $G_2$ is still dominated by the set of vertices in $G_1$ (since $g'(V_1) = g(V_1) \geq 1$), and since every vertex in $G_1$ is still dominated by the set of vertices in $G_2$ (since $g'(V_2) \geq 1$). Thus, the function $g$ is not minimal, contradicting the hypothesis that $g$ is a minimal dominating function on $G_1 + G_2$. Therefore, $g(V_2) \leq 1$.

Now if $g(V_2) = 1$, then by the same argument as in the preceding paragraph, it follows that $g(V_1) = 1$. Assuming $G_1$ is not complete, $2 \leq \beta(G_1) \leq \Gamma_f(G_1) \leq \Gamma_f(G_1 + G_2) = g(V_1) + g(V_2) = 2$, so that $\Gamma_f(G_1) = 2$. Also since $\Gamma_f(G_2) \leq \Gamma_f(G_1 + G_2) = 2$, we have that $\Gamma_f(G_1 + G_2) = \max(\Gamma_f(G_1), \Gamma_f(G_2))$. The result also holds if $G_1$ is complete and $G_2$ is not complete.

Therefore, let us assume that $g(V_2) = a < 1$. Thus $g(V_1) > 1$. Define a new function $h: V \rightarrow [0, 1]$ as follows:

\[ h(u) = g(u)/(1 - a) \quad \text{for } u \in V_1, \]

\[ h(v) = 0 \quad \text{for } v \in V_2. \]

We claim that $h$ is in fact a minimal dominating function on $G_1 + G_2$, and its restriction to $G_1$ is a minimal dominating function on $G_1$. To show this we use the minimality of $g$.

Let us first show that $h$ is a dominating function on $G_1 + G_2$. Notice that every vertex in $G_2$ is still dominated by the set of vertices in $G_1$ since $h(V_1) = 1/(1 - a)g(V_1) \geq g(V_1) > 1$. Also every vertex $v$ in $G_1$ is still dominated because

\[ 1 \leq g(N[v]) - g(N[v] \cap V_1) + a, \]

\[ 1 - a \leq g(N[v] \cap V_1), \]

\[ 1 \leq 1/(1 - a)g(N[v] \cap V_1) = h(N[v]). \]

We now claim that $h$ is a minimal dominating function on $G_1 + G_2$ (and on $G_1$). Let $u$ be a vertex in $G_1$ for which $h(u) > 0$. Then since $g(u) > 0$ and $g$ is minimal, there exists a vertex $w$ in $N[u] \cap V_1$ such that $g(N[w]) = 1 = a + g(N[w] \cap V_1)$. (Note: $w \notin V_2$ since $g(V_2) > 1$.) Therefore $h(N[w]) = 1/(1 - a)*(g(N[w]) \cap V_1) = (1 - a)/(1 - a) = 1$.

Finally, we observe that if $a > 0$

\[ \Gamma_f(G_1 + G_2) = g(V_1) + a = 1/(1 - a)*g(V_1) - a/(1 - a)*g(V_1) + a \]

\[ < 1/(1 - a)*g(V_1) - a/(1 - a) + a < h(V_1). \]

But this is impossible, so that $a = 0$ and $g = h$.

Hence, $\Gamma_f(G_1 + G_2) = g(V_1) = \Gamma_f(G_1) = \max(\Gamma_f(G_1), \Gamma_f(G_2))$. \qed
We are not aware of any additional classes of graphs for which equality of the parameters can be implied from the above result.

Acknowledgement

The authors thank the members of the algorithms group at Clemson University (especially Steve and Sandra Hedetniemi) for many useful discussions related to this work.

References