The Dichotomy for Conservative Constraint Satisfaction Problems Revisited

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Fixed template CSPs

Γ template fixed set of relations on a finite set (domain) A

Definition (CSP(Γ) - Constraint Satisfaction Problem over Γ)

**INPUT:** Formula of the form

\[(x_1, x_2) \in R_1 \ \& \ (x_3, x_1, x_3, x_4) \in R_2 \ \& \ x_7 \in R_3 \ \& \ldots\]

where each \(R_i\) is in Γ (\(R_1\) binary, \(R_4\) 4-ary, \(R_3\) unary)

(i.e. a conjunction of atomic formulas over Γ)

**QUESTION:** Is the formula satisfiable?

**Examples:** Various forms of SAT, (Di)graph reachability, Equations over . . .

Alternative formulation (if Γ is finite): the homomorphism problem with a fixed target relational structure
The dichotomy conjecture

Conjecture (Feder, Vardi 93, generalized version)
For every $\Gamma$, $\text{CSP}(\Gamma)$ is tractable or NP-complete.

Recent (2000 – ) highlights:

(0) It is a universal algebraic problem Bulatov, Jeavons, Krokhin

(1) Conjecture is true when $|A| \leq 3$ Bulatov

(2) Conjecture is true if $\Gamma$ contains all unary relations on $A$
   (so called conservative CSPs) Bulatov

(3) Applicability of “Gaussian elimination like” methods
    characterized Dalmau, Bulatov, Berman, Idziak, Marković,
    McKenzie, Valeriote, Willard

(4) Applicability of local consistency methods characterized Barto,
    Kozik

(5) A couple of nice tricks Maróti
(1) Conjecture is true when $|A| \leq 3$

(2) Conjecture is true for conservative templates
   ▶ Proofs use heavy universal algebraic machinery
     ($\Rightarrow$ hard to understand for a non-specialist)
   ▶ Long and complicated
     ($\Rightarrow$ hard to understand for a specialist)
   ▶ Techniques very specific for the problem

(3) Applicability of “Gaussian elimination like” methods

(4) Applicability of local consistency methods
   ▶ Proofs don’t use any heavy machinery
   ▶ Bring new general notions and results, applicable elsewhere

To move on we need to understand (1),(2) better.
Good times, bad times \textbf{Led Zeppelin}

(1) Conjecture is true when $|A| \leq 3$
(2) Conjecture is true for conservative templates
(3) Applicability of “Gaussian elimination like” methods
(4) Applicability of local consistency methods
(5) A couple of nice tricks

Fortunately (1),(2) are consequences of (3),(4),(5):

(1') Conjecture is true when $|A| \leq 3$ (4?) \textit{Marković et al}
(2') Conjecture is true for conservative templates \textit{Barto}

Also...

(4') Applicability of local consistency methods \textit{Bulatov}
  - Using similar techniques as original proofs of (1) and (2)
polymorphism of $\Gamma$ ... an operation on $A$ compatible with all relations in $\Gamma$

**Theorem (Bulatov, Jeavons, Krokhin)**

*If $\Gamma$ has no "nice" polymorphisms, then $\text{CSP}(\Gamma)$ is NP-complete*

Where "nice" for core $\Gamma = \text{e.g. cyclic...} \ t(x, \ldots, x) = x, \ t(x_1, x_2, \ldots, x_n) = t(x_2, \ldots, x_n, x_1)$ Barto, Kozik

**Conjecture (Bulatov, Jeavons, Krokhin)**

*If $\Gamma$ has a "nice" polymorphism, then $\text{CSP}(\Gamma)$ is tractable.*

Similar conjectures for finer complexity classification.
Theorem

If $\Gamma$ is a conservative template which has a “nice” polymorphism, then $\text{CSP}(\Gamma)$ is tractable.

Proof: Algorithm for domains of size $k \rightarrow \text{alg}$ for doms of size $k + 1$ (simplified, but not too much):

(Step 1) Transform the instance to an equivalent instance which is consistent enough

(Step 2) Find a small restriction which is still consistent enough (4)

(Step 3) Use the algorithm for smaller domains (to certain restricted instances). Either we find a solution, or we can delete some elements and repeat, or

(Step 4) If you cannot delete anything, use (3)
Let $\Gamma$ be a fixed conservative template (on the domain $A$).

**Definition**

Let $C \subseteq A$.

A subset $B \subseteq C$ is an *absorbing subuniverse* of $C$, if there exists a polymorphism $t$ of $\Gamma$ such that

$t(a_1, \ldots, a_n) \in B$

whenever all $a_i$'s are in $C$ and

all $a_i$'s but at most one are in $B$

- Start with a proper absorbing subuniverse
- Walk until you stabilize
- Restrict
- Repeat
A conversation

CS guy: Hi, I have this conservative tractable template $\Gamma$. Give me the P-time algorithm for solving CSP over $\Gamma$!

me: Hi, first you have to give me a list of all absorbing subuniverses of all subsets of $A$.

CS guy: ??????????? ok, how do I find them?

me: I don’t know. I don’t know whether it’s decidable that a given set is an absorbing subuniverse of $A$ for a given set $\Gamma$ of relations on $A$ (or of a given algebra)...

CS guy: So you proved that a P-time algorithm exists without providing the algorithm????

me: Yes.

CS guy: I don’t like it.

me: I love it.

CS guy: See you.

me: See you.
A note on binary constraints

Using (Hell, Rafiey or Kazda) and ((2) or (4)):

**Theorem**

*If $\Gamma$ is conservative and contains only at most binary relations, then $\text{CSP}(\Gamma)$ is solvable by local consistency methods, or NP-complete.*

Thank you!!!