Brief paper

Two families of semiglobal state observers for analytic discrete-time systems

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A R T I C L E  I N F O

Article history:
Received 17 February 2011
Received in revised form 7 October 2011
Accepted 7 January 2012
Available online 26 June 2012

Keywords:
State observers
Nonlinear systems
Discrete-time systems
Analytic approximations

A B S T R A C T

Two families of observers for discrete-time nonlinear systems are presented in this paper, whose design is based on the Taylor approximation of the inverse of the observation map. Semiglobal convergence results are provided under the assumption that the system observation map is a globally analytic diffeomorphism. The performances of the observers in the two families are compared both from theoretical and practical points of view.

1. Introduction

The problem of state reconstruction for nonlinear systems from input and output measurements has been widely investigated in the literature, and many techniques exist for the design of asymptotic state observers. One method consists in finding a nonlinear change of coordinates and an output injection that recast the system into some canonical form, suitable for a linear observer design. In the discrete-time framework, first papers dealing with this approach are Lee and Nam (1991) and Lin and Byrnes (1995), where autonomous systems are only considered. More recent papers are Xiao, Kazantzis, Kravaris, and Krener (2003) and Xiao (2006). The case of systems with input is considered by Besançon and Bornard (1995), Besançon, Hammouri, and Benamor (1998), Califano, Monaco, and Normand-Cyrot (2003, 2009). In general, the appropriate coordinate transformation exists under quite restrictive conditions and its computation is a very difficult task. An interesting technique for the construction of observers with linear error dynamics for systems admitting a differential/difference representation is in Monaco, Normand-Cyrot, and Barbot (2007).

Another approach exploits dynamic inversion of suitably defined observation maps to achieve asymptotic state reconstruction without the need of any coordinate transformation (Ciccarella, Dalla Mora, & Germani, 1993, 1995). Local convergence of these observers is proved under standard Lipschitz assumptions. The use of the Extended Kalman Filter as a local observer has been investigated in Boutayeb and Aubry (1999), Boutayeb, Rafaralahy, and Darouach (1997) and Reif and Unbehauen (1999), while in Germani and Manes (2008) the convergence of the Polynomial Extended Kalman Filter (Germani, Manes, & Palumbo, 2005), when used as an observer, is studied. Observers for the case of nonlinear systems with linear measurements are considered in Abbasszadeh and Marquez (2008), Boutayeb and Darouach (2000) and Ibrir (2007). An $H_\infty$ observer design approach is followed by Zemouche, Boutayeb, and Bara (2008), Zemouche and Boutayeb (2009a,b). Another approach is the Moving Horizon Estimation technique, as in Kang (2006), which allows to consider also uncertainties and disturbances, as in Alessandri, Baglietto, and Battistelli (2008).

This paper presents two families of semiglobal observers, based on high order Taylor approximations of the inverse of the observation map, that improve the local observers in Ciccarella et al. (1993, 1995), based on the first order Taylor approximation. The degree $\nu$ of the approximating polynomial defines the order of the observer in the families. Our approach takes inspiration from Germani, Manes, Palumbo, and Sciacchitano (2006), where a root-finding method has been developed by suitably exploiting Taylor polynomials of degree $\nu > 1$ to get higher convergence rates than the Newton–Raphson method. The main feature of the presented observer families is that, for any given bound on
Throughout the paper, for a given vector $Y$, if such a map is invertible, then the state reconstruction from functions $V_r, V_{(r-1)}$, will denote the function defined as

$$f^r(x, V) = f(x, V), \quad \text{and } f^0(x) = x.$$

Alternatively, $f^r(x, V) = f^{r-1}(f(x, V_{(r+1)}), V_{(r+1)}), \quad r > 1$.

The symbol $h \circ f^r$ will denote the function defined as

$$h \circ f^r(x, V) = h(f^r(x, V_{(r+1)}), V_{(r+1)}), \quad V \in \Omega'.$$

The $n$ functions $h \circ f^r(x, V) = h(f^r(x, V_{(r+1)}), V_{(r+1)}), \quad r = 1, \ldots, n$, can be stacked into a square map $\Phi(x, V) = \Phi(x, V), \quad V \in \Omega'$, as follows:

$$\Phi(x, V) = \begin{bmatrix} h \circ f^{n-1}(x, V_{(1)}) \\ \vdots \\ h \circ f^1(x, V_{(n)}) \\ h \circ f^0(x, V_{(1)}) \end{bmatrix}.$$

Given the input and output sequences $u(t)$ and $y(t)$, let us define the vectors $U_t \in \Omega'$ and $Y_t \in \mathbb{R}^n$ as

$$Y_t = \begin{bmatrix} y(t+1) \\ \vdots \\ y(t+n-1) \end{bmatrix}, \quad U_t = \begin{bmatrix} u(t+1) \\ \vdots \\ u(t+n-1) \end{bmatrix},$$

so that the following relation holds for any $t \in \mathbb{Z}$:

$$Y_t = \Phi(\Phi(x(t), U_t)).$$

The function $z = \Phi(x; V)$ defined in (4) is a square map from $x \in \mathbb{R}^n$ to $z \in \mathbb{R}^n$, where $V \in \Omega'$ is a vector of known parameters. If such a map is invertible, then the state reconstruction from the knowledge of the input and output sequences $(Y_t, U_t)$ is theoretically possible. For this reason, the following definitions are given.

**Definition 1.** The map $\Phi : \mathbb{R}^n \times \Omega' \mapsto \mathbb{R}^n$ defined in (4) is called the observation map of the system (1), and its Jacobian $\nabla_x \Phi(x, V)$ is called the observability matrix.

**Definition 2.** The nonlinear system (1) with $u(t) \in \mathbb{R}$ is said to be uniformly observable in a subset $\Omega \subset \mathbb{R}^n$ if its observation map (4) is invertible in $\Omega$ for any $V \in \Omega'$. If $\Omega = \mathbb{R}^n$, then the system (1) is said to be globally observable. If $\Omega = 0$, the system is said to be drift-observable.

The inverse of the observation map is symbolically written as $x = \Phi^{-1}(z, V)$.

**Remark 1.** The invertibility of $\Phi(x, V)$ may depend on the set $\mathbb{R}$ of admissible inputs. When $\mathbb{R}$, uniform observability in $\Omega \subset \mathbb{R}^n$ is equivalent to observability for any input (see Gauthier, Hammersou, & Othman, 1992, for continuous-time systems). This is a rather strong property, even stronger when $\Omega = \mathbb{R}^n$ (global uniform observability), because in general the inverse of a nonlinear map is only locally well-defined, and often admits bifurcation points (see e.g. Barbot, Belmouhoub, & Boutat-Baddas, 2006). However, uniform observability for any input in a subset $\Omega \subset \mathbb{R}^n$ can be a much weaker property, because $\mathbb{R}$ can be small enough to keep out bad inputs. In Dalla Mora, Germani, and Manes (2000), it is shown that any drift-observable system admits a bounded set $\mathbb{R}$ such that the system is uniformly observable for any $u(t) \in \mathbb{R}$.

When the uniform observability assumption for any input in $\mathbb{R}$ is satisfied, the presence of the parameter $V$ in the observation map (4) and in the r-steps transition functions (2) and output functions (3) does not add any theoretical complexity to the state reconstruction schemes here presented. Thus, in order to have simpler notations, the case of uniform discrete-time systems is considered at first:

$$x(t+1) = f(x(t), u(t)), \quad t \in \mathbb{Z},$$

$$y(t) = h(x(t)), \quad t \in \mathbb{Z},$$

so that $f^0(x) = x$ and

$$y^{r+1}(x) = (f \circ f^r)(x) = f^r(x), \quad r \geq 0,$$

$$h \circ f^r(x) = h(f^r(x)).$$

The observation map takes the simpler form

$$\Phi(x) = [h \circ f^{n-1}(x), \ldots, h \circ f^1(x), h(x)]^T,$$

and the output sequence is a function of the state only

$$Y_t = \Phi(x(t)).$$

**Definition 3.** The nonlinear system (7) is said to be observable in a subset $\Omega \subset \mathbb{R}^n$ if its observation map (9) is invertible in $\Omega$. If $\Omega = \mathbb{R}^n$, then the system (7) is said to be globally observable.

By Definition 3, a system is observable if for any $t \in \mathbb{Z}$ the output sequence in the interval $[t, t+n]$ univocally determines the state $x$ at time $t$, formally expressed as a function of the inverse map

$$x(t) = \Phi^{-1}(Y_t).$$

In (11), the current state $x(t)$ is written as a function of future observations. The causal computation of $x(t)$ as a function of current and past observations $Y_{t-n+1}$ can be made in two steps (ideal exact state reconstruction):

1a. compute the state at time $t-n+1$ as

$$x(t-n+1) = \Phi^{-1}(Y_{t-n+1}).$$

2a. compute the current state $x(t)$ as

$$x(t) = f^n(x(t-n+1)).$$
The dead-beat observer in Glad (1983) follows exactly the steps 1a–2a. However, nonlinear vector maps very seldom admit closed-form inverses, and therefore numerical iterative algorithms are needed in general to compute $x(t - n + 1)$ at step 1a. Since the number of iterations of any numerical scheme in real-time applications is limited by the sampling interval, at each time-step the state reconstruction may well not be exact. As a general scheme, a causal approximate state reconstruction at time $t$ is made in two steps:

1b. try to solve for $\chi$ the equation

$$Y_{t-n+1} = \Phi(\chi) = 0,$$

(14)

2b. compute the estimate of $x(t)$ as

$$\hat{x}(t) = f^{n-1}(\chi).$$

(15)

If $\chi$ provided by step 1b is the exact solution of (14), then $\chi = x(t - n + 1)$, as in 1a, and the state reconstruction (15) at step 2b is exact, as in 2a. If $\chi$ is an approximate solution of (14), then $\chi$ is an estimate of $x(t - n + 1)$, and the step (15) does not provide the exact reconstruction of $x(t)$. In principle, any root-finding algorithm can be used to find the solution of (14). The problem can be even formulated as an optimization problem over the moving horizon $[t - n + 1, t]$ (see e.g. Kang, 2006). However, when the solution of (14) is not exact, the convergence of the state reconstruction algorithm 1b–2b is not obvious, and may depend on the root-finding algorithm used at step 1b, on the choice of its starting point, and on the number of iterations allowed within a sampling interval.

Both families of observers presented in this paper are based on the general scheme 1b–2b, where the computation of $\chi$ at step 1b is made with a one-step formula, without iterations, by exploiting the Taylor polynomial approximation of the unknown inverse map $\Phi^{-1}(\cdot)$. In Ciccarella et al. (1993), the first order Taylor approximation was exploited to solve the problem, while the use of higher order Taylor approximations in a state observation algorithm has been proposed for the first time in Germani and Manes (2009). The convergence results here presented require the following assumptions.

$\text{H}_p_1$: the system (7) is globally observable, the observation matrix $\nabla_x \Phi$ is globally invertible and the inverse $(\nabla_x \Phi)^{-1}$ is uniformly bounded in $\mathbb{R}^n$.

$\text{H}_p_2$: the functions $f$ and $h$ in (7) are globally analytic and uniformly Lipschitz in $\mathbb{R}^n$.

Remark 2. Recall that a function defined in $\mathbb{R}^n$ is globally analytic if it is analytic in all compact sets of $\mathbb{R}^n$. Well known results of real analysis establish that the composition of analytic functions is analytic. Thus, assumption $\text{H}_p_2$ implies that the observation map $z = \Phi(x)$ defined in (9) is globally analytic. Moreover, the uniform bound assumed in $\text{H}_p_1$ on the inverse of the Jacobian $\nabla_x \Phi$ implies that the inverse $x = \Phi^{-1}(z)$ is globally analytic too.

Remark 3. Assumptions $\text{H}_p_1$ and $\text{H}_p_2$ are rather strong, and $\text{H}_p_2$ is often impossible to check, when the system dynamics is not simple and the state variables are not few. However, these assumptions allow to provide a straightforward proof of semiglobal convergence of the proposed observers. Weaker assumptions (e.g., observability and Lipschitz properties in a bounded set $\Omega \subset \mathbb{R}^n$), would only provide local convergence results.

3. Taylor approximation of the inverse map

The Kronecker formalism is used here as a compact notation for Taylor polynomials and for derivatives of vector functions (see the Appendix). According to this formalism, given $x \in \mathbb{R}^n$, the symbol $x^{(k)} \in \mathbb{R}^{nk}$ denotes its $k$-th order Kronecker power, recursively defined as $x^{(0)} = x \otimes x^{(k-1)}$, for $k > 1$, where $x^{(0)} = 1$. For a given vector function $F(x) : \mathbb{R}^n \mapsto \mathbb{R}^m$, the derivative of order $k$ is denoted $\nabla_x^{(k)} F(x)$, and is an $m \times n^k$ matrix function. According to this formalism, the symbol $\nabla_x \otimes F(x)$ denotes the standard Jacobian $\nabla_x F(x)$.

Consider the observation map $z = \Phi(x)$ and its inverse $x = \Phi^{-1}(z)$, which exists by assumption $\text{H}_p_1$, but is not available in a closed form. Let $G_0(z)$ denote the $k$-th order derivatives of $\Phi^{-1}(z)$:

$$G_0(z) = \nabla_z^{(k)} \otimes \Phi^{-1}(z), \quad k = 0, 1, \ldots$$

(16)

Note that $G_0(z) = \Phi^{-1}(z)$ and $G_1(z)$ is the Jacobian $\nabla_z \Phi^{-1}(z)$.

Assume that for all $k$ the derivative $G_k(z)$ exists in all $\mathbb{R}^n$ and is uniformly bounded:

$$\Gamma_k = \sup_{z \in \mathbb{R}^n} \|G_k(z)\|.$$  

(17)

Then, the Taylor theorem states the following identity

$$\Phi^{-1}(z) = \sum_{k=0}^{\infty} \frac{1}{k!} G_k(\tilde{z})(z - \tilde{z})^k + R_{k+1}(z, \tilde{z}),$$

(18)

where the remainder $R_{k+1}$ is infinitesimal of order $v + 1$ w.r.t. $(z - \tilde{z})$, i.e.

$$\|R_{k+1}\| \leq \frac{\Gamma_{k+1}}{(v + 1)!} \|z - \tilde{z}\|^{v+1}$$

(19)

(from the Lagrange form of the remainder). The exponents $[k]$ in the Taylor formula (18) denote the Kronecker $k$-th powers. The following assumption on the coefficients $\Gamma_k$ defined in (17) is needed in this paper:

$$\text{H}_p_3: \quad \forall k \in \mathbb{N}, \exists \Gamma_k > 0: \|G_k(z)\| \leq \Gamma_k, \forall z \in \mathbb{R}^n \text{ and } \forall \rho > 0: \lim_{k \to \infty} \frac{\Gamma_k^k}{k!} = 0.$$  

(20)

Remark 4. The assumption $\text{H}_p_3$ is weaker then assuming a uniform bound on the sequence $\Gamma_k$, and the existence of a $\Gamma > 0$ such that $\Gamma_k \leq \Gamma$, $\forall k \in \mathbb{N}$, trivially implies $\text{H}_p_3$. Note that the assumption of existence of the constants $\Gamma_k$ that satisfy the condition (20) does not mean that such constants can be known or computed. The computation of the sup in (17) is impossible, in general, because the derivatives $G_k(z)$ cannot be computed, being $\Phi^{-1}(z)$ unknown.

Proposition 5. If assumption $\text{H}_p_3$ holds, then, for any given compact set $S \subset \mathbb{R}^n$, the Taylor series of $\Phi^{-1}(z)$ centered at any $\tilde{z} \in S$ (i.e., the summation in (18)), uniformly converges in all $S$.

Proof. Let $\rho > 0$ denote the diameter of the given compact set $S \subset \mathbb{R}^n$, so that for any pair $z, \tilde{z}$ in $S$ it is $\|z - \tilde{z}\| < \rho$. From (19) it follows that

$$\|R_{v+1}(z, \tilde{z})\| \leq \frac{\Gamma_{v+1}}{(v + 1)!} \rho^{v+1}, \quad \forall z, \tilde{z} \in S.$$  

(21)

By assumption $\text{H}_p_3$, for any $\epsilon > 0$ there exists a degree $v_\epsilon$ such that $\frac{\Gamma_{v+1}}{(v + 1)!} \rho^{v+1} < \epsilon$ for all $v \geq v_\epsilon$, and therefore $\|R_{v+1}\| < \epsilon$ for all $v \geq v_\epsilon$, and this proves the uniform convergence of the Taylor series in $S$. □

Let $G_k(x), k \in \mathbb{N}$, denote the $k$-th derivative of the inverse observation map computed at $z = \Phi(x)$:

$$G_k(x) = G_k \circ \Phi(x) = (\nabla_x^{[k]} \otimes \Phi^{-1}(z))_{z=\Phi(x)}.$$  

(22)

Let $\tilde{z} = \Phi(\tilde{x})$ and $x = \Phi^{-1}(z)$. Observing that $G_k(z) = \tilde{x}$, because $\nabla_x^{[k]} \otimes \Phi^{-1}(\tilde{z}) = \Phi^{-1}(z) = \tilde{x}$, the Taylor formula (18) becomes

$$x = \tilde{x} + \sum_{k=1}^{\infty} \frac{1}{k!} G_k(\tilde{z})(z - \Phi(\tilde{x}))^k + R_{k+1}(z, \Phi(\tilde{x})).$$  

(23)
where, observing that $F_k = \sup_{z \in \mathbb{R}^n} \|G_k(z)\| = \sup_{z \in \mathbb{R}^n} \|\tilde{G}_k(x)\|$, it is
\[
\|r_{v+1}\| \leq \frac{\Gamma_{v+1}}{(v+1)!} \|z - \Phi(\tilde{x})\|^{v+1}.
\] (24)

An important fact is that the coefficients $\tilde{c}_k(x)$ defined in (22) and used in (23) can be computed without the explicit knowledge of the inverse map $x = \Phi^{-1}(z)$. In Germani and Manes (2009) it has been shown that the $\tilde{G}_k(x)$ can be recursively computed by combining derivatives of the direct map as follows
\[
\tilde{G}_1(x) = (\nabla x \Phi(x))^{-1},
\]
\[
\tilde{G}_k(x) = -\left(\sum_{h=1}^{k-1} \tilde{G}_h(x) F_{h,k}(x)\right)(\tilde{G}_1(x))^{[k]},
\] (25)
for $k = 2, 3, \ldots$, where the matrix functions $F_{h,k}(x)$ are recursively defined as
\[
F_{h,k}(x) = \nabla x \otimes F_{h,k-1}(x) + (\nabla x \Phi(x)) \otimes F_{h-1,k-1}(x),
\] (26)
initialized with $F_{0,k}(x) = 0$, for all $k \in \mathbb{N}$ (see Germani and Manes (2009) and the Appendix).

Remark 6. In the observer equations of Section 4 the matrices $\tilde{G}_k$ must be computed at some points $x \in \mathbb{R}^n$ (state predictions). It is important to stress that to this aim the symbolic computation of the functions $\tilde{G}_k(x)$ is not required, in that the recursive equations (25) easily allow the numerical computation of the matrices $\tilde{G}_k$. Only the functions $F_{h,k}(x)$ need to be symbolically computed by means of the recursion equations (26), and need to be evaluated at the given $x$ before starting the numerical iterations equation (25). Thus, the inverse of the Jacobian $\nabla x \Phi$ in (25) is a numerical inverse, and not a symbolic one, and so are the sums, the products and the powers of matrices in (25).

Remark 7. Other formalisms exist for writing the Taylor series of the inverse of a vector map, although most of them are less effective than (23). Consider for instance the Lie Series method developed by Gröbner and Knapp (1967). Using the Gröbner method, the symbolic inverse of the Jacobian $\nabla x \Phi(x)$ in (25) is needed, and its derivatives up to the desired degree must be computed in order to construct the approximating series of the inverse vector map.

4. Two families of semiglobal observers

The first family of observers based on the Taylor approximation of the inverse of the observation map is the one presented in Germani and Manes (2009). The observer of order $v$ is the one that exploits the Taylor polynomial of degree $v$, and is denoted $\Delta$-observer of degree $v$:
\[
\tilde{x}(t) = f(\chi(t-1)),
\] (27)
\[
\chi(t) = \tilde{x}(t) + \sum_{k=1}^{v} \frac{1}{k!} \tilde{G}_k(\chi)(\Delta(t))^{[k]},
\]
where $\Delta(t) = K(y(t-n) - h(\tilde{x}(t-1))) + B_0(y(t) - h \circ f^{n-1}(\tilde{x}(t)))$,
\[
\tilde{x}(t) = f^{n-1}(\chi(t)).
\] (29)

The sequence $\tilde{x}(t)$ provided by this algorithm asymptotically converges to the true state sequence $x(t)$ (see Theorem 9). The matrix functions $\tilde{G}_k(\cdot)$ in (28) are those defined in (25). In (29) $B_0 = [1 \ 0 \ \cdots \ 0]_{n \times p}$, and the gain vector $K \in \mathbb{R}^n$ is chosen so that matrix $A_0 - KC_0$ is stable ($A_0, C_0$ is an observable Brunovsky pair, see Germani and Manes (2009)). The $\Delta$-observer (27)–(30) starts at a time $t_0$ when all the observations in the interval $(t_0 - n, t_0)$ are available. The starting value $\chi(t_0)$ is any a priori estimate of $x(t_0 - n + 1)$.

The new family of observers here presented is made of algorithms denoted $\eta$-observers of degree $v$:
\[
\tilde{x}(t) = f(\chi(t-1)),
\] (31)
\[
\chi(t) = \tilde{x}(t) + \sum_{k=1}^{v} \frac{1}{k!} \tilde{c}_k(\tilde{x}(t))(\eta(t))^{[k]},
\]
where $\eta(t) = Y_{t-n+1} - \Phi(\tilde{x}(t))$.
\[
\tilde{x}(t) = f^{n-1}(\chi(t)).
\] (33)

The convergence of $\tilde{x}(t)$ to the true state $x(t)$ is studied in Theorem 10.

Remark 8. Both $\Delta$- and $\eta$-observers have a classical prediction–correction structure, where the correction terms are made of polynomials of the vector variables $\Delta(t)$ and $\eta(t)$, respectively. Obviously, the higher is the degree $v$, the higher is the complexity of the observation algorithm and the computations needed. However, this is the price to pay to get larger convergence regions w.r.t. other schemes, such as Ciccarella et al. (1993, 1995), where the correction terms are linear and the convergence is local.

The semiglobal convergence properties of the family of $\Delta$-observers are stated in the following.

Theorem 9 (Germani and Manes (2009)). Consider system (7) and system (27)–(30), where $K$ is such that all eigenvalues of $A_0 - KC_0$ are real and distinct in the interval $(-1, 1)$. Assume that hypotheses Hp1, Hp2 and H3 hold.

Then, for any pair of real numbers $\alpha \in (0, 1)$ and $\delta_x > 0$, there exist a degree $v \geq 1$ of the observer (27)–(30), and a positive real $\mu$ such that
\[
\|x(t) - \tilde{x}(t)\| \leq \mu |\alpha^{t-t_0}|x(t_0 - n + 1) - \tilde{x}(t_0)|,
\] for all pairs $x(t_0 - n + 1), \tilde{x}(t_0) \in \mathbb{R}^n$ such that
\[
\|x(t_0 - n + 1) - \tilde{x}(t_0)\| \leq \delta_x.
\] (35)

The theorem below shows that under the same assumptions of Theorem 9 the $\eta$-observer has an interesting additional convergence property.

Theorem 10. Consider system (7) and system (31)–(34). Assume that hypotheses Hp1, Hp2 and H3 hold and let $\gamma_{f^{n-1}}$ denote the Lipschitz constant of $f^{n-1}(x)$. Then, for any pair of real numbers $\alpha \in (0, 1)$ and $\delta_x > 0$, there exists a degree $v \geq 1$ of the observer (31)–(34) such that
\[
\|x(t) - \tilde{x}(t)\| \leq \gamma_{f^{n-1}}|\alpha^{t-t_0}|x(t_0 - n + 1) - \tilde{x}(t_0)|,
\] for all pairs $x(t_0 - n + 1), \tilde{x}(t_0) \in \mathbb{R}^n$ such that
\[
\|x(t_0 - n + 1) - \tilde{x}(t_0)\| \leq \delta_x.
\] (37)

Moreover, for any pair $\beta \in (0, 1)$ and $\delta_y > 0$, there exist a degree $v \geq 1$ of the observer (31)–(34) and a positive real $\mu$ such that
\[
\|x(t) - \tilde{x}(t)\| \leq \mu |\beta^{t-t_0}|Y_{t-n+1} - \Phi(\tilde{x}(t))|,
\] for all pairs $x(t_0 - n + 1), \tilde{x}(t_0) \in \mathbb{R}^n$ satisfying
\[
\|\Phi(x(t_0 - n + 1)) - \Phi(\tilde{x}(t_0))\| \leq \delta_y.
\] (39)

Proof. Let $\gamma_f$ and $\gamma_\Phi$ denote the Lipschitz constants of the functions $f$ and $\Phi$, respectively. Thanks to assumptions Hp1 and Hp2, the observation map $\Phi$ and its inverse $\Phi^{-1}$ are globally analytic.
(see Remark 2), and therefore act as a global change of coordinates. Let \( \xi(t) \in \mathbb{R}^n \) and \( z(t) \in \mathbb{R}^n \) be defined as
\[
\xi(t) = x(t - n + 1), \quad z(t) = \Phi(\xi(t)),
\]
so that \( z(t) = Y_{t-n+1} \). Let
\[
\hat{z}(t) = \Phi(\hat{x}(t)), \quad \tilde{z}(t) = \Phi(\tilde{x}(t)),
\]
which are estimates of \( z(t) \) (recall that \( \hat{x}(t) = f(\chi(t-1)) \), Eq. (31)) and define the following errors
\[
\eta(t) = z(t) - \hat{z}(t), \quad \epsilon(t) = \xi(t) - \chi(t).
\]
Note that \( \eta(t) = \Phi(\xi(t)) - \Phi(\tilde{x}(t)) = Y_{t-n+1} - \Phi(\tilde{x}(t)) \). Using (43), the inverse of the map in (41) is written as
\[
\xi(t) = \Phi^{-1}(z(t)) = \Phi^{-1}(\hat{z}(t) + \eta(t)).
\]
The Taylor formula for \( \Phi^{-1}(z(t)) \) centered at \( \hat{z}(t) = \Phi(\hat{x}(t)) \) gives
\[
\xi(t) = \Phi^{-1}(\hat{z}(t)) + \sum_{k=1}^{v} \frac{1}{k!} \xi_{\hat{z}}(\hat{z}(t)) (\eta(t))^{[k]} + r_{v+1}(t),
\]
with \( \| r_{v+1}(t) \| \leq \frac{\Gamma_{v+1}}{(v + 1)!} \| \eta(t) \|^{v+1} + 1 \). Since \( \hat{x}(t) = \Phi^{-1}(\hat{z}(t)) \), by (32), (45) can be written as
\[
\xi(t) = \chi(t) + r_{v+1}(t).
\]
Recalling that \( \epsilon(t) = \xi(t) - \chi(t) \), it trivially follows that \( \epsilon(t) = r_{v+1}(t) \), and therefore
\[
\| \epsilon(t) \| = \| r_{v+1}(t) \| \leq \frac{\Gamma_{v+1}}{(v + 1)!} \| \eta(t) \|^{v+1}.
\]
From this, the implications below easily follow
\[
\| \epsilon(t) \| \leq \delta_k \implies \| \epsilon(t) \| \leq \alpha \| \epsilon(t-1) \|, \quad \| \epsilon(t_0) \| \leq \delta_k \implies \| \epsilon(t) \| \leq \alpha^{t-t_0} \| \epsilon(t_0) \|.
\]
Recalling that \( x(t) = f_{n-1}(\xi(t)) \) and \( \hat{x}(t) = f_{n-1}(\chi(t)) \), it is
\[
\| x(t) - \hat{x}(t) \| = \| f_{n-1}(\xi(t)) - f_{n-1}(\chi(t)) \| \leq \gamma_{n-1} \| \xi(t) - \chi(t) \| \leq \gamma_{n-1} \| \epsilon(t) \|.
\]
From this and from (56)
\[
\| x(t) - \hat{x}(t) \| \leq \gamma_{n-1} \alpha^{t-t_0} \| \epsilon(t_0) \|.
\]
so that inequality (37) is proved (recall that \( \epsilon(t_0) = x(t_0 - n + 1) - \hat{x}(t_0) \)).

In order to prove (39), consider the inequality (48), written between \( t + 1 \) and \( t \)
\[
\| \eta(t + 1) \| \leq \gamma \eta \| \xi(t) - \chi(t) \|.
\]
and recall that \( \xi(t) - \chi(t) = \epsilon(t) = r_{v+1}(t) \), so that, using (47),
\[
\| \eta(t + 1) \| \leq \gamma \gamma \| \eta(t) \|^{v+1}.
\]
For a given a positive real \( \delta_y \), rewrite (60) as
\[
\| \eta(t + 1) \| \leq \gamma \gamma \| \eta(t) \|^{v+1} \leq \delta_y \| \eta(t) \|^{v+1}.
\]
By assumption \( H_{p_2} \), for any given \( \beta \in (0, 1) \) there exists \( v \) such that
\[
\gamma \gamma \| \eta(t) \|^{v+1} \leq \delta y \| \eta(t) \|^{v+1} \leq \beta < 1.
\]
With such a choice for \( v \), inequality (61) implies that
\[
\| \eta(t + 1) \| \leq \beta \| \eta(t) \|^{v+1} - \delta y.
\]
Following the same steps of inequalities (54)–(56) one obtains the implication
\[
\| \eta(t_0) \| \leq \delta y \implies \| \eta(t) \| \leq \beta^{t-t_0} \| \eta(t_0) \|.
\]
Consider now (57). Taking into account inequality (47) yields
\[
\| x(t) - \hat{x}(t) \| \leq \gamma_{n-1} \| r_{v+1}(t) \| \leq \gamma_{n-1} \frac{\Gamma_{v+1}}{(v + 1)!} \| \eta(t) \|^{v+1}.
\]
Substitution of (64) into (65) gives
\[
\| \eta(t_0) \| \leq \delta y \implies \| x(t) - \hat{x}(t) \| \leq \gamma_{n-1} \frac{\Gamma_{v+1}}{(v + 1)!} \beta^{t-t_0} \| \eta(t_0) \|^{v+1}.
\]
which is exactly the inequality (39) with \( \mu = \gamma_{n-1} \frac{\Gamma_{v+1}}{(v + 1)!} \beta^{t-t_0} \). \( \square \)

**Remark 11.** Eqs. (53) and (63) show that the convergence of the \( \eta \)-observer is faster than any exponential rate (convergence of order \( v + 1 \)). The same is true for the \( \Delta \)-observer.
Remark 12. The majorant (39) of the state observation error depends on $\|Y_{0-n+1} - \Phi(\hat{x}(t_0))\|$, which is a computable measure of the observation error as seen from the output. Conversely, the majorants given by inequality (37) (inequality (35) for the $\Delta$-observer) cannot be computed, being $x(t_0 - n + 1)$ unknown.

5. $\Delta$- and $\eta$-observers for forced systems

In the case of systems forced by an external input, i.e., those described by the model (1), the matrix functions $\bar{G}_k$ defined in (22), and recursively computed using (25) and (26), depend also on the parameter vector $\bar{V} \in \bar{U}$ that is present in the definition of the map $\Phi(x; \bar{V})$. As a consequence, the matrices $\bar{G}_k$ that appear in the summations both in $\Delta$- and $\eta$-observer equations, depend also on the stacked input $U_{t-n+1}$. Also the one-step and $(n-1)$-steps transition functions used for prediction depend on the input sequence. As a consequence, the $\Delta$- and $\eta$-observer equations for systems with input are as follows:

$\Delta$-observer of degree $\nu$

$$\hat{x}(t) = f(x(t-1), u(t-n)),$$

$$\chi(t) = \hat{x}(t) + \sum_{k=1}^{\nu} \frac{1}{k!} \bar{G}_k(\hat{x}(t), U_{t-n+1})(\Delta(t))^{[k]}$$

$$\Delta(t) = K(y(t-n) - h(\chi(t-1), u(t-n))) + B_0y(t) - h \circ f^{\nu-1}(\hat{x}(t), U_{t-n+1}),$$

$$\hat{x}(t) = f^{\nu-1}(\hat{x}(t), U_{t-n+1-1}).$$

$\eta$-observer of degree $\nu$

$$\hat{x}(t) = f(x(t-1), u(t-n)),$$

$$\chi(t) = \hat{x}(t) + \sum_{k=1}^{\nu} \frac{1}{k!} \bar{G}_k(\hat{x}(t), U_{t-n+1})(\eta(t))^{[k]}$$

where $\eta(t) = Y_{t-n+1} - \Phi(\hat{x}(t), U_{t-n+1}),$

$$\hat{x}(t) = f^{\nu-1}(\hat{x}(t), U_{t-n+1-1}).$$

It is straightforward to verify that the convergence results proved in Theorems 9 and 10 remain unchanged, provided that the three assumptions $H_p, \bar{H}_p$ hold true for any input sequence in its definition set $\bar{U}$ (i.e., uniformly in $\bar{U}$).

6. Simulation results

Both $\Delta$- and $\eta$-observers have been tested on a chaos synchronization problem with uncertain parameters (see De Angeli, Genesio, & Tesi, 1995; Millerioux & Daafouz, 2001). Consider the Hénon system (Hénon, 1976)

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 - a x_2^2(t) + x_2(t) \\ b x_1(t) \end{bmatrix},$$

where $a = 1.4$ and $b = 0.3$, so that the system is chaotic and its trajectories in the strange attractor obey the bounds $|x_1(t)| \leq 1.5$ and $|x_2(t)| \leq 0.4$. We consider here the state reconstruction problem when only $x_1$ is measured and the parameter $b$ is assumed to be unknown, but in a known interval $(\bar{b}, \bar{b})$. In order to meet the global Lipschitz assumption $H_p$, the term $x_2^2$ in (75) is replaced by the function $\sigma_1(x_1)$ defined as

$$\sigma_1(x_1) = dx_1 \tanh(x_1/d),$$

where $d = 10$, so that $|x_2^2 - \sigma_1(x_1)| < 0.017$ for $|x_1| \leq 1.5$, and the system behavior in the strange attractor does not change.

The bounded parameter $b$ is written as a function of a new (constant) state variable $x_3$ as

$$b = \sigma_2(x_3) = \frac{\bar{b} + \bar{b} - b}{2} \tanh(x_3)$$

so that $\sigma_2 : \mathbb{R} \rightarrow (\bar{b}, \bar{b})$. The values $b = 0, \bar{b} = 0.4$ and $b = 0.3$ have been considered in the simulations. The system equations considered for the construction of the observers are

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} 1 - 1.4 \sigma_1(x_1(t)) + x_2(t) \\ x_1(t) \sigma_2(x_3(t)) \\ x_3(t) \end{bmatrix},$$

$$y(t) = x_2.$$

$\Delta$- and $\eta$-observers of orders $\nu = 1, 2, 3$ have been simulated (the matrix functions $F_{\nu,k}(x)$, defined in (26), for $k = 1, 2, 3$ and $h = 1, 2, 3$, have been computed using the symbolic toolbox of MATLAB<sup>®</sup>, by reproducing exactly the recursive computations in (26)). The gain matrix $K$ of the $\Delta$-observer used in the simulations assigns eigenvalues $\lambda = (0.1, 0.12, 0.14)$ to the Brunovsky pair $(A_0, C_0)$ of dimension $n = 3$. As a general result, the simulations confirm that higher order observers obtain faster convergence than lower order ones. In some cases of large initial observation error, low order observers may not converge. Moreover, the $\eta$-observer always achieves a faster convergence. In most cases, the performance of the $\eta$-observer of order $\nu = 2$ is significantly better than the observer of order $\nu = 1$, and an almost perfect state reconstruction is achieved after only two or three time steps. Figs. 1–4 report simulations where the initial states of the system (78) at $t_0 - n + 1$ (with $n = 3$) and of the observers at $t_0$ are

$$x(t_0 - 2) = \begin{bmatrix} 0.8 \\ 0.15 \\ 0 \end{bmatrix}, \quad \hat{x}(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

where $\hat{x}_3 = \sigma_{\nu-1}^{-1}(b) = 0.5493$ (recall that $b = 0.3$). Figs. 1–2 report the true and observed state components $x_1$ (the unmeasured state) and $x_3$ (the constant parameter to be estimated) for the $\Delta$-observers of orders $\nu = 1, 2, 3$, while Figs. 3–4 report the same data for the $\eta$-observers. The faster convergence of the $\eta$-observer is evident, and the increasing convergence rate is also clear in Figs. 3–4.

7. Conclusions

Two families of observers for discrete-time nonlinear systems are presented and compared in this paper. The first one
(Δ-observers) has been proposed by the authors in a pre- previous work, while the second family (η-observers) is a new improved version. Both observer structures are based on the Taylor approximation of a given degree ν of the inverse of the observation map. It is shown that, under suitable assumptions of global analyticity of the system, for any given arbitrary bound on the initial observation error, an observer in both families can be chosen, with ν large enough to ensure the error decay at a desired exponential rate (semiglobal exponential convergence). A computable majorant of the state observation error is available for the η-observers. Computer simulations show that η-observers provide better performances in terms of convergence rates and of sizes of the convergence regions.

Appendix. Kronecker formalism in Taylor series

The formalism of the Kronecker algebra can be used for representing derivatives of matrix functions of several variables, and provides a compact notation for multivariable Taylor series (more details can be found in Germani and Manes (2009)). Let \( M(x) \) be a smooth matrix function of \( x \in \mathbb{R}^n \), i.e.: \( M : \mathbb{R}^n \to \mathbb{R}^{m \times n} \), and let \( \nabla_x \otimes M : \mathbb{R}^n \to \mathbb{R}^{n \times m \times n} \) denote the following matrix of derivatives:

\[
\nabla_x \otimes M = \begin{bmatrix}
\frac{\partial M}{\partial x_1} & \cdots & \frac{\partial M}{\partial x_n}
\end{bmatrix}.
\]

The derivative (A.1) is a formal Kronecker product of the vector \( \nabla_x = [\partial / \partial x_1 \cdots \partial / \partial x_n] \) with the matrix \( M(x) \). Higher order derivatives can be recursively defined as

\[
\nabla^{[i]}_x \otimes M = M, \quad \nabla^{[i+1]}_x \otimes M = \nabla_x \otimes (\nabla^{[i]}_x \otimes M), \quad i \geq 0.
\]

Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a smooth square nonlinear map. In this case \( \nabla_x \otimes F \) is the standard Jacobian of \( F \), and can be written simply as \( \nabla_x F \).

Using the Kronecker notation, the Taylor expansion of \( x = F(z) \) can be written as

\[
F(z) = \sum_{k=0}^{\nu} \frac{1}{k!} (\nabla^{[k]}_z \otimes F(z)) z_0^k (z - z_0)^k + r_{\nu+1}(z, z_0),
\]

where the remainder is such that \( \|r_{\nu+1}\| \leq \sigma_{\nu+1} \|z - z_0\|^{\nu+1} \) in a closed neighborhood \( S_0 \) of \( z_0 \), where

\[
\sigma_{\nu+1} = \frac{1}{(\nu + 1)!} \sup_{z \in S_0} \|\nabla^{[\nu+1]}_z \otimes F(z)\|.
\]

If \( F(z) \) is the inverse of the observation map (9), i.e. \( F(z) = \Phi^{-1}(z) \), the expansion (A.3) is written as in (18). The matrix derivatives (16) can be recursively defined as

\[
G_0(z) = \Phi^{-1}(z), \quad G_{\nu+1}(z) = \nabla_z \otimes G_\nu(z), \quad k = 0, 1, \ldots
\]

By repeatedly differentiating the identity \( G_0 \circ \Phi(x) = x \) (i.e., formally multiplying by \( \nabla_x \otimes \)), the following identities are obtained (see Germani & Manes, 2009):

\[
\nabla_x \otimes (G_0 \circ \Phi(x)) = (G_1 \circ \Phi) F_{1,1} = I_n,
\]

\[
\nabla^{[k]}_x \otimes (G_0 \circ \Phi(x)) = \sum_{h=1}^{k} (G_0 \circ \Phi) F_{h,k} = 0,
\]

where the matrix functions \( F_{h,k}(x) \) are

\[
F_{h,k}(x) = \nabla_x \otimes F_{h,k-1}(x) + (\nabla_x \Phi(x)) \otimes F_{h-1,k-1}(x),
\]

\[
h = 1, \ldots, k - 1,
\]

\[
F_{k,k}(x) = (\nabla_x \Phi(x))^{[k]}, \quad k = 1, 2, \ldots
\]

initialized with \( F_{0,k}(x) = 0 \), for all \( k \in \mathbb{N} \).
The identities (A.6)–(A.6) can be derived by repeatedly applying the following differentiation rules:
\[ \nabla_x \otimes (A(x) \cdot B(x)) = (\nabla_x \otimes A)(I_x \otimes B) + A(\nabla_x \otimes B), \]
\[ \nabla_x \otimes (H \circ \Phi) = (\nabla_x \otimes H) \circ \Phi(\nabla_x \otimes \Phi^{-1}) \cdot I_x, \]
where \( A, B \) and \( H \) are matrix functions, \( H \circ \Phi \) denotes the composition \( H(\Phi(x)) \), and the term \( (\nabla_x \otimes H) \circ \Phi(x) \) denotes the derivative of \( H \) with respect to \( x \), computed at \( x = \Phi(x) \) (detailed computations are reported in Germani and Manes (2009)).

Eqs. (A.6) and (A.7) can be solved for \( G_x \circ \Phi(x) \) to provide the recursion (22), that shows how the coefficients of the Taylor expansion of the inverse map \( \Phi^{-1}(z) \) around a point \( \bar{z} = \Phi(x) \) can be computed without the explicit knowledge of the map \( \Phi^{-1} \).

References

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