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Free choice permission as resource-sensitive reasoning*

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Abstract Free choice permission is a long-standing puzzle in deontic logic and in natural language semantics. It involves what appears to be a conjunctive use of *or*: from *You may eat an apple* ***or*** *a pear*, we can infer that *You may eat an apple* and that *You may eat a pear*—though not that *You may eat an apple* ***and*** *a pear*. Following Lokhorst (1997), I argue that because permission is a limited resource, a resource-sensitive logic such as Girard's Linear Logic is better suited to modeling permission talk than, say, classical logic. A resource-sensitive approach enables the semantics to track not only that permission has been granted and what sort of permission it is (i.e., permission to eat apples versus permission to eat pears), but also *how much* permission has been granted, i.e., whether there is enough permission to eat two pieces of fruit or only one. The account here is primarily semantic (as opposed to pragmatic), with no special modes of composition or special pragmatic rules. The paper includes an introduction to Linear Logic.

Keywords: Free choice, permission, linear logic, deontic, implicature, resource-sensitive, substructural

1 The resource-sensitivity of permission talk

Since Ross 1941, it has been clear that the logic of obligation and permission behaves dramatically differently than other sorts of ordinary reasoning:

(1) a. You may eat an apple or a pear.
    b. You may eat an apple.
    c. You may eat a pear.

If (1a) is true, then it is certainly true that you may eat an apple. Likewise, it is equally true that you have it within your power to safely eat a pear. So an

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adequate account of the meaning of (1a) must explain how it comes to imply (1b) and (1c).

This pattern is by no means the usual case. Consider a variation on (1) in which the permissive modal *may* is omitted:

(2) a. You ate an apple or a pear.
b. You ate an apple.
c. You ate a pear.

In this case, (2a) certainly does not imply either (2b) or (2c). So something about permission talk correlates with the unusual implications we are concerned with here.

The puzzle posed by the facts in (1) is known as the free choice permission problem (Kamp (1973) attributes the choice of name to von Wright).

Since (1a) implies both (1b) and (1c), (1b) and (1c) are therefore both equally true. Thus in many discussions, (1a) is said to imply (3a), since (3a) is merely the conjunction of (1b) and (1c):

(3) a. You may eat an apple and you may (also) eat a pear.
b. You may eat an apple or you may (*also) eat a pear.

Crucially, however, (3a) has an interpretation on which it furnishes permission to eat more than one piece of fruit. This interpretation is the one compatible with adding *also* in the second conjunct. Now, although (1a) may be consistent with a situation in which the addressee is allowed to eat more than one piece of fruit (as we will see below), the truth of (1a) alone is never sufficient to guarantee that more than one piece of fruit may be eaten. As a result, (3b) is a better candidate for a paraphrase of (1a): it, too (surprisingly!) implies (1b) and (1c), but, like (1a), it does not ever justify eating more than one piece of fruit. This is why *also* is never appropriate in the second disjunct in (3b) on the intended reading.

What I am suggesting is that a complete characterization of permission sentences must not only tell us whether permission exists and what type of permission it is (i.e., permission to eat an apple versus permission to eat a pear), it must also characterize *how much* permission has been granted. Thus it must predict that (1a) and (3b) guarantee permission only to eat one piece of fruit, but that (3a) can be used to provide permission to eat two pieces of fruit.

The key insight that I would like to develop in this paper first appears, as far as I know, in unpublished work of Lokhorst (Lokhorst(1997)): that permission and obligation is a resource-sensitive domain, so that logics based on (resource-insensitive) classical logic are not appropriate. Lokhorst suggests using Girard's (1987) Linear Logic instead, and I will follow the technical details of his proposal closely. The contribution of this paper will be to introduce Lokhorst's work to a linguistic audience, to evaluate it with respect to competing linguistic analyses, and to investigate the implications of adapting Lokhorst's proposal for the theory of natural language semantics and pragmatics.

Resource-sensitive ('substructural') logics are already familiar in linguistics as tools for building syntax/semantics interfaces (e.g., Moortgat 1997 or Dalrymple 2001). As far as I know, however, no one has yet suggested that natural language connectives such as *or* or *and* can have uses in which they behave semantically like connectives in a substructural logic, as I am suggesting here.

Kamp (1973, 1978) discusses free choice permission not just as a puzzle for modeling reasoning about obligation (deontic logic), but as a puzzle for the composition of natural language expressions. From the point of view of natural language semantics, the interesting thing about the free choice permission problem is that it appears to require not only making assumptions about the meaning of certain uses of modal expressions such as *may*, but about the meaning of the corresponding uses of the coordinating conjunctions *and* and *or*. This will be true of the solution I offer below.

Many solutions to the free choice permission problem rely on pragmatic mechanisms for much of the heavy lifting, including Kamp 1978, Zimmermann 2000, Fox 2007, and others. The arguments that free choice implications are pragmatic, and more specifically are scalar implicatures, stem from discussions of indefinites in Kratzer and Shimoyama 2002, as developed by Alonso-Ovalle (2006) and Fox (2007). The main evidence that free choice implications may be scalar implicatures turns on the behavior of negated permission sentences (*You may not eat an apple or a pear*); I show how the analysis here can explain the behavior of such sentences in section 5.

In contrast to the pragmatic approaches, I will argue that the main free choice implications, including especially the implications from (1a) to (1b) and to (1c), are matters of entailment. To the extent that the analysis here is viable, it calls into question whether free choice implications are indeed implicatures. I discuss other entailment approaches (e.g., Aloni 2007) in section 6.2.

2 Classical logic versus Linear Logic

The account of free choice given below will depend on understanding the basics of Linear Logic at a fairly deep level. Since Linear Logic is unfamiliar to most semanticists, this section will present the basics of Linear Logic.

2.1 Classical logic

I will only introduce the elements of classical logic that will be relevant for comparison with Linear Logic in the discussion below. This will include conjunction, disjunction, negation, and Weakening, but not, for example, quantification.

Formulas. There is a set of atomic formulas $a, b, c, ...$, and a set of variables over formulas $A, B, C, ...$ Assume A and B are formulas. Then the classical negation of A, written $\neg A$, is a formula; the classical conjunction of A and B, written $A \wedge B$, is a formula; and the classical disjunction of A and B, written $A \vee B$, is a formula. In addition, the classical implication of A and B, written, $A \rightarrow B$ is defined as an abbreviation of $(\neg A) \vee B$.

Sequents. A sequent $A, B, ..., M \vdash N, O, ..., Z$ consists of two multisets of formulas joined by a turnstyle ('$\vdash$'). Classical sequents are interpreted as asserting that whenever all of the formulas in the leftmost multiset hold, then at least one of the formulas in the rightmost multiset must also hold. Saying that a sequent contains multisets rather than lists of formulas means that the order in which formulas are written is immaterial. Thus A, B and B, A represent the same multiset, but A, B is a different multiset than A, A, B, since the second multiset contains two instances of the formula A.

Capital Greek letters (Δ, Γ, ...) schematize over (possibly empty) multisets of formulas. The turnstyle can occur in any position, and there can be more than one formula on the right hand side, so that the expression '$\Delta \vdash A, B$', the expression '$\Delta \vdash$', and the expression '$\vdash \Delta$' are all legitimate sequents.

Negation. The following pair of inference rules characterize classical negation:

$$\frac{\Delta, A \vdash \Gamma}{\Delta \vdash \neg A, \Gamma}\,\neg_1 \qquad \frac{\Delta \vdash A, \Gamma}{\Delta, \neg A \vdash \Gamma}\,\neg_2$$

Beginning with $\neg_1$, the inference rule on the left: if Γ follows from the formulas in Δ along with A (this is what the sequent above the horizontal line expresses), then from Δ alone we can conclude that either some member of Γ is still true, or else A must be false (the sequent below the horizontal line). Similar reasoning applies for the inference rule on the right, $\neg_2$.

Proofs. A proof that a sequent is valid begins with trivial tautologies, here, that $A \vdash A$:

$$\cfrac{\cfrac{A \vdash A}{\vdash \neg A, A}\,\neg_1}{\neg\neg A \vdash A}\,\neg_2$$

As long as each subsequent inference step instantiates a valid inference rule, the proof guarantees that the final sequent will also be valid. A sequent at the bottom of such a proof is called a theorem of the logic.

Reading from top to bottom, the first step of the proof here is an instantiation of the inference rule $\neg_1$. This step concludes that either A or its negation must be true (a version of the law of excluded middle); the second step (labeled $\neg_2$) proves that two adjacent negations cancel out (the law of double negation). Proving that $A \vdash \neg\neg A$ is equally easy.

Conjunction. The inference characterizing classical conjunction has two premises:

$$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \wedge B} \wedge$$

If the assumptions in Δ allow you to prove that A is true (i.e., if $\Delta \vdash A$), and the very same set of assumptions also allow you to prove that B is true, then you are certainly in a position to assert that the classical conjunction of A and B must be true.

Disjunction. For disjunction, we have a matched pair of inferences:

$$\frac{\Delta \vdash A}{\Delta \vdash A \vee B} \vee_1 \quad \frac{\Delta \vdash B}{\Delta \vdash A \vee B} \vee_2$$

If the assumptions in Δ allow you to prove that some proposition A is true, you can conclude that the classical disjunction of A and B is true. After all, if you know that Ann arrived, then you know that either Ann arrived or Bill arrived. The reason we need a pair of rules is that disjunction is symmetric, i.e., we are free to add the new disjunct either on the left or on the right.

The classical duality of conjunction and disjunction. The following equivalences hold:

(4) a. $\neg\neg A \equiv A$
  b. $\neg(A \wedge B) \equiv \neg A \vee \neg B$
  c. $\neg(A \vee B) \equiv \neg A \wedge \neg B$

The last two (DeMorgan's laws) express the logical interrelationship between disjunction and conjunction. These equivalences can be thought of as bi-directional inference rules. In any case, I will freely replace formulas with forms deemed equivalent by (4).

Weakening. Weakening allows assumptions to be discarded.

$$\frac{\Delta \vdash \Gamma}{\Delta, A \vdash \Gamma} \text{Weak}$$

If Γ follows from Δ, then Γ certainly still follows if A also happens to be true, no matter what A happens to express. The assumption A is gratuitous, but harmless. Weakening allows us to pick and choose among evidence as we focus on different parts of an argument.

Implication as a form of disjunction. Recall that in the definitions of well-formed formulas, we defined classical implication $A \rightarrow B$ as an abbreviation of $\neg A \vee B$. The inference rule that characterizes implication is Modus Ponens, which says that $A, A \rightarrow B \vdash B$ is valid. We can prove Modus Ponens as follows. The main aspect of the proof that is relevant for comparison with Linear Logic is the role of Weakening.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \neg B \vdash A}\text{Weak} \quad \dfrac{\neg B \vdash \neg B}{\neg B, A \vdash \neg B}\text{Weak}}{\neg B, A \vdash A \wedge \neg B}\wedge}{A, \neg(A \wedge \neg B) \vdash \neg\neg B}\neg_1, \neg_2}{A, A \rightarrow B \vdash B}\equiv$$

Wadler (1993) uses classical modus ponens in the following proof to emphasize the differences between classical logic and Linear Logic:

$$\dfrac{\dfrac{A \vdash A}{A, A \rightarrow B \vdash A}\text{Weak} \quad \dfrac{[\text{see previous proof}]}{A, A \rightarrow B \vdash B}}{A, A \rightarrow B \vdash A \wedge B}\wedge$$

Weakening allows us to make use of assumption A twice: once to justify the left conjunct of the conclusion, and once to support modus ponens in order to derive the right conjunct of the conclusion. We will see that Linear Logic requires careful accounting: each assumption can be used exactly once, so this proof will not go through.

Finally, completing the $\neg, \wedge, \vee$ fragment of classical logic requires Contraction: from $\Delta \vdash A, A$, infer $\Delta \vdash A$. In Linear Logic, Contraction is also rejected, but Contraction does not play a role in the exposition here.

2.2 Linear Logic

Formulas. Once again there is a set of atomic formulas $a, b, c, ...$, and a set of variables over formulas $A, B, C, ...$. However, since none of the Linear Logic connectives mean what their classical counterparts mean, Linear Logic uses a completely distinct set of connective symbols. Assume A and B are formulas. Then the linear negation of A, written $A^{\perp}$, is a formula; the additive conjunction of A and B, written $A \,\&\, B$ (pronounced "A with B") is a formula; the multiplicative conjunction of A and B, written $A \otimes B$ (pronouns "A times

B") is a formula; the additive disjunction of A and B, written $A \oplus B$ (pronounced "A plus B") is a formula; and the multiplicative disjunction of A and B, written $A \parr B$ (pronounced "A par B") is a formula. (Many things in natural language semantics are called 'additive'. The Linear Logic notions of 'additive' and 'multiplicative' do not line up with any of them.) In parallel with the definition of classical implication above, linear implication, written $A \multimap B$ (pronounced "A lollipop B"), is defined as an abbreviation for $A^{\perp} \parr B$.

Sequents. A sequent $\Delta \vdash \Gamma$ says that whenever the multiplicative conjunction of Δ holds, then the multiplicative disjunction of Γ must hold.

Fragment of Linear Logic for the free choice permission problem. Here is the complete set of rules of Linear Logic that we will use in the discussion of the free choice permission problem, gathered together:

Fragment of Linear Logic for FCP

$$\frac{\Delta, A \vdash \Gamma}{\Delta \vdash A^{\perp}, \Gamma}\ \perp_1 \qquad \frac{\Delta \vdash A, \Gamma}{\Delta, A^{\perp} \vdash \Gamma}\ \perp_2$$

$$\frac{}{A \vdash A}\ \text{Axiom}$$

$$A \multimap B \equiv A^{\perp} \parr B$$

$$A^{\perp\perp} \equiv A$$

$$(A \,\&\, B)^{\perp} \equiv A^{\perp} \oplus B^{\perp} \qquad (A \otimes B)^{\perp} \equiv A^{\perp} \parr B^{\perp}$$

$$(A \oplus B)^{\perp} \equiv A^{\perp} \,\&\, B^{\perp} \qquad (A \parr B)^{\perp} \equiv A^{\perp} \otimes B^{\perp}$$

$$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \,\&\, B}\ \& \qquad \frac{\Delta \vdash A \qquad \Gamma \vdash B}{\Delta, \Gamma \vdash A \otimes B}\ \otimes$$

$$\frac{\Delta \vdash A}{\Delta \vdash A \oplus B}\ \oplus_1 \qquad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B}\ \oplus_2 \qquad \frac{\Delta \vdash A, B}{\Delta \vdash A \parr B}\ \parr$$

Linear conjunction and disjunction. The rules for & and $\oplus$ (the 'additive' connectives) look exactly like the classical rules for $\wedge$ and $\vee$, except for the substitution of & for $\wedge$ and of $\oplus$ for $\vee$. However, as a result of how they interact with the rest of the logic, the linear logic additives behave differently from their classical counterparts. For instance, the law of the excluded middle is valid for classical disjunction: $\vdash (\neg A) \vee A$. In Linear Logic, the law of excluded middle is not valid for additive disjunction, despite the fact that the inference rule for additive disjunction has the same form as the inference rule for classical disjunction: $\not\vdash A^{\perp} \oplus A$. However, the excluded middle is valid for multiplicative disjunction ($\vdash A^{\perp} \parr A$).

Linear negation. We have direct analogs to the classical rules for pushing a formula across the turnstyle, namely, $\perp_1$ and $\perp_2$. Since we now have two kinds of conjunctions and two kinds of disjunctions, there are more duality equivalences; however, each conjunction is still dual to a disjunction, and vice-versa.

Linear implication. Once again, we have defined implication in terms of disjunction. Now, interestingly, we can prove the linear version of Modus Ponens without using Weakening (which is a good thing, since Weakening is not allowed in Linear Logic):

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B^{\perp} \vdash B^{\perp}}{A, B^{\perp} \vdash A \otimes B^{\perp}}\otimes}{A, (A \otimes B^{\perp})^{\perp} \vdash B^{\perp\perp}}\perp_1, \perp_2}{A, A \multimap B \vdash B}\equiv$$

Because the inference rule for $\otimes$ splits up the resources (that is, the formulas) into those used to prove A and those used to prove B, there is no need to ignore gratuitous assumptions via Weakening.

If we try to reproduce Wadler's classical proof from the previous section, we're out of luck:

$$\dfrac{?? \vdash A \qquad ?? \vdash B}{A, A \multimap B \vdash A \otimes B}\otimes$$

We could take some of the resources to the left of the turnstyle to prove A, and we could take some (actually, we would need all) of the resources to prove B, but no matter how we divide up the left-hand formulas, we'll fall short of proving one or the other of the conjuncts. Linear Logic requires strict accounting of assumptions, and we can't make use of A twice, the way we could in the classical proof.

2.3 Choice

Since free choice permission is about making choices, what does Linear Logic have to say about choice?

The critical connectives will be the additive conjunction '&' and its (also additive) disjunctive dual, '$\oplus$'. The relevant inference rules are repeated here:

$$\frac{\Delta \vdash A \qquad \Delta \vdash B}{\Delta \vdash A \mathbin{\&} B}\,\& \qquad\qquad \frac{\Delta \vdash A}{\Delta \vdash A \oplus B}\,\oplus_1 \quad \frac{\Delta \vdash B}{\Delta \vdash A \oplus B}\,\oplus_2$$

Imagine yourself in the role of the prover. Then the assumptions on the left of the turnstyle are what your environment gives you to work with, and the conclusion on the right of the turnstyle is what you return as the result of your labors (perhaps to be used as an assumption in a larger proof).

So here is what the & inference says: if the resources in Δ allow you to provide A, and if the same resources allow you to provide B, then you can certainly offer to provide either A or B. Furthermore, since you are prepared to provide either alternative, you can leave the choice up to whoever might be interested in making use of the conclusion. Thus & conjoins two equally viable alternatives.

Though both alternatives are equally viable, the consumer is forced to choose between them. For instance, imagine that Δ contains a certain amount of sugar and a certain number of eggs. Using the resources provided, you can construct either a meringue or else an angel food cake, but you don't have enough ingredients to cook both. Being as flexible and gracious as possible, you offer "meringue & cake" for dessert, and you let your guest choose. Tellingly, "meringue & cake" is pronounced "meringue **or** cake" in idiomatic English (this is a point that we will return to in section 7.3).

In the context of granting permission, the consumer is the entity to which permission has been granted: we shall see that (unembedded) & corresponds to free choice on the part of the entity given permission.

Continuing with our investigation of choice in Linear Logic, turning to the $\oplus_1$ inference rule, if the resources in Δ allow you to provide A, then you can certainly offer to provide either $A \oplus B$—as long as you remain in control of which of the alternatives is chosen. You may only know how to make one dessert, perhaps. You can truthfully promise that dessert will either be meringue or else Baked Alaska, although you know in advance that it will have to be meringue. (Analogously with the roles reversed for $\oplus_2$.)

In the context of granting permission, offering $A \oplus B$ does not give the grantee free choice.

In order to complete the picture of the dualities of & and $\oplus$, we must consider what happens on the other side of the turnstyle. Hopping across the turnstyle involves negation, which exchanges & for $\oplus$ (and vice versa).

$$\frac{A \vdash \Delta}{A \mathbin{\&} B \vdash \Delta} \quad \frac{B \vdash \Delta}{A \mathbin{\&} B \vdash \Delta} \qquad\qquad \frac{A \vdash \Delta \qquad B \vdash \Delta}{A \oplus B \vdash \Delta}$$

These rules follow from the official inference rules by applications of $\perp_1$ and $\perp_2$.

If A alone is enough to enable you to provide Δ, then if someone promises you $A \,\&\, B$, you can certainly commit to providing Δ: just select A when they give you your choice. (Similarly for the other rule introducing & on the left of the turnstyle.)

Finally, if having A is enough for you to be able to offer Δ, and if having B is likewise enough for you to be able to offer Δ, then you're in a position to promise Δ even if all you can count on is $A \oplus B$. All you know is that you'll get either an A or a B, and that which one you get will be someone else's choice. However, since you are prepared to cope with either possibility, you can commit to providing Δ.

The bottom line is that & and $\oplus$ are two perspectives on a single choice, differing only in who has the power to make the selection: & provides two equally legitimate alternatives, but forces an unconstrained (free) choice between them; $\oplus$ also provides two alternatives, but reserves the choice for whoever is providing the resource.

3 Strong permission versus weak permission

Standard deontic logics introduce unary modalities representing obligation ($\Box$) and permission ($\Diamond$), and add axioms that characterize an appropriate set of entailments, usually including at least K and D, though there is considerable variation; see McNamara 2006 or Portner 2009a for an introduction to deontic logic. Lokhorst (1997) chooses instead a strategy attributed independently to Anderson and to Kanger called DEONTIC REDUCTION. Deontic reduction depends on a special proposition δ (pronounced "yay"), glossed as 'the good thing', or 'all things are as required'. Thus δ is roughly analogous to Kratzer's (e.g., 1991) notion of an ordering source, that is, the set of propositions that characterize how things ought to be.

Then A is obligatory iff $\delta \multimap A$: if A follows from the state where all things are as required, then A is required. Dually, a weak version of permission is often defined as $(\delta \multimap A^\perp)^\perp$: if the negation of A is not obligatory, then A is at least not forbidden. However, there is a difference between weak permission, which is the absence of prohibition, and strong permission, i.e., a permissive norm (as discussed in, e.g., Hansen et al. 2007), which is the assertion that some action is explicitly ok.

Lokhorst (1997) renders strong permission as $A \multimap \delta$. Viewed from the linguistics tradition, it is not so easy to make sense out of this as a statement of permission (as discussed in Portner 2009a:60). It is important to bear in mind that the 'strong' part of 'strong permission' does not mean that merely eating an apple will guarantee that everything is ok, no matter what else happens. If only permission could be that strong! Rather, the difference between 'weak' and 'strong' here is the difference between a system in which we have only obligation and its negation (in which everything that is not forbidden is permitted), and a more articulated system in which some things are permitted ($A \multimap \delta$), some things are forbidden ($(A \multimap \delta)^{\perp}$), and some things are neither permitted nor forbidden. If I explicitly give you permission to eat an apple, and I explicitly forbid you to eat a pear, what about eating a banana? Is it permitted or forbidden? Maybe yes, maybe no.

There is not much discussion of weak permission versus strong permission in the linguistics literature, but at least Asher and Bonevac (2005) conclude that free choice permission involves strong permission. Certainly if we want to distinguish between explicit permission and the absence of prohibition, then we need a logic that can express strong permission. Since I have claimed that *You may eat an apple or a pear* crucially neither permits nor forbids eating both an apple and a pear, we must use strong permission here.

But what exactly does $A \multimap \delta$ assert, if not that eating an apple will guarantee the good thing? The key is to consider when $A \multimap \delta$ will be true. We will be in a situation in which $A \multimap \delta$ just in case eating an apple in that situation is compatible ('cotenable' in the terminology of Relevant Logic) with all obligations being fulfilled. There are two kinds of such situations: situations in which eating an apple happens to be obligatory, in which case we can only conform to obligations by eating the apple (after all, everything that is obligatory is at least permitted); and situations in which we're already in compliance, but eating an apple is optional and does not disturb our happy state. But if we are otherwise in compliance, and we decide to eat an apple (A), and we decide to simultaneously kill the postman (K), the fact that apple eating is permitted will not save us: because of the resource-sensitivity of linear logic, in particular, the absence of Weakening, we can't ignore the dead postman. As a result, the combination of eating an apple and killing the postman will land us in a situation that is far from ok: $A, K, A \multimap \delta \nvdash \delta$.

A fuller understanding of linear implication, and therefore of strong permission, will emerge from the model theory developed in section 8.

One major expository advantage of the reduction strategy is that it enables us to talk about permission without complicating the logic with inference rules for $\Box$ and $\Diamond$. Note that we do not necessarily give up anything by omitting the unary connectives: McNamara (2006) and Lokhorst (2006) show that under appropriate additional assumptions, deontic reduction characterizes all the theorems of standard deontic modal logics.

Not that replicating standard deontic logic should be our goal; after all, standard deontic logic has $\Box A \to \neg\Box\neg A$ as a tautology, which imposes a kind of consistency on the set of deontic obligations. In the linguistics tradition, a number of people (notably Kratzer (1991)) have argued that this is not appropriate for describing natural language modality, and that we should instead allow for inconsistent laws. However, I'm not aware of any reason why deontic reduction is incompatible with Kratzer's characterization of deontic modality.

I should note that deontic reduction is not an innocent choice for the empirical phenomena under consideration here. As I will explain shortly, because linear implication is defined as $A \multimap B \equiv A^{\perp} \parr B$, the formula for which permission is granted (i.e., A) occurs in a downward-entailing position. This will be crucial in deriving the desired entailments. For all I know, however, it is possible that if a suitable notion of strong permission were defined in a standard deontic framework (i.e., one based on unary operators like $\Box$), similar entailments would go through.

I intend for deontic reduction to be a convenient expository choice, and not an essential feature of a resource-sensitive approach to free choice permission. Nevertheless, there may be some empirical support for the naturalness of deontic reduction. After all, in addition to being able to use a modal verb to express permission and obligation, English can also deploy a conditional: *It's ok if you eat* 'You may eat'. In fact, in Japanese there is no modal verb that expresses permission, and permission normally can only be conveyed by means of a conditional construction (Clancy 1985, Akatsuka 1992): *tabe-temo ii* 'eat-even.if good', 'It's ok if you eat'.

4 Free choice permission

We can now suppose that *or* has among its meanings $\oplus$, so that *You may eat an apple or$_{\oplus}$ a pear* translates as $(a \oplus p) \multimap \delta$: the additive disjunction of a and p is explicitly permitted. Then the desired free-choice implication follows directly from simple linear reasoning. Generalizing slightly by using variables over formulas (A, B) instead of atomic formulas (a, p), we have:

$$
\begin{prooftree}
\AxiomC{$\vdash A, A^{\perp}$}
\RightLabel{$\oplus_1$}
\UnaryInfC{$\vdash A \oplus B, A^{\perp}$}
\AxiomC{$\vdash \delta^{\perp}, \delta$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash (A \oplus B) \otimes \delta^{\perp}, A^{\perp}, \delta$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash (A \oplus B) \otimes \delta^{\perp}, A^{\perp} \parr \delta$}
\AxiomC{$\vdash B, B^{\perp}$}
\RightLabel{$\oplus_2$}
\UnaryInfC{$\vdash A \oplus B, B^{\perp}$}
\AxiomC{$\vdash \delta^{\perp}, \delta$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash (A \oplus B) \otimes \delta^{\perp}, B^{\perp}, \delta$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash (A \oplus B) \otimes \delta^{\perp}, B^{\perp} \parr \delta$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash (A \oplus B) \otimes \delta^{\perp}, (A^{\perp} \parr \delta) \,\&\, (B^{\perp} \parr \delta)$}
\RightLabel{$\perp_2, \equiv$}
\UnaryInfC{$(A \oplus B) \multimap \delta \vdash (A \multimap \delta) \,\&\, (B \multimap \delta)$}
\end{prooftree}
$$

This theorem is noted in Lokhorst 1997:6.[1]

What the speaker provides when she utters *You may eat an apple or*$_{\oplus}$ *a pear* is justification for assuming either that eating an apple is permitted, or that eating a pear is permitted. She is not providing enough resources to prove both, so if her utterance is to provide the justification for action, a choice must be made. However, since the resources allow proof of either alternative, the consumer is free to choose whichever of the alternatives he prefers. That is how the addressee has permission to eat an apple, or else permission to eat a pear, but normally (and certainly not by virtue of the utterance of (1a)) does not have permission to eat two pieces of fruit.

This result depends on only two assumptions: that *or* can express additive disjunction, and that it is reasonable to represent strong permission using the deontic reduction strategy. The assumption that *or* can express additive disjunction is essential, and is the heart of the explanation offered here. Deontic reduction is a well-established approach to deontic logic motivated entirely independently of any concern with the free choice permission problem. Whether it can be replaced with a modal system more familiar to linguists (if desired) remains for future work.

It is worth emphasizing that the basic free choice meaning is purely semantic, without requiring any silent pragmatically-triggered type shifting operators (as in, e.g., Fox 2007), or other pragmatic enrichment.

5 Prohibition

The behavior of permission under negation plays an important role in recent discussions. As mentioned above, Alonso-Ovalle (2006) and Fox (2007) argue that the fact that free-choice implications seem to disappear under negation

1 Strictly speaking, since the inference rules given above in section 2.2 are written with a single formula on the right-hand side, many of the steps given in this proof (for example, the $\oplus_1$ inference) require shuffling extra formulas across the turnstyle, applying the inference rule of interest, then shuffling them all back.

shows that free choice implications are likely to be implicatures. Since I am claiming that the relevant free choice implications are entailments, it is important to carefully examine negated cases.

Whatever is not permitted is forbidden: just as in English, Lokhorst renders (strong) prohibition as negated (strong) permission. Thus if $(A \multimap \delta)^{\perp}$, then A is prohibited. (It is a well-known property of English that *may not* is always construed with negation taking scope over *may*.)

(5) a. You may not eat this apple or this pear.
 b. You may not eat this apple.
 c. You may not eat this pear.

The main fact to be explained is that (5a) implies (perhaps entails) (5b) and (5c). Unlike positive free choice implications, we can usually infer that (5b) and (5c) hold simultaneously. That is, you cannot comply with (5a) by merely refraining from eating apples. Apparently, permission is a scarce resource, but prohibition is all too abundant. I will call this construal of (5a) the double-prohibition reading, and I will suggest that it arises as a standard Gricean implicature.

As with most stories about scalar implicatures, we will be concerned with the epistemic state of the discourse participants.

(6) a. You may not eat this apple or this pear.
 b. You may not eat this apple or you may not eat this pear.
 c. $((A \oplus B) \multimap \delta)^{\perp} \vdash (A \multimap \delta)^{\perp} \oplus (B \multimap \delta)^{\perp}$

The translation of (6a) entails the translation of (6b) (that is, (6c) is a theorem), so we predict that (6a) ought to have an interpretation on which it guarantees that (6b) is true. Such an interpretation is widely attested in the literature, and usually is described as favoring the continuation *... but I don't know which.* I'll call this the ignorance reading.

Note, by the way, if a forgetful babysitter utters (6) to the child she is babysitting, if the child behaves rationally, he will not eat either piece of fruit, since he can't be sure which action is safe—exactly the same behavior as if both actions had been explicitly forbidden.

So far, so good. Next, consider a situation in which the speaker is not ignorant. Exactly one of the alternatives is prohibited, and this time the speaker knows which one it is. Let's say that apple-eating is forbidden, but pear eating is fine. If the speaker were being fully cooperative, then she would normally choose to simply say (5b), and certainly would not choose to say (5a). In Gricean terms, adding a superfluous disjunct would violate either the maxim of Quantity, or the maxim of Manner, or both.

There are nevertheless situations in which this kind of uncooperative statement might be used. For instance, if a father tells an older sister the rules ("apples forbidden, pears ok"), she might later uncooperatively tell her younger brother

(7) You may not eat this apple or this pear ... but I won't tell you which.

Once again, the rational course of action on the part of the younger sibling will be to refrain from eating either piece of fruit. Presumably this is exactly the outcome the unkind sister is aiming for. (I'm indebted to Sven Lauer for this scenario; see also Simons 2005:273n.4.)

In both the ignorance scenario and the uncooperative scenario, at least one of the disjuncts holds, but the choice of which fruit is prohibited belongs to the master, not the slave. The subject of the prohibition must plan for the worst, and therefore can't safely commit to either alternative.

Finally, imagine that the speaker is neither ignorant nor uncooperative. She may be an expert (perhaps she just received full instructions from the parents) or she may be herself the source from which permission flows; in any case, she is fully opinionated about what is forbidden. Crucially, although (6) guarantees only one disjunct, it is consistent with situations in which both disjuncts hold. As just argued, if exactly one disjunct held, the speaker would simply have said so. We can deduce, therefore, that both disjuncts must hold.

There is one more step to complete the Gricean explanation. If the speaker intends to convey double prohibition, why not use *and*?

(8) You may not eat an apple and a pear.

Although this sentence may have the desired double-prohibition reading, it certainly also has a reading on which it prohibits (only) complex events that involve eating both an apple and a pear. Uttering (8), then, leaves in play the possibility that eating a single piece of fruit may be permitted. The speaker uses a weak form in (6) to express a stronger meaning in order to avoid misinterpretation.

Thus the assumption that the speaker is opinionated and cooperative derives the implicature that both disjuncts are prohibited via ordinary Gricean reasoning, without the need to stipulate any special uniformity or distributivity axioms (as in Alonso-Ovalle 2006) or Zimmermann's 2000:286 Authority Principle.

6 Comparisons with other accounts

6.1 Implicature accounts

A number of authors, including Schulz (2005) and Fox (2007), suggest that free choice implications are implicatures that arise in contexts in which the speaker is opinionated about which options are permitted and which are not. Fox (2007) reasons as follows: if a speaker utters a disjunction when she could have made a stronger statement, this could naturally lead to a Quantity implicature that she did not have sufficient evidence to assert the stronger statement. If those ignorance implicatures are implausible, as when the speaker is describing permissions in a situation in which their judgment is authoritative, the implausibility can trigger a repair strategy under which the disjunction is pragmatically enriched by the application of a predicate `exh` (for "exhaustive"). For instance, if an authoritative speaker says *You may eat an apple or a pear*, it may be implausible that she doesn't know whether you may eat an apple, or whether you may eat a pear. Therefore the statement $\Diamond(\mathsf{A} \vee \mathsf{P})$ can be strengthened (given a number of additional assumptions) to an exhaustive meaning equivalent to the proposition $\Diamond\mathsf{A} \wedge \Diamond\mathsf{P} \wedge \neg(\Diamond(\mathsf{A} \wedge \mathsf{P}))$. This asserts that you may have an apple, and you may have a pear, but you may not both have an apple and a pear.

I will discuss three potential problems with these accounts. The first problem is that the free-choice reading can survive even in the presence of manifest ignorance on the part of the speaker:

(9) I don't know whether you may have an apple or a pear.

Since exhaustivity is supposed to be triggered by contexts that are incompatible with ignorance, (9) should only have a reading on which it means 'I don't know whether you may have an apple or whether you may have a pear'. But (9) robustly also has a free-choice reading on which it means 'I don't know whether you may eat a piece of fruit, where the fruit is your choice between an apple or a pear'.

(10) If it turns out that John may have an apple or a pear, he'll choose the pear.

Likewise, as Kamp (1978:279) notes, free choice interpretations remain available for the antecedent of a conditional, where it is far from clear how assumptions about complete knowledge of the alternatives could enter in.

The second problem is that if free choice implications were implicatures, we should expect them to be generally cancelable:

(11) You may eat an apple or a pear, although in fact you may not eat an apple.

Probably (11) has a non-free choice reading on which it is at least logically consistent. If this were the basic semantic meaning of (11), then we would expect it to emerge whenever the free-choice implication is cancelled. The puzzling thing is that if we assume the speaker is opinionated, (11) gives a strong impression of contradiction rather than of a cancelled implicature.

Chemla (2009a, 2009b) proposes a pragmatic principle that he calls symmetry, which says that the epistemic attitude of the speaker must be uniform across disjuncts. Symmetry correctly predicts that (11) should be infelicitous, since it implies that the speaker holds a different attitude towards one disjunct than towards the other. However, symmetry alone cannot explain why (11) sounds contradictory.

One possibility is that performativity is interfering. Portner (2009b) suggests that performative uses (see section 7.2 below) force, or at least strongly promote, a free choice interpretation. If so, then what (11) shows is that at least when an utterance is performative, free choice implications cannot be cancelled.

The third problem applies to Fox's account, though not to Schulz's: as Fox himself notes, the proposed implicatures for the free-choice reading do not match intuitions about the meanings of the sentences in question. Fox's `exh`-enhanced truth conditions assert that eating an apple is permitted, and eating a pear is permitted, but eating an apple and a pear is forbidden. But as Simons 2005 and others observe, free choice is compatible with joint permission. For instance,

(12) [You may eat as much fruit as you want, so]
You may (certainly) eat an apple or a pear.

On Fox's account, (12) should be contradictory on a free-choice reading of the final clause. However, although (12) may be mildly redundant, there is no hint of contradiction.

Franke 2009:8 and van Rooij 2010:18 derive results similar to Fox's by using a particular game-theoretic technique ("Iterated Best Response") to compute implicatures. One advantage of their approach is that the proposition that eating both an apple and a pear is forbidden arises as an implicature only when certain alternatives are salient, correctly predicting that (12) need not be a contradiction.

On the account here, of course, the explanation for the fact that (12) is not a contradiction is particularly simple and direct: *You may eat an apple or a pear* entails that you may eat an apple, and that you may eat a pear, but refrains from saying anything about whether it's ok to eat both an apple and a pear. It neither grants permission to eat two pieces of fruit, nor forbids it.

Van Rooij frames the comparison between exhaustivity and game theory as part of the debate about embedded implicatures: if free choice implications can be handled using iterated best response, then free choice no longer provides an argument that implicatures must be calculated locally (i.e., in embedded contexts). The resource-sensitive approach here weakens the argument that free choice motivates embedded implicatures even further, by calling into question whether free choice implications are implicatures in the first place.

6.2 Alternative set semantics

Zimmermann 2000 proposes that disjunction contributes a set of exhaustive epistemic alternatives, so that *You may eat an apple or you may eat a pear* expresses the claim that it is possible that you may eat an apple and it is possible that you may eat a pear. Novel pragmatic principles (notably his Authority Principle) strengthen this conjunction into an assertion that you may eat an apple and you may eat a pear.

Geurts 2005 elaborates on Zimmermann's analysis, arguing that disjunctive alternatives should not always be epistemic. Rather, disjunction "fuses" with nearby modal operators, so that *You may eat an apple or a pear* means that you may eat an apple and you may eat a pear without needing to invoke any special pragmatic principle.

Neither Zimmermann's nor Geurts' analyses explain why the free-choice *or* differs from an overt *and* (i.e., *You may eat an apple and you may eat a pear*) in failing to guarantee that two pieces of fruit may be eaten. In addition, as Geurts 2005:406 briefly discusses, it is not clear how either analysis accounts for negated free choice (discussed above in section 5).

Zimmermann's idea that disjunction introduces a set of alternatives has been implemented in a variety of ways. I will mention three here.

Kratzer and Shimoyama 2002 propose that indefinites contribute a set of alternatives, one for each way of resolving the indefinite. This requires in turn a modification of the basic compositional semantics, since it is necessary to allow for composition with sets of meanings instead of single meanings. This is done pointwise using "Hamblin semantics", so that an embedded indefinite can give rise to a set of alternatives at higher compositional levels (see Shan 2004 for discussion of the complexities of pointwise composition). Alonso-Ovalle 2006 extends this strategy from indefinites to disjunction, explicitly addressing the free choice problem.

Aloni's (2007) approach manages disjunction-alternatives within a dynamic semantics based on Dekker 2002, supplemented with structured propositions.

Van Rooij (2008:309) sketches yet a third implementation, on which alternatives are built into the definition of a minimal extension of a world. Then a world in which you eat only an apple might qualify as a minimal extension of the world we are in, but not a world in which you eat both an apple and a pear. In order to deliver free choice implications, it is necessary for the propositions expressed by a disjunction to always be among those used for articulating minimal extensions, though this requirement is not guaranteed by the formal analysis.

In these approaches, free choice effects arise when certain operators explicitly manipulate alternative sets. For instance, Aloni stipulates that *may*(Φ) is true (where Φ is a set of alternatives) just in case the ordinary meaning of *may* is true of each alternative. Thus *You may eat an apple or a pear* involves applying *may* to the set of alternatives corresponding to the addressee eating an apple and the addressee eating a pear. The sentence will be true, then, just in case *You may eat an apple* is true and *You may eat a pear* is true.

The account here resembles Aloni's alternatives account in two important respects. First, free choice implications are entailments rather than implicatures. As we saw in section 6.1, the fact that free choice implications do not always seem to be cancelable argues in favor of theories on which they are treated as entailments.

Second, because alternative-taking *may* requires that ordinary *may* must be true of every alternative, it is a downward-entailing operator with respect to the disjunction that gives rise to the alternatives. Aloni points out that this explains why (so-called free choice) *any* is licensed (e.g., *You may eat anything*), and since the antecedent of linear implication is likewise a downward-entailing position (as noted above), the same explanation carries over here. (Of course, there is more to free choice than placing an indefinite in a downward entailing context. For instance, a referee observes that in some Romance languages, some free-choice indefinites are licenced under permission, but not in the antecedent of conditionals or in other downward entailing contexts.)

One important difference between the approach here and alternative-based analysis, including Aloni's, is the integration with the larger compositional system. The alternative-set approach in effect creates unbounded dependencies in the semantics: *or* introduces alternatives which the compositional system must track until an alternative-aware operator collapses the alternatives back into to a single proposition. The account here adjusts only the denotations of the logical connectives, leaving the compositional system entirely undisturbed. (Not that I had provided a compositional analysis, though I trust that appropriate details can easily be supplied.)

7 Issues

7.1 Free choice effects apart from permission

It is widely assumed that whatever explains free choice implications for deontic modals should be the same thing that explains the similar behavior of epistemic modals:

(13) a. John might be in Aarhaus or in Boston.
 b. John might be in Aarhaus.
 c. John might be in Boston.

In parallel with the permission cases, the disjunction in (13a) entails (13b) and (13c).

The simplest way to extend the account here to epistemic cases would be to add to our logic a new atomic formula ϵ, which is true just in case everything that is epistemically known holds. Then *You might be in Aarhaus* would translate as $A \multimap \epsilon$, and the desired entailments follow as a matter of logic.

Adding an epsilon to the logic is more than a superficial change. It is important to keep track of what the logic claims to be modeling. Classical logic promises to preserve truth: if the assumptions are true, the conclusion will be true. Since truth is not resource sensitive (if something is true once, it is true again and again), that is why it is legitimate to duplicate and discard assumptions. Linear Logic promises to preserve resources: whatever resources the assumptions provide, that is exactly what resources will appear in the conclusion. In our deontic application, the critical resource is permission: if the assumptions provide enough permission to eat exactly one piece of fruit, then the conclusion will provide the same amount of permission. In the epistemic case, the critical resource is epistemic commitment: whatever commitments are made by the assumptions, the conclusion will make exactly the same commitments.

There are other important differences between deontic logic and epistemic logic. For instance, it is generally considered desirable for an epistemic logic to guarantee that if you know that A is true, then A is true ($\Box A \vdash A$). But deontically, you would not want to conclude from the fact that A is obligatory that A must hold, since obligations are all too often not fulfilled. More relevantly, there are empirical dis-analogies between the free choice behavior of deontic uses of modals versus epistemic modals. For instance, Kamp (1978), Zimmermann (2000), and Aloni (2007) note that it is significantly more difficult to construe epistemic modals as having a *...but I don't know which* interpretation (though it is still possible—see especially Simons 2005:274).

I'm not aware of any reason why a reduction strategy could not be part of a more complete analysis of epistemic modality; nevertheless, it would be prudent to be cautious about assuming that any deontic analysis should automatically extend to epistemic cases.

In addition to the possibility that free choice effects may occur in other modalities, Fox (2007) argues that free choice effects can be discerned in non-modal contexts that involve existential quantifiers.

(14) There's beer in the fridge or in the cooler out back.

Especially when (14) is heard as an implicit permissive, (14) entails both that there is beer in the fridge and that there is beer in the cooler out back. Both alternatives are guaranteed to be true, and the consumer of the information has free choice of which one is relevant for forming a plan of action.

Klinedinst (2007) suggests that free choice effects are present with some existential quantifiers, but only when the quantificational DP is plural:

(15) a. Some passengers got sick or had difficulty breathing.
 b. A passenger got sick or had difficulty breathing.

In (15a), there is a reading on which some passengers got sick, and some had difficulty breathing. On such a reading, at least some of the passengers must have gotten sick, and at least some of the passengers must have had difficulty breathing. But in (15b), there is no guarantee that both of the properties must be instantiated.

Having mentioned these facts, I will not attempt a discussion here of the interaction of free choice with quantifiers or with plurals. See Chemla 2009a for experimental evidence and relevant discussion.

7.2 Performativity

Kamp (1978) draws a distinction between granting permission versus describing permission, where granting permission is a performative action. When a parent says *You may eat an apple or a pear* in the right circumstances, fruit-eating options may come into being that were not present before the utterance. But when a sibling comments later *Apparently, you may eat an apple or a pear*, they are merely describing the current situation, and no new options come into being. Van Rooij (2008) and Portner (2009b) develop a dynamic semantics for permission on which a permission sentence performatively changes the set of what is allowed.

One of the main arguments that performativity is important relies on correlations between performative uses and the availability of free choice interpretations. Certainly descriptive uses (such the sibling's comment) can have a free choice interpretation or not. Performatives, however, strongly prefer a free choice interpretation. Yet it may still be possible for a performative to have a non-free choice interpretation:

(16) You may pillage city X or city Y. But first take counsel with my secretary.

Kamp (1973:67; see also Kamp 1978:279) says of this example that "[t]he second part of this statement makes it clear that the vassal should not infer from the first part that he may make his own choice of city. Which one he may loot ultimately depends on the secretary's advice, the tenour of which—we may assume—is at this point unknown to king and vassal alike." To be sure, nothing specific has been permitted, and the vassal cannot form a complete plan of action. If we conceive of a performative as something that enlarges what an agent may safely do, we might therefore suppose that (16) is a merely descriptive use, since it does not by itself allow the vassal to act. Yet something must have been permitted: where does the disjunctive permission that the sentence describes come from, if not from the performance of (16)?

As far as the current paper is concerned, it is enough for permission sentences to characterize what is allowed. Then whether an utterance expands the sphere of permissibility depends on the interaction of the truth conditions with the normal range of factors that influence how a discourse participant decides to react to an utterance. Whether this minimalist strategy is viable, or whether it will ultimately be necessary to provide a special role for performativity remains to be seen. (See Kamp 1978 for extensive, but ultimately inconclusive, discussion.)

7.3 Is there a conjunctive use of *or* after all?

Geurts (2005) and Simons (2005) emphasize the importance of explaining how free choice implications arise when *or* takes scope over the permission modal.

(17) a. You may eat an apple or a pear.
  b. You may eat an apple or you may eat a pear.

The account of free choice given so far does not explain why (17b) also has a free choice interpretation.

Simons proposes an across-the-board LF movement operation on which the sentence with unembedded *or* is predicted to be logically equivalent to *You may* [*eat an apple or eat a pear*]. That approach is compatible with the account of free choice here.

However, there is an alternative explanation that may be worth some consideration: perhaps resource-sensitive *or* is ambiguous between $\oplus$ (the translation we've given it so far) and &.

After all, there is no other lexical item that is a candidate for expressing &. For instance, as mentioned above, if you have ingredients for either meringue or angel food cake, but only enough to make one recipe, and someone asks 'What's for dessert?', the answer is *meringue* $or_{\&}$ *cake*, never *meringue* $and_{\&}$ *cake*.

A second intriguing clue comes from conditionals. In Linear Logic, strengthening of the antecedent is valid for & but not for $\otimes$. That is, we have $\mathsf{A} \multimap \mathsf{C} \vdash (\mathsf{A} \mathbin{\&} \mathsf{B}) \multimap \mathsf{C}$ but $\mathsf{A} \multimap \mathsf{C} \nvdash (\mathsf{A} \otimes \mathsf{B}) \multimap \mathsf{C}$. The observation that *and* never expresses & explains why trying to strengthen an antecedent using *and* in English does not work: *If John left, we could all play bridge* does not entail *If John left and Mary left, we could all play bridge*. But if *or* has a conjunctive use, then we could explain why the inference does seem valid if we use or: *If John left* $or_{\&}$ *Mary left, we could all play bridge*.

If *or* can express &, then the ability of (17b) to serve as a paraphrase of (17a) is immediately explained: it translates directly as $(\mathsf{A} \multimap \delta) \mathbin{\&} (\mathsf{B} \multimap \delta)$, and it is easy to prove that $(\mathsf{A} \multimap \delta) \mathbin{\&} (\mathsf{B} \multimap \delta) \vdash (\mathsf{A} \oplus \mathsf{B}) \multimap \delta$.

Of course, if *or* had such a conjunctive use, we would expect it to occur in embedded position too, for example, *You may eat an apple* $or_{\&}$ *a pear*. But this is harmless, and merely gives a different route to the *... but I don't know which* reading, which we derived above by giving (disjunctive) *or* wide scope.

More problematically, we would also expect a conjunctive *or* to be available in non-modal sentences. Then saying that *John left* $or_{\&}$ *Mary left* would offer the addressee free choice of which disjunct to believe, yet would license belief in at most one of the disjuncts. Such a meaning does not appear to be available.

Put another way, non-modal uses of *or* appear to always be classical disjunction (this is hardly surprising). One notable feature of Linear Logic is that the classical connectives are easily expressible, given the addition of the so-called exponential operators, ! (pronounced 'of course') and ? ('why not?'): from $\Delta \vdash !\mathsf{A}$ infer $\Delta \vdash \mathsf{A}, !\mathsf{A})$; from $?\mathsf{A} \vdash \Delta$ infer $\vdash \Delta$. These operators allow a richer control over resources in which assumptions can be used repeatedly, as in contraction, or ignored, as in weakening. Given Linear Logic with exponentials, we can choose a more relaxed classical resource management regime, or a more fussy pure Linear Logic regime, as needed. For instance, the classical disjunction of A and B can be expressed as $!\mathsf{A} \oplus !\mathsf{B}$.

So there is no problem allowing Linear reasoning to peacefully coexist with classical reasoning, as long as we can reliably tell which kind of resource management to use in any given context. To a first approximation in English, linear resource management appears to be relevant only for untensed clauses with bare verb forms, as in *You may eat an apple or eat a pear*, in which *or* takes scope over the untensed bare verb phrases *eat an apple* and *eat a* *pear*. Then we could suppose the reason that *John left or Mary left* does not have a conjunctive interpretation is because the tensed clauses trigger (only) a classical interpretation of *or*.

Figuring out how to regulate the distribution of an ambiguous *or* would be a major undertaking, so I leave this issue unresolved for now.

8 Semantics for linear logic

The discussion in the main text is conducted entirely in terms of inference rules and proofs. It is unusual these days, though not unheard of, to express the meaning of natural language using proof theory without giving a model theory. More often, of course, we have the opposite situation, in which semantic analyses provide models without any proof theory.

The most complete picture, however, emerges when proof theory and model theory complement each other. Therefore I will discuss models for Linear Logic here, with a detailed illustration of a free choice example.

There are a number of semantic approaches to Linear Logic. Girard's (1987, 1995) original semantics in terms of coherence spaces and in terms of phase spaces would not be directly helpful here. There are other semantic approaches, however, that have tantalizing associations with the granting and denying of permission. I will mention three. First, Petri nets describe the movement of tokens through a network. Lokhorst (1997) uses Petri nets as models of his Linear Logic treatment of deontic reasoning. (Think of the tokens as lumps of permission moving from one location to another.) Second, in game semantics a Proponent and an Opponent take turns making choices, and I have argued that tracking choice is central to understanding permission talk. See, e.g., Accorsi and van Benthem 1999 for a discussion of game semantics for Linear Logic. Third, there are computational models of Linear Logic that make an explicit connection between the additives and choice. For example, Abramsky's (1993) computational semantics for intuitionistic Linear Logic interprets $\mathsf{A} \otimes \mathsf{B}$ as an ordered pair $\langle \mathsf{A}, \mathsf{B} \rangle$ both of whose elements will be used in further computation (eager evaluation); $\mathsf{A} \mathbin{\&} \mathsf{B}$, on the other hand, denotes an ordered pair only one of whose elements will ever be used (lazy evaluation), and of course $\mathsf{A} \oplus \mathsf{B}$ delivers a projection function that chooses one or the other of the elements in a & pair. Unfortunately for our purposes here, Abramsky's computational interpretation of classical Linear Logic involves parallel distributed processing, which would take us too far afield.[2]

2 Though it is intriguing to think that the meaning of some natural language expressions might be appropriately modeled by a distributed process. Perhaps some permission sentences denote programs which the recipient can execute in various environments in order to produce

Most reassuringly familiar for linguists, Allwein and Dunn (1993) provide a kosher Kripke-style possible worlds semantics, and that is the approach that I will present here.

Following Allwein and Dunn, the expository strategy will be to begin with an algebraic model that is faithful to the inference rules, then show how to reconstruct that algebra in terms of worlds.

8.1 An algebraic semantics

The algebraic model contains three main components: a lattice for modeling the additive connectives, a unary operation for modeling negation, and a binary operation for modeling the multiplicative connectives.

Additives: let A, $\wedge$, and $\vee$ form a bounded lattice with partial order $\leq$ and top and bottom elements. The lattice can be finite or non-finite, and it can be distributive or non-distributive.

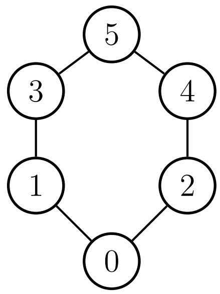
Negation: now let $\sim$ be a DeMorgan negation on that lattice. This means that $\sim$ must be order-reversing (for all x, y in A, $x \leq \sim y$ iff $y \leq \sim x$), and it must be involutive (for all x in A, $\sim\sim x \leq x$).

Multiplicatives: we add a commutative, associative binary operation $\circ$ with identity element t (that is, $t \circ a = a = a \circ t$ for all a in A). Thus A,$\circ$, and t form a commutative monoid. Note that t may be distinct from the top of the lattice. The monoid operation must distribute over the join operation, that is, for all $a, b, c \in A : a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$. It must also be compatible with negation in the sense that for all $a, b \in A : a \circ b \leq c$ iff $a \circ \sim c \leq \sim b$ ("antilogism").

The points in the lattice model formulas. Given a valuation v mapping atomic formulas onto elements of A, we extend v to complex formulas as follows: $v(A^{\perp}) = \sim v(A)$; $v(A \,\&\, B) = v(A) \wedge v(B)$; $v(A \oplus B) = v(A) \vee v(B)$; $v(A \otimes B) = v(A) \circ v(B)$; $v(A \parr B) = \sim(\sim v(A) \circ \sim v(B))$; and $v(A \multimap B) = \sim(v(A) \circ \sim v(B))$.

As an example, I will present a six-element, non-distributive lattice:

whichever certificate of permission is required. Then a free choice permission sentence denotes a program whose execution is blocked until it receives an external choice (a selection of which alternative to deploy).



$\sim$	
0	5
1	3
2	4
3	1
4	2
5	0

$\circ$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	1	2	5
2	0	2	1	2	1	5
3	0	1	2	3	4	5
4	0	2	1	4	3	5
5	0	5	5	5	5	5

The Hasse diagram on the left gives the lattice order in the usual way, so that $0 \leq 1$, $1 \leq 3$, and so on. In addition, since $\leq$ is reflexive and transitive, we also have $0 \leq 0$, $0 \leq 3$, etc.

Since meet ($\wedge$) in a lattice is the unique greatest lower bound, it can be read off the Hasse diagram, e.g., $5 \wedge 5 = 5$, $4 \wedge 5 = 4$, $4 \wedge 3 = 0$, and so on (dually for the join operation $\vee$).

It is easy to see by inspection that the negation relation $\sim$ is involutive (e.g., $\sim\sim 3 = 3$) and order reversing (e.g., along with $0 \leq \sim 3$ we have $3 \leq \sim 0$).

Note that 3 serves as the identity element t of the monoid. Since the monoid operation is commutative, the matrix is symmetric across the top-left to bottom-right diagonal (e.g., $4 \circ 2 = 2 \circ 4$). Furthermore, mechanical checking will confirm that the monoid operation is associative (e.g., $(4 \circ 2) \circ 1 = 4 \circ (2 \circ 1)$) , that it distributes over the join operation (e.g., $3 \circ (1 \vee 4) = (3 \circ 1) \vee (3 \circ 4)$), and that it respects the antilogism requirement (e.g., $4 \circ 2 \leq 3 \equiv 4 \circ \sim 3 \leq \sim 2$).

A sequent Γ semantically entails Δ (written '$\Gamma \models \Delta$') just in case the valuation of the multiplicative conjunction of the formulas in Γ is dominated by the valuation of the multiplicative disjunction of the formulas in Δ. For instance, since $x \wedge y \leq x$ for all x, y in A by the definition of meet in a lattice, we have that $A \,\&\, B \models A$.

To illustrate how these tables provide a model of the logic, recall that we have the following three theorems discussed in previous sections and one non-theorem:

(18) a. $(A \multimap \delta) \,\&\, (B \multimap \delta) \vdash (A \oplus B) \multimap \delta$
b. $(A \oplus B) \multimap \delta \vdash (A \multimap \delta) \,\&\, (B \multimap \delta)$
c. $(A \multimap \delta) \oplus (B \multimap \delta) \vdash (A \,\&\, B) \multimap \delta$
d. $(A \,\&\, B) \multimap \delta \nvdash (A \multimap \delta) \oplus (B \multimap \delta)$

If the given algebra is a faithful model of Linear Logic, we expect that for every valuation ν assigning a lattice element to the propositional symbols δ, A, and B, the valuation of the left hand side of any theorem will be dominated (in the sense of the lattice order $\leq$) by the valuation of the right hand side. This is the case for (18) (a) through (c), but we have a countermodel for (18d): if

$\nu(\delta) = 0$, $\nu(A) = 1$, and $\nu(B) = 2$, then $\nu((A\&B) \multimap \delta) = \nu(((A\&B)\otimes\delta^{\perp})^{\perp}) = {\sim}((\nu(A)\wedge\nu(B))\circ{\sim}\nu(\delta)) = {\sim}((1\wedge 2)\circ{\sim}0) = 5$. But $\nu((A \multimap \delta)\oplus(B \multimap \delta)) = 0$, and $5 \not\leq 0$.

There are (infinitely) many other possible choices for a lattice, and for any given lattice, there may be many choices for a suitable negation and for a suitable monoid operation. For instance, Restall (2000:170) gives an even simper (but still instructive) model of (distributive) Linear Logic based on a four-element lattice. Since Linear Logic is sound and complete with respect to the class of algebraic models given here, a sequent is a theorem iff its left hand side semantically entails its right hand side for every valuation in every model.

8.2 A possible-worlds semantics

The algebraic semantics is simple and straightforward, in part because it merely recapitulates the inference rules; for the same reason, it may not add any insight beyond what is already evident from the inference rules themselves. Constructing a Kripke-style possible worlds semantics is a bit more complicated, but may allow natural language semanticists to transfer some of their intuitions from more familiar sorts of semantics for natural languages. We shall see that one particularly intriguing feature of the Kripke semantics for Linear Logic is that there will be three possibilities for the status of a formula at a world: it may be true, false, or neither true nor false, which is exactly what makes Linear Logic suitable for modeling actions that may be permitted, forbidden, or neither permitted nor forbidden.

Allwein and Dunn associate each element in A with a particular set of worlds. The construction goes as follows. Consider pairs of the form $\langle F, I\rangle$, where F and I are sets of points in the lattice. We require $\langle F, I\rangle$ to satisfy the following four requirements: first,

(w1): F and I must be disjoint.

Second,

(w2): F must be closed upward under $\leq$, so that for all $a \in F$ and for all $b \in A : (a \leq b)$ implies $b \in F$. Dually, I must be closed downward under $\leq$, so that for all $a \in A$ and for all $b \in I : (a \leq b)$ implies $a \in I$. In particular, F always contains the top element, and I always contains the bottom element of the lattice.

Third, F and I must be closed under meets and joins, respectively. That is:

(w3): for all $a, b \in F : a \wedge b \in F$; and for all $a, b \in I : a \vee b \in I$.

In other words, conditions (w2) and (w3) require that F must be a filter, and that I must be an ideal.

Finally, there is a maximality condition:

Maximality: A filter/ideal pair $\langle \mathsf{F}, \mathrm{I} \rangle$ satisfying (w1), (w2), and (w3) satisfies maximality only if there is no other distinct pair of sets $\langle \mathsf{F}', \mathrm{I}' \rangle$ also satisfying (w1), (w2) and (w3) that properly includes the first, i.e., such that $\mathsf{F} \subseteq \mathsf{F}'$ and $\mathrm{I} \subseteq \mathrm{I}'$.

Here are a few of the possible pairs of subsets that fail to satisfy the requirements:

$\langle \{1,2\}, \{1,3\} \rangle$	violates w1
$\langle \{3\}, \{0\} \rangle$	violates w2
$\langle \{4,3,5\}, \{0\} \rangle$	violates w3
$\langle \{4,5\}, \{0\} \rangle$	violates Maximality

In fact, in this model there are exactly four maximal disjoint filter/ideal pairs:

World a: $\langle \{4,5\}, \{0,2\} \rangle$
World b: $\langle \{3,5\}, \{0,1\} \rangle$
World c: $\langle \{2,4,5\}, \{0,1,3\} \rangle$
World d: $\langle \{1,3,5\}, \{0,2,4\} \rangle$

These pairs will stand in one to one correspondence with our possible worlds. For each world $w = \langle \mathsf{F}, \mathrm{I} \rangle$, we will interpret F as the set of points that are true at w, and I as the set of points that are false at w.

For worlds c and d, every point in the lattice is either true or false. But for world a, points 1 and 3 are neither true nor false. Similarly, for world b, points 2 and 4 are neither true nor false. In terms of permission talk, there may be situations in which some things are permitted, some things are forbidden, and some things are neither permitted nor forbidden.

The next step is to associate each point in the lattice with a set of worlds. If w is a world associated with the pair of sets of points $\langle \mathsf{F}, \mathrm{I} \rangle$, let w_1 indicate F and w_2 indicate I. Then we can define a map β that takes each point p in the lattice onto the set of worlds w such that $p \in w_1$:

$\beta(0) = \{\}$
$\beta(1) = \{d\}$
$\beta(2) = \{c\}$
$\beta(3) = \{b, d\}$
$\beta(4) = \{a, c\}$
$\beta(5) = \{a, b, c, d\}$

In other words, we map each point in the lattice to the set of worlds that make it true.

We now need to define relations over sets of worlds that will allow us to reconstruct the logical operations we want to model: $\wedge$, $\vee$, $\sim$, and $\circ$.

The meet operation is straightforward. We extend β in the following way:

$\beta(p \wedge q) = \beta(p) \cap \beta(q)$. So meet corresponds to simple set intersection. Thus $4 \wedge 2 = 2$, and $\beta(4 \wedge 2) = \beta(4) \cap \beta(2) = \{a, c\} \cap \{c\} = \{c\} = \beta(2)$.

The join operation is not quite so straightforward. We cannot represent join as set union. To see why, note that $3 \vee 2 = 5$, but $\beta(3) \cup \beta(2) = \{b, d\} \cup \{c\} = \{b, c, d\} \neq \beta(5)$. The solution is to exploit the information present in the second element in the pair of sets that define the worlds. To do this, we define two operations on sets of worlds. Let W be our set of worlds, and let C be any subset of W:

$$l(C) = \{x | \text{for all } y \in W, x_1 \subseteq y_1 \text{ implies } y \notin C\}$$

$$r(C) = \{x | \text{for all } y \in W, x_2 \subseteq y_2 \text{ implies } y \notin C\}$$

Although l and r are defined over all subsets of W, we will only need to apply them in the following cases:

$$\begin{array}{ll}
r(\beta(0)) = r(\{\}) & = \{a, b, c, d\} \\
r(\beta(1)) = r(\{d\}) & = \{b, c\} \\
r(\beta(2)) = r(\{c\}) & = \{a, d\} \\
r(\beta(3)) = r(\{b, d\}) & = \{c\} \\
r(\beta(4)) = r(\{a, c\}) & = \{d\} \\
r(\beta(5)) = r(\{a, b, c, d\}) & = \{\}
\end{array}$$

For instance, the reason a is not in $r(\beta(1))$ is because $a_2 \subseteq d_2$, but $d \in \beta(1)$. Allwein and Dunn show that for all points p in the lattice, $l(r(\beta(p))) = \beta(p)$.

We can now define join by shifting the conjuncts using r, then taking their intersection, then shifting back using l: $\beta(p \vee q) = l(r(\beta(p)) \cap r(\beta(q)))$. For instance, we have $\beta(1 \vee 3) = l(r(\beta(1)) \cap r(\beta(3)) = l(\{b, c\} \cap \{c\}) = l(\{c\}) = \beta(3)$. Trying the problematic case given above, $\beta(3 \vee 2) = l(r(\beta(3)) \cap r(\beta(2))) = l(\{c\} \cap \{a, d\}) = l(\{\}) = \{a, b, c, d\} = \beta(5)$, as desired.

At this point, β, l, and r allow us to fully simulate the structure of the lattice in terms of sets of worlds.

Representing negation: $\beta({\sim}p) = \{x | \langle {\sim}x_2, {\sim}x_1 \rangle \in r(\beta(p))\}$ (where applying $\sim$ to a set of points returns the set resulting from applying $\sim$ to each member of the original set). For instance, we have $\beta({\sim}1) = \{\langle {\sim}\{0, 1\}, {\sim}\{3, 5\} \rangle,$ $\langle {\sim}\{0, 1, 3\}, {\sim}\{2, 4, 5\} \rangle\} = \{b, d\} = \beta(3)$.

Note that linear negation expresses something about provability, not about falsity. One way to see this is to observe that in this model, 3 and its negation ${\sim}3 = 1$ are both true at world d.

Representing the tensor relation $\circ$ proceeds in two steps. In the usual Kripke semantics, unary modal operators are characterized by an accessibility relation, a two-place relation over worlds. Because the multiplicatives are two-place connectives, we will need a three-place relation.[3]

$$\mathsf{S}xyz \text{ iff } \forall p, q : (p \circ q \in z_2 \text{ and } q \in y_1) \text{ implies } p \in x_2$$

The strategy here is a generalization of the Routley-Meyer semantics for Relevant Logic. The goal is for the relation S to capture all of the information present in the monoid operation $\circ$. In order to do this, it needs to take advantage of both sets of points that define the worlds: the set of propositions that are true at a world as well as those that are false at that world.

Conceptually, S models modus ponens, in which x plays the role of antecedent, y plays the role of the implication, and z plays the role of the consequent. If the implication is true at y, and the consequent is false at z, S guarantees that the antecedent must be false at x. For instance, since 3 (role: the implication) is true at b and $1 \circ 3$ (the consequent) is false at c, but 1 (the antecedent) is not false at a, S does not hold of a, b, and c. We do have $\mathsf{S}aba$, however. The complete relation is aab, aba, baa, bbb, caa, cad, cbb, cbc, cca, ccd, cdb, cdc, dab, dac, dba, dbd, dcb, dcc, dda, ddd.

Once we have constructed S as a function of $\circ$, we can define multiplicative conjunction purely in terms of relations over worlds:

$$\beta(p \circ q) = l(\{z | \forall x, y : \mathsf{S}xyz \text{ and } y \in \beta(q) \text{ implies } x \in r(\beta(p))\})$$

This definition unpacks S in order to reconstruct the original relation $\circ$.

8.3 Understanding linear implication

What does the multiplicative conjunction of two formulas mean? Since we now have both an algebraic and a possible worlds semantics in correspondence, we can move back and forth between the two semantics in search of insight.

Begin with the algebra. We can keep track of the state of our reasoning process by picking out a point in the lattice. Assume that I have good reason to believe we are located at lattice position 1. This is a highly specific situation: I know that we are located on world d, since that is the only world at which 1 is true.

Now assume that I learn something: that you have eaten a pear. Call this fact B, and associate it with lattice point 4 (i.e., let $\nu(B) = 4$). To find out where we are now, I compute $1 \circ 4 = 2$. Since $\beta(1 \circ 4) = \beta(2) = \{c\}$, we are

3 Lambek grammars (e.g., Moortgat 1997) also use a three place relation to give a semantics for a multiplicative conjunction, where the conjunction is used to model concatenation of linguistic expressions. For an example of modus ponens in type-logical grammar, $\mathsf{DP} \otimes \mathsf{DP}\backslash\mathsf{S} \vdash \mathsf{S}$.

now on world c. Learning that you have eaten an apple changes our location from world d to world c.

This may initially seem somewhat distressing. In the usual Stalnakerian system, adding information is typically a monotonic process of eliminating possible worlds. If we've already narrowed the set of live options to a single world d, there is no way to end up on a distinct world c. Because $\circ$ is non-monotonic in this sense, it may be better to think of what we have been calling worlds as classes of worlds. Sometimes the term 'set-up' is used instead of 'world'. I will use the term 'situation'. Then learning that you have eaten an pear changes the current situation into a different situation, one in which the consequences of having eaten a pear obtain.

Let's continue to reason. We pick a point in the lattice to serve as A, the situation in which you eat an apple, and a separate point to serve as δ, the situation in which all obligations are fulfilled. Say that $\nu(A) = 2$, $\nu(\delta) = 3$, and $\nu(B)$ is still 4. Now consider the proposition that eating an apple is permitted: $A \multimap \delta$. Then $\nu(A \multimap \delta) = \nu((A \otimes \delta^{\perp})^{\perp}) = {\sim}(\nu(A) \circ {\sim}\nu(\delta)) = {\sim}(2 \circ {\sim}3) = {\sim}(2 \circ 1) = {\sim}2 = 4$. Apparently, in this model, the situation in which you eat a pear is modeled by the same situation in which you are permitted to eat an apple. (This sort of coincidence is unavoidable in such a tiny model, in the same way that a valuation for classical logic will be forced to map very different formulas to the same truth value.)

So let's say that I know we're in a situation in which you are permitted to eat an apple (say, point 4), and then I learn that you have eaten an apple. Perhaps I watch you eat it. This changes things: I compute $4 \circ 2 = 1$. Thanks to your eating an apple, we're now in situation 1. And since $1 \leq 3$, things are as they are supposed to be. In terms of worlds, δ is modeled by worlds (situations) b and d; and since point 1 corresponds to (a singleton set containing only) world d, we must be in a δ-world.

So, what if you are permitted to eat an apple or a pear? That's ${\sim}((2 \vee 4) \circ {\sim}3) = 4$. We just saw that if we start at 4 and you an apple, we land on a δ-world. And indeed, if we're at point 4 and you eat a pear instead, $4 \circ 4 = 3$, and once again we're in a δ-situation.

But what if you eat an apple and you eat a pear? $4 \circ 4 \circ 2 = 2$. Situation 2 is not a δ situation, so things are not ok. Having permission to eat an apple or a pear is not the same thing as having permission to eat an apple and a pear. Likewise, if killing the postman is modeled by situation 4 (i.e., $\nu(K) = 4$), then eating an apple and killing the postman will definitely not leave us in a δ-situation. (This small model is somewhat unrealistic, however, in that there are situations in which eating an apple, killing the postman, and then eating another apple is perfectly permissible.)

However, as emphasized above, having permission to eat an apple or a pear is compatible with also having permission to eat both. Making use of the same model, if we have $\nu(A) = \nu(\delta) = \nu(B) = 3$, then $\nu((A \mathbin{\&} B) \multimap \delta) = \nu((A \otimes B) \multimap \delta) = 3$. With this valuation, eating apples and pears is truly optional: you can eat an apple and stop, or you can eat a pear and stop, or you can eat an apple and you can eat a pear, and in all three cases you'll end up in a δ-situation.

9 Conclusions

On the view presented here, understanding free choice hinges on recognizing that permission is a scarce resource, and so permission talk requires a resource-sensitive semantics. Following Lokhorst (1997), I propose Linear Logic as a way of tracking permission: not only what kind of permission has been granted, but how much. Then primary free choice implications (given *You may eat an apple or a pear*, infer *You may eat an apple* & *You may eat a pear*) follow merely from expressing permission using the (independently-motivated) Anderson/Kanger deontic reduction strategy. Double prohibition (from *You may not eat an apple or a pear* infer *You may not eat an apple* and *You may not eat a pear*) follows from standard Gricean reasoning, without any need to postulate special pragmatic mechanisms.

The implications of this view are fairly dramatic. The claim is that natural language expressions can differ in the resource management schemes they impose. At the least, alethic modes impose classical resource mangement, and deontic modes impose linear resource management (and quite likely, other modes as well).

Linear Logic is one of the better known resource-sensitive logics. Other logics may be worth considering instead. Similarly, the Anderson/Kanger deontic reduction strategy was adopted in part for ease of exposition, and work remains to integrate the account here within a more general framework of modality in natural language. But apart from the advantages of Linear Logic specifically or the deontic reduction, I would like to suggest a more general conclusion: that we may be able to gain new and valuable insights into long-standing puzzles in natural language semantics if we allow ourselves to consider richer logical approaches than standard classical logic.

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