The Shapley value of Cooperative Games under Fuzzy Settings: A Survey*

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Abstract

We survey the recent developments in the studies of cooperative games under fuzzy environment. The basic problems of a cooperative game in both crisp and fuzzy contexts are to find how the coalitions form vis-à-vis how the coalitions distribute the worth. One of the fuzzification processes assumes that the coalitions thus formed are fuzzy in nature having only partial participations of the players. A second group of researchers fuzzify the worths of the coalitions while a few others assume that both the coalitions and the worths are fuzzy quantities. Among the various solution concepts of a cooperative game, the Shapley value is the most popular one-point solution concept which is characterized by a set of rational axioms. We confine our study to the developments of the Shapley value in fuzzy setting and try to highlight the respective characterizations.

keywords: Cooperative game, Fuzzy coalitions, Vague expectation, Shapley mapping.

*In memory of Milan Mareš whose sudden demise is a great loss to the Game Theory and Fuzzy Quantities Communities and in honor of L. Shapley, the 2012 Nobel Laureate in Economics.
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1 Introduction

In this paper, we survey the recent developments of Cooperative Game Theory under fuzzy environment. We confine our study only to a well known one point solution concept in different model formulations: The Shapley value. Since first proposed by L. Shapley [41] in his 1953 Ph.D Thesis, it has received widespread attention from the research community. Such popularity is attributed to the fact that it is characterized by a set of reasonable, intuitive and easy to understand axioms. The situation where some economic agents make up a cooperative relationship called a coalition, and get more gains than those if they do not do so is described in the literature. In such situations, one of their interests is how much share each of them would get by forming the coalition. The Shapley value shows a vector whose components are agent’s shares derived from these various reasonable bases, see [43].

Game theory is a branch of mathematics that studies strategic decision making problems including both cooperation and non-cooperation. The second half of the last century has witnessed enormous use of Game Theory in various spheres of decision sciences involving military battles, political campaigns, elections, advertising and marketing campaigns, socio-economic issues, sustainable utilizations of natural resources to name a few. The basic concepts of Cooperative Game Theory and their elementary properties were developed by von Neumann and Morgenstern [44]. Since then the subject has gradually developed into an important tool for Mathematical Modeling in various disciplines. A game is defined as a situation between two or more persons that involves their individual or joint activities according to a set of rules and finally each person receives some benefit or satisfaction or suffers loss (negative benefit). The set of rules defines the game. Based on the nature of the situation and how the players in the game behave, we have two forms of games: Non-cooperative and cooperative. Non-cooperative games are situations where players play individually without any cooperation among themselves. In such games each
player has a strategy set with a set of possible payoffs. In equilibrium, a player receives her payoff based on her best response against the rest of the players. On the contrary, cooperative games are played under some binding agreements where individual strategies of the players are not considered and rather a strategy is treated as a joint or collective activity among the constituent players. Such a game is represented by the players’ set and a characteristic function. In our current survey, we deal only with cooperative games in characteristic function form. The concept and its elementary properties were initially developed by von Neumann and Morgenstern [44] in 1953. After then, the subject has evolved very rapidly to cover almost all disciplines and a lot of research works has been carried out in developing and analyzing solution concepts of various types of cooperative games. In a nutshell, most of the literature dealing with cooperative games in characteristic function form characterize solution concepts as a distribution of the total output of the game generated by all the players together, given that the same accrued by smaller groups satisfy some rationality conditions. Thus the challenge now rests in forming either the grand coalition or in its absence, a stable coalition structure and finding a suitable distribution of the worth or profit of a coalition among its players. Solution concepts for cooperative games are broadly divided into two categories, namely single point and multiple point. Core, Weber set, Bargaining set etc. are multiple point solutions while Shapley values, compromise values, minimum norm solutions for example are single point solutions in nature [41, 39, 36, 34, 12, 33, 19].

Cooperative Game Theory in characteristic function form relies upon the notion of coalitions. A coalition is a group of persons or players who hold different cooperation possibilities. In the n-person case (n > 2), there are 2^n possible coalitions including the empty and the singleton coalitions and this implies that if a coalition is to form and remain for sometimes, the different members of the coalition must reach some sort of equilibrium or stability. It follows that the basic model of a cooperative game is based on the assumption of the existence of a universal and linear representative of utility which can be used for the
distribution of the coalition’s total profit among its members without any deformation of the utility value. Therefore, due to this model, each coalition wins some utility which can be distributed among its members with respect to a coalitional agreement, and the sum of utilities obtained by the coalition members preserves the value of a distributed common income. Such games are termed as Transferable Utility games or in short T.U. games, see [30]. Throughout this exposition we consider only T.U. games, however we preserve the more traditional term, cooperative games. In real life situations, synergy of self-interested players under binding agreements has been well explained and justified by the theory of crisp cooperative games. Negotiation and compromise among players is another important aspect in forming coalitions.

In general, we deal with problems in terms of systems that are constructed as models of either some aspects of reality or some desirable man-made objects. The objective of constructing models of the former type is to understand some real phenomena, making appropriate predictions, learning how to control the phenomenon in any desirable way and utilizing all these capabilities for various ends; models of the later type are constructed for the purpose of prescribing operations by which conceived artificial object can be constructed in such a way that desirable objective criteria are satisfied within given constraints. In constructing a model, we always attempt to maximize its efficiency, which is closely connected with the relationship among three key characteristics of every system model; complexity, credibility and uncertainty. We know that uncertainty (predictive, prescriptive etc.) has a pivotal role in any efforts to maximize the usefulness of system models. Allowing more uncertainty tends to reduce complexity and increases credibility of the resulting model. Many research works show that probability theory is capable of representing only one of several distinct types of uncertainty. In a published paper by Lotfi A. Zadeh [47], a new concept was introduced where the objects- fuzzy sets are sets with boundaries that are not precise. The membership in a fuzzy set is a matter of a degree. The crisp set defined in a way aims to dichotomize the individuals in some given
universe of discourse into two groups: members (which certainly belong to the set) and nonmembers (which certainly do not). A clear distinction exists between the members and non members of the set. However, many classification concepts we commonly employ and express in natural language describe sets that do not exhibit these characteristics (sharp boundary). Examples are the set of tall people, expensive cars, numbers much greater than one, sunny days. We perceive these sets as having imprecise boundaries that facilitate gradual transitions from membership to non membership and vice-versa. The capability of the fuzzy sets to express gradual transitions from membership to non membership and vice-versa has a wide utility. The fuzzy set provides us both a meaningful and powerful representation of measurements of uncertainties as well as a meaningful representation of vague concepts (e.g., about 10, cloudy sky etc.) expressed in natural language, see [26].

If a player wants to participate in more than one coalition simultaneously, with her resources (or power) available at her hand, it is possible to provide only fractions of her full resource(power) to those coalitions, which leads us to define the concept of fuzzy coalitions. A fuzzy coalition is defined as a fuzzy subset of $N$, and represented by an $n$-tuple, where its $i$th component signifies the degree of participation (or fraction of the power) of player $i$ ranging between 0 and 1. A crisp coalition can also be viewed as a special type of fuzzy coalition where the degree of participation of the players in it is either 1 or 0. Aubin [3], Butnariu [14], Branzei et al. [12], Tsurumi et al. [43], Butnariu and Kroupa [15] have well developed the theory of fuzzy cooperative games and thereby justified the fuzzification in terms of the players' participation in a coalition. Mareš [30], Mareš and Vlach [32] have fuzzified the worth of the coalition and provided a new dimension to the theory. Borkotokey [11] has combined these two ideas and introduced cooperative games with fuzzy coalitions and fuzzy characteristic function. In both crisp and fuzzy environment, it is desirable to determine how a fuzzy coalition structure is formed vis-à-vis how a suitable payoff distribution be proposed to the players accordingly. In [1, 2, 3], Aubin suggested about variable levels of participation in coalitions giving the notion of fuzzy coalitions. He
discussed various solution concepts such as core, equilibria and Shapley value for fuzzy cooperative games as well. Branzei et al. [12] have studied in details cooperative games with fuzzy coalitions and its properties followed by corresponding solution concepts. In [7], Azrieli and Lehrer have dealt with properties of fuzzy cooperative games and their interrelations. They have discussed convex games, exact games, games with large core, extendable games and games with a stable core. Butnariu [14] re-defined core and Shapley value and investigated the consistency of these value for a fuzzy cooperative game. In his study he has proved that the core is consistent for any $n$-persons fuzzy cooperative game. It is further shown that the Shapley value for any fuzzy game with proportional values, exists and is unique. In [43], Tsurumi et al. gave an alternative definition of the Shapley value for fuzzy cooperative games, and indentified several rational properties. Subsequently Li and Zhang [28] have developed a simplified expression of the Shapley mapping for fuzzy cooperative games, which generalizes the Shapley values discussed in [14] and [43]. Mareš in his book [30] has extensively discussed cooperation with vague expectation (for cooperative games with side payments and without side payments). Borkotokey [11] has formulated a model of cooperative games with fuzzy coalitions and fuzzy characteristic function and obtained the Shapley value for such games. In [46], a new approach is adopted to define cooperative games with fuzzy coalitions and fuzzy characteristic function and the corresponding Shapley value is obtained using Hukuhara difference.

The rest of the paper is organized as follows. Section 2 deals with the models of cooperative games and preliminary ideas on Fuzzy Set Theory; Section 3 deals with cooperative games with fuzzy coalitions. In Section 4, we discuss cooperative games under vague expectations, Section 5 deals with games with fuzzy coalitions and vague expectations followed by the concluding remarks in Section 6.
2 Preliminaries

In this section we compile the basic notions of crisp cooperative games focusing mainly on the corresponding Shapley value of such games and few elementary ideas about Fuzzy set theory from [12, 25, 41, 23, 15].

2.1 Cooperative Games

Let $N$ be a non-empty set of players or agents who consider different cooperation possibilities. Each subset $C$ of $N$ is referred to as a crisp coalition. The set $N$ is called the grand coalition and $\emptyset$ is called the empty coalition. The collection of coalitions, i.e., the set of all subsets of $N$ is denoted by $2^N$.

**Definition 2.1.** A cooperative game in characteristic function form is an ordered pair $< N, v >$ consisting of the players set $N$ and the characteristic function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$.

The real number $v(C)$ can be interpreted as the maximal worth or cost savings that the members of $C$ can obtain when they cooperate. According to Butnariu and Kroupa [15], the worth $v(C)$ measures the utility of forming the coalition $C$. The set of all cooperative games with the player set $N$ is denoted by $G^N$.

**Definition 2.2.** A game $v \in G^N$ is said to be monotonic if

$$v(C_1) \leq v(C_2) \text{ for all } C_1, C_2 \in 2^N \text{ with } C_1 \subset C_2.$$ 

**Definition 2.3.** A game $v \in G^N$ is said to be additive if

$$v(C_1 \cup C_2) = v(C_1) + v(C_2) \text{ for all } C_1, C_2 \in 2^N \text{ with } C_1 \cap C_2 = \emptyset.$$ 

Definition 2.4. A game $v \in G^N$ is said to be superadditive if

$$v(C_1 \cup C_2) \geq v(C_1) + v(C_2)$$

for all $C_1, C_2 \in 2^N$ with $C_1 \cap C_2 = \emptyset$.

By $G_C(N)$, we denote the class of all superadditive crisp games with player set $N$.

Definition 2.5. A payoff vector is an $n$-tuple $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, where for each $i \in N$, $y_i$ represents the payoff to be given to the $i$-th player.

Definition 2.6. A payoff vector $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ is said to satisfy individual rationality if $y_i \geq v(\{i\})$ for all $i \in N$.

In the following, we give some definitions related to the rationality conditions of a payoff vector as a solution concept.

Definition 2.7. For the game $v \in G^N$,

- A player $i \in N$ is called a null player if for every coalition $S \subseteq N$, we have $v(S) = v(S \setminus i)$.

- Let $\pi : N \rightarrow N$ be any permutation on $N$. Then for every game $v \in G^N$, by the permutation game $\pi v \in G^N$, we mean the game such that for every coalition $S = \{i_1, ..., i_k\}$,

$$\pi v(\pi S) = v(S),$$

where $\pi S = \{\pi(i_1), ..., \pi(i_k)\}$

Definition 2.8. The Shapley mapping is a linear function $\Phi : G^N \rightarrow \mathbb{R}^n$ which associates to each cooperative game $v \in G^N$ a vector $\Phi(v) = (\Phi_1(v), ..., \Phi_n(v))$ and satisfies the following conditions:

- Efficiency : For every $v \in G^N$, $\sum_{i \in N} \Phi_i(v) = v(N)$.

- Null Player Property : For every game $v \in G^N$ it holds that $\Phi_i(v) = 0$ for every null player $i \in N$ in the game $v$. 

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• Symmetry: For every permutation $\pi : N \to N$, $\Phi_{\pi(i)}(\pi v) = \Phi_i(v)$.

• Additivity: For all games $v, w \in G^N$ and every player $i \in N$: $\Phi_i(v + w) = \Phi_i(v) + \Phi_i(w)$.

We call $\Phi(v)$ the Shapley value for $v$ and each component $\Phi_i(v) \in \mathbb{R}$ the Shapley value of player $i$ in $v \in G^N$. A formal expression of $\Phi_i(v)$ is given as follows.

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{(n-|S|-1)!}{n!} \frac{|S|!}{|S|!} [v(S \cup i) - v(S)]$$  \hspace{1cm} (2.1)

**Theorem 2.9.** (Shapley’s Theorem) The Shapley value is the unique value on $G^N$ that satisfies efficiency, the null player property, symmetry and additivity.

Interested reader may find a detailed proof of this theorem in any standard book on cooperative game theory and therefore it is omitted here, see for example [41, 37, 36].

### 2.2 Fuzzy Quantities

Fuzzy quantities arise naturally when we try to deal with impreciseness. Mathematical models based on fuzzy set theory and logic have received wide attention in the last few decades due to their flexibility and proximity to the human behaviors. A fuzzy set mimics the characteristic function of a crisp set providing the membership value of an element from the interval $[0, 1]$ (or from any bounded lattice under a more general framework). Note that the characteristic function of a crisp set assigns values 0 and 1 depending on whether an element is a member or non-member of the set. Throughout this paper, we assume that every fuzzy set maps into the lattice $([0,1], \wedge, \vee)$, where $\wedge$ and $\vee$ are the minimum and maximum operators respectively, i.e., the union and intersection of fuzzy sets is defined point-wise by means of the operations $\wedge$ and $\vee$ respectively. The following definitions are taken from [30], see also [48].

**Definition 2.10.** A fuzzy subset $A$ of $\mathbb{R}$ ($A : \mathbb{R} \to [0, 1]$) satisfying:
(a) there exists $x_A \in \mathbb{R}$ such that $A(x_A) = 1$.

(b) there exist $x^1_A, x^2_A \in \mathbb{R}$ such that $x^1_A \leq x_A \leq x^2_A$ with $A(x) = 0$ for all $x \not\in [x^1_A, x^2_A]$

(c) $A$ is non-decreasing on $[x^1_A, x_A]$ and non-increasing on $[x_A, x^2_A]$.

is called a fuzzy quantity and the number $x_A$ is called the modal value of $A$. The expression $L(\mathbb{R})$ denotes the set of all fuzzy quantities of $\mathbb{R}$.

**Definition 2.11.** If ‘∗’ denotes a binary algebraic operation on $\mathbb{R}$, then it can be extended to act on fuzzy quantities from $L(\mathbb{R})$, and then for $A, B \in L(\mathbb{R})$, $A * B$ is defined as,

$$[A * B](z) = \lor_{x*y}{A(x) \land B(y)} \quad \forall z \in \mathbb{R}$$ \hspace{1cm} (2.2)

We use $\oplus$ to denote the algebraic sum of two fuzzy quantities, $\ominus$ to denote the difference of two fuzzy quantities and so on. Therefore, in particular the algebraic sum of two fuzzy quantities $A, B \in L(\mathbb{R})$ is defined as

$$[A \oplus B](z) = \lor_{x+y}{A(x) \land B(y)} \quad \forall z \in \mathbb{R}$$ \hspace{1cm} (2.3)

Fuzzy quantities can be compared and ordered. It is worth mentioning here that there are several ordering relations found in the literature, the simplest of all being defined as follows. For fuzzy quantities $A$ and $B$ on $\mathbb{R}$, $A \succeq B$ if and only if $A(x) \geq B(x)$ for all $x \in \mathbb{R}$. The fuzzy quantities $A$ and $B$ are equal, denoted by $A \approx B$ if and only if $A \succeq B$ and $B \succeq A$. A second definition used by Mareš and discussed in [23] is based on the idea that the validity of some relation between quantities with vague values is also generally vague. Thus the ordering relation between fuzzy quantities can also be thought of as a fuzzy relation which is for every pair $A$ and $B$ of fuzzy quantities valid with some possibility. Thus we have the following definition.

**Definition 2.12.** For any pair of fuzzy quantities $A$ and $B$ the possibility of the relation $A \succeq B$ is defined as a real number $\mu_\succeq(A, B) \in [0, 1]$ such that

$$\mu_\succeq(A, B) = \sup_{x,y \in \mathbb{R}\atop x \geq y}[\min(A(x), B(y))]$$ \hspace{1cm} (2.4)
Definition 2.13. The equivalence (similarity) relation between two fuzzy quantities, denoted by \( A \sim B \) is a fuzzy relation which is valid with the possibility

\[
\mu_{\sim}(A, B) = \sup_{x \in \mathbb{R}} \min(A(x), B(x))
\]  

(2.5)

Observe that if \( \mu_{\geq}(A, B) = \mu_{\geq}(B, A) = 1 \), then \( \mu_{\sim}(A, B) = 1 \) also. In general however \( \mu_{\sim}(A, B) \geq \min(\mu_{\geq}(A, B), \mu_{\geq}(B, A)) \). The following definition of a fuzzy number given by Yu and Zhang [46] is more restrictive than that of a general fuzzy quantity.

Definition 2.14. A fuzzy number, denoted by \( A \), is a fuzzy subset of \( \mathbb{R} \) satisfying the following conditions:

(a) there exists at least one number \( a_0 \in \mathbb{R} \) such that \( A(a_0) = 1 \);

(b) \( A(x) \) is non-decreasing on \( (-\infty, a_0] \) and non-increasing on \( [a_0, \infty) \);

(c) \( A(x) \) is upper semi-continuous, i.e.,

\[
\lim_{x \to x_0^+} A(x) = A(x_0) \text{ if } x_0 < a_0 \text{ and } \lim_{x \to x_0^-} A(x) = A(x_0) \text{ if } x_0 > a_0;
\]

(d) \( \text{Supp}(A) \) is compact, where \( \text{Supp}(A) = \{ x \in \mathbb{R} | A(x) > 0 \} \).\(^1\)

Let \( <0> \) denote the fuzzy set in which every element of \( \mathbb{R} \) has zero membership value.

Definition 2.15. For a fuzzy number \( A \) and a \( \lambda \in (0, 1] \), the \( \lambda \)-level set is defined as \( [A]_\lambda = \{ x \in \mathbb{R} | A(x) \geq \lambda \} \). The 0-level set \( [A]_0 \) is defined as the closure of \( \text{Supp}(A) \). It follows that each of the \( \lambda \)-level sets of \( A, \lambda \in (0, 1] \) is an interval number, which we denote by \( A_{\lambda} = [A^L_{\lambda}, A^R_{\lambda}] \) where \( A^L_{\lambda} \) and \( A^R_{\lambda} \) denote respectively the lower and the upper bounds of \( [A]_\lambda \).

\(^1\)This assumption is however not standard. The most standard assumption found in the literature is that closure of the support is compact. Few authors alternatively accept that the support needs only to be bounded. In the literature, some more alternatives such as the fuzzy set \( A \) is convex etc. are also found.
Yu and Zhang [46] mentioned that algebraic operations on fuzzy numbers defined in Definition 2.11 are not easy to perform and therefore following Dubois et al. [23] they have used alternatively the following interval algebra using the level sets of fuzzy numbers.

(i) \( [A + B]_\lambda = [A]_\lambda + [B]_\lambda = [A^L_\lambda + B^L_\lambda, A^R_\lambda + B^R_\lambda] \).

(ii) \( [A - B]_\lambda = [A]_\lambda - [B]_\lambda = [A^L_\lambda - B^R_\lambda, A^R_\lambda - B^L_\lambda] \).

(iii) \( [mA]_\lambda = m[A]_\lambda = [mA^L_\lambda, mA^R_\lambda] \), for all \( m \in \mathbb{R}, m > 0 \).

Fuzzy numbers are ordered using interval logic [23] as follows. For any two fuzzy numbers \( A \) and \( B \),

(a) \( A \succeq B \) if and only if \( A^L_\lambda \geq B^L_\lambda \) and \( A^R_\lambda \geq B^R_\lambda \) for all \( \lambda \in (0, 1] \);

(b) \( A = B \) if and only if \( A \succeq B \) and \( B \succeq A \);

(c) \( A \subseteq B \) if and only if \( A^L_\lambda \geq B^L_\lambda \) and \( A^R_\lambda \leq B^R_\lambda \) \( \forall \lambda \in (0, 1] \)

The Hukuhara difference between fuzzy numbers following Dubois et al. [23] is given as follows.

**Definition 2.16.** Let \( A \) and \( B \) be fuzzy numbers. If there exists a fuzzy number \( C \) such that \( A = B \oplus C \), then \( C \) is called the Hukuhara difference, denoted by \( C = A \ominus_H B \).

Note that the Hukuhara difference between two fuzzy numbers need not always exist. In [23], its properties are extensively discussed.

### 3 Cooperative Games with Fuzzy Coalitions

As already mentioned in the Introduction, cooperative games where players have put only partial participations in various coalitions have been studied by many researchers. In this section, we shall discuss few particular classes of cooperative games with fuzzy coalitions.
derived from their crisp counterparts. It is worth mentioning here that most of the studies in this direction are made on such particular classes of games which are dependent on another class of crisp games. Only Azriely and Lehrer [7] have so far discussed a general class of fuzzy games independent to any class of crisp games. However, our primary focus being on the Shapley value, we keep our concern to games having Shapley value as a solution concept in the literature. Generally speaking, it is not easy to give the explicit form of a Shapley function on any class of fuzzy games. Butnariu [14] introduced a limited class of fuzzy games and obtained the explicit form of a Shapley function. Any fuzzy game in that class is completely specified by a crisp game. Because a fuzzy game which has no relation with any crisp games is not reasonable, the class can be considered reasonable in this sense. However, at a later stage, it was found not to have adequate properties. Tsurumi et al. [43] gave a new class of fuzzy games which is more natural than Butnariu’s. Any fuzzy game in the new class can be also derived from a crisp game. In order to define these two classes of games (i.e., by Butnariu and Tsurumi et al.), we need the following notions. Let \( N = \{1, 2, 3, ..., n\} \) be the set of players.

**Definition 3.1.** A fuzzy coalition \( S \) is a fuzzy subset of \( N \), that can be identified as a real valued function \( S : N \to [0, 1] \). Thus, for a fuzzy coalition \( S \) and a player \( i \) of \( N \), \( S(i) \) denotes the membership grade of \( i \) in \( S \). We also call \( S(i) \) to be the rate of the \( i \)th player’s participation in the coalition \( S \). If \( S(i) = s_i \), then \( S \) can also be expressed as an \( n \)-tuple of the form \( s = (s_1, s_2, ..., s_n) \). The empty coalition where no player has any non-zero membership is denoted by \( \emptyset \).

If \( S \) is a fuzzy coalition of \( N \), its support is denoted as \( Supp(S) \) and is defined by \( Supp(S) = \{ i \in N | S(i) > 0 \} \). A fuzzy coalition \( S \) can also be seen as a partition of the sets of players into the coalitions \( S_t = \{ i \in N | S(i) = t \} \) for \( t \in [0, 1] \), see [15]. For any level \( \lambda \in [0, 1] \) and any fuzzy coalition \( S \), the \( \lambda \)-level set \( [S]_\lambda \) is defined as \( \{ i \in N | S(i) \geq \lambda \} \).\(^2\)

\(^2\)This definition includes the case \( \lambda = 0 \) and differs from the one given in Definition 2.15 as the fuzzy
The class of all fuzzy subsets of a coalition $U$ ($U$ may be crisp or fuzzy) is denoted by $L(U)$. Note that by a fuzzy subset $S$ of a fuzzy set (coalition) $U$, we mean a fuzzy set $S$ satisfying $S(i) \leq U(i), \forall i \in N$ and denote by $S \preceq U$.

**Definition 3.2.** [12] A cooperative game with fuzzy coalitions or simply a fuzzy game is a pair $(N, v)$ in which the function

$$v : L(N) \rightarrow \mathbb{R}$$

is such that $v(\emptyset) = 0$. $G_F(N)$ denotes the class of all fuzzy games $(N, v)$. A game can also be represented by its characteristic function (crisp or fuzzy) only, when there is no ambiguity in the players’ set $N$.

**Definition 3.3.** A game $v \in G_F(N)$ is superadditive, if

$$v(S \cup T) \geq v(S) + v(T), \forall S, T \in L(N) : S \cap T = \emptyset.$$

**Definition 3.4.** A game $v \in G_F(N)$ is convex, if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \forall S, T \in L(N).$$

In what follows we define the class of games with fuzzy coalitions given by Butnariu [14]

**Definition 3.5.** Given a crisp game $v \in G^N$ and $S \in L(N)$. A game $v^p \in G_F(N)$ is said to be “with proportional values” if and only if,

$$v^p(S) = \sum_{t \in [0,1]} v(S_t) t \forall S \in L(N).$$

Note that the sum occurring here is well-defined as all but finitely many terms of it are zero. The set of all fuzzy games with proportional values is denoted by $FG_F(N)$.
Observe that a sufficiently large number of fuzzy games in $FG_P(N)$ is neither monotone nondecreasing nor continuous with respect to the rates of players’ participation. In the following we describe the fuzzy game with Choquet Integral form defined by Tsurumi et al. [43] as a follow up to such criticisms.

**Definition 3.6.** [43] Given a superadditive crisp game $v \in G_C(N)$ and $S \in L(N)$, let $Q(S) = \{ S(i) | S(i) > 0, i \in N \}$ and let $q(S)$ be the cardinality of $Q(S)$. The elements of $Q(S)$ in increasing order can be written as $h_1 < ... < h_{q(S)}$. A game $v^c \in G_F(N)$ is said to be a fuzzy game with Choquet integral form if and only if the following holds:

$$v^c(S) = \sum_{l=1}^{q(S)} v([S]_{h_l}).(h_l - h_{l-1})$$

for any $S \in L(N)$, where $h_0 = 0$. The class of all fuzzy games with Choquet Integral form is denoted by $FG_C(N)$. Tsurumi et al. have shown that the fuzzy games with Choquet Integral form are monotonic non-decreasing with respect to players’ participations and continuous under the metric $d$ over $L(N)$ given by $d(S, T) = \max_{i \in N} \{|S(i) - T(i)|\}$, $\forall S, T \in L(N)$ along with the usual metric on $\mathbb{R}$.

Recently Butnariu and Kroupa [15] have proposed a new class of games namely “games with weights” which is a generalization of Butnariu’s [14] class of games with proportional values and obtained the corresponding Shapley value. This value coincides with the Shapley value of the games with proportional values when the weights are taken as the membership values of the players in a coalition. In the following we define the aforementioned class of games, its Shapley value and the related theorems for existence, as proposed by Butnariu and Kroupa [15].

**Definition 3.7.** Given a crisp cooperative game $v \in G^N$ and a function $\psi : [0, 1] \rightarrow \mathbb{R}$ that satisfies $\psi(0) = 0$ and $\psi(1) = 1$, a game $v^\psi \in G_F(N)$ is called a fuzzy game with weight function $\psi$ if $v^\psi$ satisfies

$$v^\psi(S) = \sum_{r \in [0,1]} \psi(r).v(S_r)$$

(3.1)
Note that the sum occurring here is well-defined since all but finitely many terms of it are zero. The set of games with weight functions is denoted by \( \text{FG}_W(N) \). Note further that \( \text{FG}_P(N) \) corresponds to \( \psi = \text{id}_{[0,1]} \) (identity on \([0,1]\)).

**Definition 3.8.** Let \( v \) be a fuzzy game and \( A \) a fuzzy coalition, then the fuzzy coalition \( B \) is called a \( v \)-carrier of \( A \) if the following two conditions are satisfied:

(i) \( B_t \subseteq A_t \) for every \( t \in (0,1] \).

(ii) if \( C \in L(N) \) and \( C_t \subseteq A_t \) for every \( t \in (0,1] \), then \( v(B_t \cap C_t) = v(C_t) \).

It is worth mentioning here that \( A \) is a trivial \( v \)-carrier of \( A \), independently of \( v \).

**Definition 3.9.** For every permutation \( \pi \) of \( N \), every \( A \in L(N) \), and a fuzzy game \( v \), we denote \( \pi A = A \circ \pi^{-1} \) and \( \pi v(A) = v(\pi^{-1}A) \). Clearly, if \( v \in \text{GF}_F(N) \), then \( \pi v \in \text{GF}_F(N) \).

Butnariu and Kroupa gave a formal definition of the Shapley mapping as follows.

**Definition 3.10.** A Shapley mapping is a linear function \( \Phi : \text{FG}_W(N) \to (\mathbb{R}^N)^L(N) \) satisfying the following conditions for any \( v \in \text{FG}_W(N) \) and any \( A \in L(N) \):

**Axiom 1.** (Coalitional Efficiency) For every \( v \)-carrier \( B \in L(N) \) of \( A \) we have

\[
\sum_{i \in N : B(i) > 0} \Phi_i(v)(A) = v(B)
\]

**Axiom 2.** (Non-Member) If for some \( j \in N \), \( A(j) = 0 \), then \( \Phi_j(v)(A) = 0 \).

**Axiom 3.** (Symmetry) If \( \pi \) is a permutation of \( N \), then \( \Phi_{\pi i}(\pi v)(\pi A) = \Phi_i(v)(A) \).

After giving the axiomatic characterization of the Shapley mapping Butnariu and Kroupa proved the following theorem.
**Theorem 3.11.** There exists a unique Shapley mapping $\Phi^W : \text{FG}_W(N) \to (\mathbb{R}^N)^L(N)$ and it is given by the following formula:

$$
\Phi^W_i(v)(A) = \begin{cases} 
\psi(r) \sum_{S \in \mathcal{P}_i(A_r)} \frac{(|S|-1)!(|A_r|-|S|)!}{|A_r|!} (v(S) - v(S \setminus \{i\})) & \text{if } A(i) = r > 0 \\
0 & \text{otherwise}
\end{cases}
$$

(3.2)

where $\mathcal{P}_i(A_r) = \{R \subseteq N | i \in R \text{ and } R \subseteq A_r\}$.

Following Theorem 3.11, we obtain a Shapley mapping $\Phi^P : \text{FG}_P(N) \to (\mathbb{R}^N)^L(N)$ for the class $\text{FG}_P(N)$ with the substitution of $\psi(r)$ by $r$ as follows.

$$
\Phi^P_i(v)(A) = \begin{cases} 
r \sum_{S \in \mathcal{P}_i(A_r)} \frac{(|S|-1)!(|A_r|-|S|)!}{|A_r|!} (v(S) - v(S \setminus \{i\})) & \text{if } A(i) = r > 0 \\
0 & \text{otherwise}
\end{cases}
$$

(3.3)

where $\mathcal{P}_i(A_r) = \{R \subseteq N | i \in R \text{ and } R \subseteq A_r\}$. Now we discuss the Shapley axioms proposed by Tsurumi et al. [43]. Prior to this let us define the following:

**Definition 3.12.** Let $\gamma$ satisfy $0 \leq \gamma < U(i)$ for a $U \in L(N)$. Player $i \in \text{Supp}(U)$ is said to be a $\gamma$-null player in $U$ for $v \in \text{GF}(N)$ if $v(S) = v(S^U_i)$ for all $S \in L(U)$, such that $S(i) > \gamma$. where the fuzzy subset $S^U_i \in L(U)$ is defined as

$$
S^U_i(j) = \begin{cases} 
U(i) & \text{if } j = i \\
S(j) & \text{otherwise}
\end{cases}
$$

(3.4)

for any $U \in L(N)$, $S \in L(U)$ and $i \in N$.

Observe that a $\gamma$-null player is an extension of a null player to fuzzy games. The definition implies that there may exist some player who cannot contribute to the coalition value further if the rate of his participation exceeds a certain rate $\gamma$.

**Definition 3.13.** Let $v \in \text{GF}(N)$ and $U \in L(N)$. A fuzzy coalition $S \in L(U)$ is called an $f$-carrier in $U$ for $v$ if it satisfies $v(S \cap T) = v(T)$ $\forall$ $T \in L(U)$. The set of all $f$-carriers in $U$ for $v$ is denoted by $\text{FC}(U|_v)$. 

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The notion of an \emph{f-carrier} indicates that no extra worth will be generated by a fuzzy coalition even one increases her participation in it than that in the \emph{f-carrier}.

Let \( U \in L(N) \) and \( i, j \in N \). For any \( S \in L(U) \), define \( S^U_{ij} \in L(U) \) by

\[
S^U_{ij}(k) = \begin{cases} 
\min\{S(i), U(j)\} & \text{if } k = i \\
\min\{S(j), U(i)\} & \text{if } k = j \\
S(k) & \text{otherwise}
\end{cases}
\]  

(3.5)

For any \( S \in L(N) \), define \( P_{ij}[S] \) by

\[
P_{ij}[S](k) = \begin{cases} 
S(j) & \text{if } k = i \\
S(i) & \text{if } k = j \\
S(k) & \text{otherwise}
\end{cases}
\]  

(3.6)

The Shapley mapping defined by Tsurumi et al. is given as follows.

\textbf{Definition 3.14.} Let \( G' \subseteq G_F(N) \). A mapping \( \Phi : G' \to (\mathbb{R}^n)^{L(N)} \) is said to be a Shapley mapping on \( G' \) if it satisfies the following four axioms.

\textit{Axiom 1.} If \( v' \in G' \) and \( U \in L(N) \) then

\[
\begin{cases} 
\sum_{i \in N} \Phi_i(v')(U) = v'(U) \\
\Phi_i(v')(U) = 0 \quad \forall i \not\in \text{Supp}(U)
\end{cases}
\]

where \( \Phi_i(v')(U) \) is the \( i \)th element of \( \Phi(v')(U) \in \mathbb{R}^n \).

\textit{Axiom 2.} If \( v' \in G' \), \( U \in L(N) \) and \( T \in FC(U|_{v'}) \), then

\[ \Phi_i(v')(U) = \Phi_i(v')(T) \quad \forall i \in N. \]

\textit{Axiom 3.} If \( v' \in G' \), \( U \in L(N) \), \( U^U_{ij} \in FC(U|_{v'}) \) and \( v'(S) = v'(P_{ij}[S]) \) for any \( S \in L(U^U_{ij}) \) then \( \Phi_i(v')(U) = \Phi_j(v')(U) \).
Axiom 4. For any \( v'_1, v'_2 \in G' \), define a game \( v'_1 + v'_2 \) by \((v'_1 + v'_2)(S) = v'_1(S) + v'_2(S)\) for any \( S \in L(N) \). If \( v'_1 + v'_2 \in G' \) and \( U \in L(N) \) then
\[
\Phi_i(v'_1 + v'_2)(U) = \Phi_i(v'_1)(U) + \Phi_i(v'_2)(U) \quad \forall i \in N.
\]

Under these four axioms, the Shapley value for the fuzzy games with Choquet Integral form is obtained according to the following theorem.

**Theorem 3.15.** The mapping \( \Phi^c : FG_C(N) \to (\mathbb{R}^n)^{L(N)} \) given by
\[
\Phi^c_i(v')(U) = \sum_{l=1}^{\eta(U)} \Phi_i^l(v)([U]_{h_l}).(h_l - h_{l-1})
\] (3.7)
where \( \Phi_i^l(v) \) is the Shapley value of the crisp game \( v \in G_C(N) \) is a Shapley mapping on \( FG_C(N) \).

Li and Zhang [28] in a recent paper argued that there may exist some fuzzy coalitions whose payoff can not be expressed by crisp coalition values and participation levels. They have given a simplified expression of the fuzzy Shapley mapping for such a general fuzzy game that is independent of any crisp counterpart and have shown that the Shapley mappings for the classes \( FG_P(N) \) and \( FG_C(N) \) are particular cases of this function. In what follows we take this general class \( G_F(N) \) of fuzzy games, define their Shapley mapping and provide its explicit expression. Preparatory to the development we need to make a slight change to our previous representation of a fuzzy coalitions and the related notions. Any fuzzy coalition \( S \) can also be viewed as an \( n \)-tuple \((s_1, s_2, ..., s_n)\) where \( s_i \) represents the membership function of player \( i \). This indeed is a makeover of our earlier notation of \( S \) so that \( S(i) = s_i \). If \( e^j \) denotes the unit vector \((x_1, x_2, ..., x_n)\) such that \( x_i = 1 \) if \( i = j \) and \( x_i = 0 \) if \( i \neq j \), then the fuzzy coalition \( S = (s_1, s_2, ..., s_n) \) can also be written as \( \sum_{j \in N} s_j e^j \). Similarly we derive a fuzzy coalition \( S_T \) from the fuzzy coalition \( S \) with respect to a crisp coalition \( T \) as \( S_T = \sum_{k \in T} s_k e^k \).

**Definition 3.16.** Given \( v \in G_F(N) \), \( S \in L(N) \), a mapping \( \Phi(S) : G_F(N) \to \mathbb{R}^n \) is called a Shapley mapping on \( G_F(N) \) based on \( S \) if it satisfies the following three axioms.
Axiom 1. If \( v \in G_F(N) \) and \( S \) is an \( f \)-carrier (Li and Zhang have used the term carrier), then
\[
\sum_{i \in N} \Phi_i(S)(v) = v(S).
\]

Axiom 2. If \( v \in G_F(N) \) and \( \pi S \) is a permutation of \( S \), then for any \( i \in N \),
\[
\Phi_{\pi_i}(\pi S)(\pi v) = \Phi_i(S)(v).
\]

Axiom 3. For any \( v_1, v_2 \in G_F(N) \), then
\[
\Phi_i(S)(v_1 + v_2) = \Phi_i(S)(v_1) + \Phi_i(S)(v_2) \quad \forall i \in N.
\]

Theorem 3.17. Given \( v \in G_F(N) \) and a fuzzy coalition \( S = (s_1, s_2, \ldots, s_n) = \sum_{i \in N} s_i.e_i \), then the value \( \Phi(S)(v) \) given by,
\[
\Phi_i(S)(v) = \sum_{i \in T \subseteq N} \frac{(|T| - 1)!(|N| - |T|)!}{|N|!} \left[ v \left( \sum_{j \in T} s_je^j \right) - v \left( \sum_{j \in T \setminus i} s_je^j \right) \right]
\]
(3.8)
is a Shapley value.

Remark that the interpretation of the Shapley mapping given in (3.8) is more intuitive and simpler than the earlier games. The following theorem relates it with the Shapley values for \( FG_P(N) \) and \( FG_C(N) \).

Theorem 3.18. Let \( S = (s_1, s_2, \ldots, s_n) = \sum_{i \in N} s_i.e^i \in L(N) \).

(a) If \( v \in FG_W(N) \), then \( \Phi(S)(v) = \Phi^P(v)(S) \).

(b) If \( v \in FG_C(N) \), then \( \Phi(S)(v) = \Phi^C(v)(S) \).

The proof follows immediately from the definitions of the two classes \( FG_P(N) \) and \( FG_C(N) \).
4 Cooperative Games with Fuzzy Characteristic Function

The pioneering works in this area are credited to Mareš [30], Nishizaki and Sakawa [35], Mareš and Vlach [29, 32], Cortes [18] and so on. All these researchers were concerned about the uncertainty of the value of the characteristic function associated with a game. In their models, the domain of the characteristic function of a game remains to be the class of crisp (deterministic) coalitions but the values assigned to them are fuzzy quantities. The underlying idea is that all players and coalitions know the expected payoffs even before the negotiation process, is evidently unrealistic. In fact, during the process of negotiation and coalition forming, the players can have only vague idea about the real outcome of the situation, and this vague expectation can be modeled by mathematical tools, see [30]). Mareš in his book [30] has mentioned that the vagueness of the expected outcomes implies the vagueness of the solution concepts such as the Shapley value. Therefore, under this formulation, the Shapley value becomes a fuzzy quantity giving membership to every vector of real numbers indicating its closeness to the modal Shapley value. In what follows we start with the definition of a fuzzy characteristic function and a corresponding cooperative game with fuzzy characteristic function.

**Definition 4.1.** Let us consider a crisp cooperative game \((N, v)\). Let the function \(w : 2^N \rightarrow L(\mathbb{R})\) be such that for each \(K \in 2^N\), \(w(K)\) is a fuzzy quantity satisfying

(a) \(w(K)(v(K)) = 1\),

(b) \(w(K)\) is non-decreasing for \(x < v(K)\) and non-increasing for \(x > v(K)\),

(c) \(w(\emptyset)(x) = 1\) for \(x = 0\) and \(w(\emptyset)(x) = 0\) for \(x \neq 0\).

then \(w\) is called a fuzzy characteristic function and the pair \((N, w)\) is called the cooperative game with fuzzy characteristic function with respect to \((N, v)\). We also call \((N, w)\) a fuzzy extension of \((N, v)\).
When the crisp game $v \in G_C(N)$ is replaced by its fuzzy counterpart $w$ the corresponding Shapley value becomes a vector of fuzzy quantities. Thus we define a Shapley value for the cooperative games with fuzzy characteristic function as follows:

**Definition 4.2.** The fuzzy quantity $\Phi_i^M(w)$ for $i \in N$ given by

$$
\Phi_i^M(w) = \bigoplus_{K \subseteq N} \frac{(n-k)!(k-1)!}{n!} [w(K) \oplus (-w(K \setminus \{i\}))]
$$

is the $i$th component of the Shapley value $\Phi^M(w)$ of $w$, where the sum $\oplus$ between fuzzy quantities is given by Definition 2.11. Thus $\Phi_i^M(w)(x)$ for $x \in \mathbb{R}$ determines the possibility with which $x$ is the value of $\Phi_i^M(w)$ for each $i$.

Mareš [30] found that $\Phi_i^M(w)$ satisfies Shapley like properties in fuzzy sense. They are as follows.

**Theorem 4.3.** $\Phi_i^M(w)$ satisfies following properties:

P1. (Symmetry) The values $\Phi_i^M(w)(x)$ do not depend on the order of player $i \in N$.

P2. (Efficiency) Denote by $\Phi_i^M(w) = \bigoplus_{i \in N} \Phi_i^M(w)$. If $(N, w)$ is a fuzzy extension of $(N, v)$ and $\Phi_i(v)$ are Shapley values of $(N, v)$ then $\sum_{i \in N} \Phi_i(v)$ is the modal value of $\Phi^M(w)$. This would imply that $\Phi^M(w) \succeq w(N)$ and $w(N) \succeq \Phi^M(w)$ and consequently $\Phi^M(w) = w(N)$.

P3. (Additivity) Let $(N, v)$ and $(N, v')$ be crisp games and $(N, w)$ and $(N, w')$ are their fuzzy extensions. Let also $\Phi_i(v)$ and $\Phi_i(v')$, $i \in N$ be their respective Shapley values. Define $v + v'$ and $w + w'$ as follows.

$$(v + v')(K) = v(K) + v'(K) \text{ and } (w + w')(K) = w(K) \oplus w'(K), \ \forall K \subseteq N,$$

If $\Phi_i(v + v')$ is the Shapley value of $v + v'$ then it is the modal value of $\Phi_i^M(w \oplus w')(K)$. 

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In [31], a more general formulation of a cooperative game with fuzzy characteristic function is proposed. An interesting part of this proposal is its independence from a crisp game and so this game is not in general an extension of a crisp game. However, the extended fuzzy game can be considered as a particular type of such games under certain conditions.

**Definition 4.4.** A cooperative game with fuzzy characteristic function and transferable utility, briefly a TU fuzzy game or a fuzzy game, is a function $w : 2^N \to L(\mathbb{R})$ such that $w(\emptyset) = <0>$, where $<0>$ denotes the fuzzy set in which every element of $\mathbb{R}$ has zero membership value and $N$ being the player set. The set of all fuzzy games with player set $N$ is denoted by $FG_M(N)$.

In view of Definition 4.4, we call a fuzzy game $w : 2^N \to L(\mathbb{R})$ a fuzzy extension of the game $v : 2^N \to \mathbb{R}$ if, for each $K \subseteq N$, $w(K)(v(K)) = 1$.

**Definition 4.5.** Fuzzy Subsets Generated by Fuzzy Games: Let $w : 2^N \to L(\mathbb{R})$ be a fuzzy game, and let $\rho_w : G^N \to [0, 1]$ be defined by

$$\rho_w(v) = \min_{K \subseteq N} w(K)(v(K))$$

In this way, each fuzzy game $w$ generates a fuzzy subset of $G^N$, namely, the fuzzy subset the membership function of which is $\rho_w$. It has further been shown that the fuzzy subsets generated by different fuzzy games are not always different.

The Shapley value of a fuzzy game in $FG_M(N)$ is defined as follows.

**Definition 4.6.** The Shapley value of a fuzzy game $w \in FG_M(N)$ is the vector $(\Phi^{MV}_i(w))_{i=1}^n$ of fuzzy quantities defined by $\Phi^{MV}_i(w)(x) = \sup_{1 \leq i \leq n} \rho_w(v), 1 \leq i \leq n$, where the supremum is taken over all games $v \in G^N$ such that $x = \Phi_i(v)$.

The Shapley value given by Definition 4.6 gives rise to the following properties similar to its counterparts in crisp environment.
Theorem 4.7. If \( w \) is a fuzzy extension of a game \( v \in G^N \), then \( \Phi_i^{MV}(w)(\Phi_i(v)) = 1 \) for each \( i \in N \).

Theorem 4.8. If \( w_1 \) and \( w_2 \) are fuzzy games with player set \( N \) such that \( w_1(K)(x) \geq w_2(K)(x) \) for each coalition \( K \) and each \( x \in \mathbb{R} \), then \( \Phi_i^{MV}(w_1)(x) \geq \Phi_i^{MV}(w_2)(x) \) for each \( x \in \mathbb{R} \) and each \( i \in N \).

Theorem 4.9. If \( w \) is a fuzzy extension of \( v \in G^N \), then there exists a point \( x = (x_1, x_2, ..., x_n) \) from \( \mathbb{R}^n \) such that \( \Phi_i^{MV}(w)(x_i) = 1 \) for each \( i \in N \), and \( \sum_{i \in N} x_i = v(N) \).

Define the sum \( w_1 \oplus w_2 \) of fuzzy games \( w_1, w_2 \in FG_M(N) \) by \( (w_1 \oplus w_2)(K) = w_1(K) \oplus w_2(K) \). Since \( < 0 > + < 0 > = < 0 > \), \( w_1 \oplus w_2 \) is also a member of \( FG_M(N) \).

Theorem 4.10. If \( w_1 \) and \( w_2 \) are fuzzy extensions of a game \( v \in G^N \), then there exist \( x, y \in \mathbb{R}^n \) such that, for each \( i \in N \),

\[
\Phi_i^{MV}(w_1)(x_i) = \Phi_i^{MV}(w_2)(y_i) = \Phi_i^{MV}(w_1 \oplus w_2)(x_i + y_i).
\]

A more general form of Theorem 4.10 is given as follows:

Theorem 4.11. If \( w_1 \) and \( w_2 \) are fuzzy extensions of games \( v_1, v_2 \in G^N \), then there exist \( x, y \in \mathbb{R}^n \) such that, for each \( i \in N \),

\[
\Phi_i(w_1)(x_i) = \Phi_i(w_2)(y_i) = \Phi_i(w_1 \oplus w_2)(x_i + y_i).
\]

5 Cooperative Games with Fuzzy Coalitions and Fuzzy Characteristic Function

In this section, we consider a cooperative game with fuzzy coalitions and fuzzy characteristic function simultaneously. Here, the characteristic value for each of the fuzzy coalitions is again a fuzzy quantity which maps the set of real numbers to the closed interval \([0, 1]\).
This idea was initiated by Borkotokey [11] who has defined a new class of games, with a fixed level of significance $\delta$. We describe this model along with the corresponding Shapley Axioms. The fuzzy counterparts of superadditivity, convexity, continuity, monotonicity of crisp games are found to be retained by the members of this new class of games. If the number of fuzzy coalitions is countable, the fuzzy characteristic function behaves as a step function. Next we describe another model of games with fuzzy coalitions and fuzzy characteristic functions put forward by Yu and Zhang [46] which is based on the Hukuhara difference. The Shapley mapping obtained in this model is characterized by the Shapley axioms proposed in [11]. In what follows we define a cooperative game with fuzzy coalitions and fuzzy characteristic function given in [11].

**Definition 5.1.** Let $(N, v)$ be a cooperative game with fuzzy coalitions. An extension $w$ of $v$ is a function $w : L(N)^R \rightarrow L([0,1])$ such that for each $K \in L(N),$

(a) $w(K)(v(K)) = 1.$

(b) $\exists \ x_1, x_2 \in \mathbb{R}$ such that $w(K)(x) = 0$ for $x \notin [x_1, x_2].$

(c) $w(K)(x) = 0$ if $K = \emptyset$ considered as a fuzzy coalition.

We call the game $(N, w)$, an extension of fuzzy game $(N, v)$ with fuzzy coalitions and fuzzy characteristic functions (vague expectations) or an extended fuzzy game in short. The class of all such games is denoted by $FG_B(N)$. Using Definition 2.11, we have the following.

**Definition 5.2.** We call a game $w \in FG_B(N)$ superadditive, if

$$w(S \cup T) \succeq w(S) \oplus w(T), \ \forall S, T \in L(N) : S \cap T = \emptyset$$

and additive, if

$$w(S \cup T) \approx w(S) \oplus w(T), \ \forall S, T \in L(N) : S \cap T = \emptyset.$$

**Definition 5.3.** For fuzzy coalitions $S$ and $T$ of $N$, and $w \in FG_B(N)$ if

$$w(S \cup T) \bigoplus w(S \cap T) \succeq w(S) \bigoplus w(T)$$

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then $w$ is said to be convex.

For two extended fuzzy games $w_1$ and $w_2$, define $w_1 \oplus w_2$ by $(w_1 \oplus w_2)(S) = w_1(S) \oplus w_2(S)$. Then $w_1 \oplus w_2$ is an extended fuzzy game. In the following we define a special class of games.

**Definition 5.4.** Let $\delta > 0$ be a real number. For an arbitrary fuzzy coalition $S$ of $N$, we define the function

$$w_\delta : L(N)^R \rightarrow L([0,1])$$

by

$$w_\delta(S)(x) = \begin{cases} 
\bigvee_{K \in L(S)} \{ \bigvee_{i \in N} K(i) : x \neq v(S) \text{ and } |v(K) - x| \leq \delta \} 
& \text{if } x = v(S) \\
1 & \text{otherwise}
\end{cases}$$

where $(N, v) \in \text{FG}_C(N)$. $w_\delta$ thus defined, satisfies all the conditions of Definition 5.1. Thus $(N, w_\delta)$ is an extended fuzzy game with fuzzy coalitions and vague expectation. We call it an extended fuzzy game with $\delta$ level of significance. The corresponding class of such games is denoted by $\text{FG}_{\delta}^B(N)$. Following remarks are important.

**Remark 5.5.** The game $(N, w_\delta)$ is superadditive whenever $(N, v) \in \text{FG}_C(N)$ is superadditive.

**Remark 5.6.** The game $(N, w_\delta)$ is convex in fuzzy sense whenever $(N, v) \in \text{FG}_C(N)$ is convex.

**Remark 5.7.** For a sufficiently small $\delta$, $w_\delta(S)$ is continuous with respect to the players’ participation under the metric $d^* : L(N) \times L(N) \rightarrow \mathbb{R}_+ \cup \{0\}$ defined by

$$d^*(S, T) = \bigvee_{M \in L(S)} d(M, K)$$

(5.1)

where $d$ is another metric on $L(N)$ given by

$$d(S, T) = \bigvee_{i \in N} |S(i) - T(i)|$$
and the metric $d_o$ on $FG_C(N)$ defined by

$$d_o(w_\delta(S), w_\delta(T)) = \sup_{x \in \mathbb{R}_+ \cup \{0\}} \{|w_\delta(S)(x) - w_\delta(T)(x)|\} \tag{5.2}$$

Let $FG_{B_0}(N)$ denote the class of all extended games with fuzzy coalitions and vague expectations which are superadditive and convex. We now propose a new definition of Shapley mapping as a solution concept for the extended game with fuzzy coalitions and vague expectations (fuzzy characteristic fictions). In order to do so, we need to extend the concepts of $f$-carrier and $\gamma-$null player defined by Tsurumi et al. [43].

**Definition 5.8.** Let us define a fuzzy set $S^U_i \in L(U)$ by:

$$S^U_i(j) = \begin{cases} U(j) & \text{if } j = i \\ S(j) & \text{otherwise} \end{cases}$$

for any $U \in L(N), S \in L(U)$ and $i \in N$. Then for $\gamma$ satisfying $0 \leq \gamma < U(i)$ , a player $i \in \text{Supp}(U)$ is said to be an extended $\gamma-$null player in $U$ for $w \in FG_B(N)$ if $w(S) = w(S^U_i), \forall S \in L(U)$, such that $S(i) > \gamma$.

**Definition 5.9.** Let $w \in FG_B(N)$ and $U \in L(N)$ . $S \in L(U)$ is called an extended $f$-carrier in $U$ for $w$ if it satisfies $w(S \cap T) = w(T) \ \forall T \in L(U)$. The set of all $f$-carrier in $U$ for $w$ is denoted by $FC(U|w)$.

Note that from Definition 5.4, it is evident that if $S$ is an extended $f$-carrier in $U$ for $w_\delta \in FG_B^\delta(N)$ then it is also an $f$-carrier in $U$ for $v \in FG_C(N)$ and similarly if $i$ is an extended $\gamma-$null player for $w_\delta \in FG_B^\delta(N)$ then it is also a $\gamma-$null player for $v \in FG_C(N)$ .

**Definition 5.10.** Given the class $FG_{B_0}(N)$ of all extended games with fuzzy coalitions and vague expectations which are superadditive and convex and $N$ has $n$ players, a mapping $\Phi : FG_{B_0}(N) \rightarrow (\mathbb{R}^n)^{L(N)}$ is said to be a Shapley mapping on $FG_{B_0}(N)$ if it satisfies the following four axioms.
Axiom 1. If \( w \in FG_{B_0}(N) \) and \( U \in L(N) \) then 
\[
\bigoplus_{i=1}^{n} \Phi_i(w)(U) = w(U).
\]

where \( \Phi_i(w)(U) \) is the \( i \)th element of \( \Phi(w)(U) \in \mathbb{R}^n \).

Axiom 2. For \( w \in FG_{B_0}(N) \), \( U \in L(N) \), \( S \in FC(U|w) \) we have \( \Phi_i(w)(U) = \Phi_i(w)(S) \forall i \in N \).

Axiom 3. Let \( i \) and \( j \) be two \( \gamma \)-null players. Then for any fuzzy coalition \( U \) of \( N \) and \( w \in FG_{B_0}(N), \Phi_i(w)(U) = \Phi_j(w)(U) \).

Axiom 4. Let \( w_1 \) and \( w_2 \) \( w \in FG_{B_0}(N) \). If \( w_1 \oplus w_2 \in FG_{B_0}(N) \) and \( K \in L(N) \) then 
\[
\Phi_i(w_1 \oplus w_2)(K) = \Phi_i(w_1)(K) \oplus \Phi_i(w_2)(K).
\]

Remark 5.11. In [46], it has been shown that the Shapley axioms given in Definition 5.10 for the class \( FG_{B_0}(N) \) hold good for the class \( FG_{M}(N) \) when the fuzzy coalitions are replaced by their crisp counterparts.

In the following, we show that there exists a Shapley mapping for every member of the class \( FG^\delta_B(N) \). Given a fuzzy coalition \( U \) of \( N \), let \( r \) denote the cardinality of \( \text{Supp}(U) \).

Construct the function \( \beta^w_{\delta, U} : \mathbb{R}^n \to [0, 1]^n \) as 
\[
\beta^w_{i, \delta_i}(x_i) = \begin{cases} 
1 & \text{if } x_i = \Phi^c_i(v)(U) \\
\frac{r.x_i}{x} & \text{whenever } 0 < |\Phi^c_i(v)(U) - x_i| \leq \delta_i \\
0 & \text{otherwise}
\end{cases}
\]

where \( \delta_i \in \mathbb{R}, 1 \leq i \leq n \) is such that \( \sum_i^n \delta_i = \delta \) and 
\[
|\Phi^c_i(v)(U) - \frac{x_i}{r}| \leq \delta_i
\]

and \( \beta^w_{i, \delta_i} \) represents the \( i \)th component of the vector \( \beta^w_{\delta, U} \). For \( z \in \mathbb{R} \), We observe that, 
\[
\bigoplus_{i=1}^{n} \beta^w_{i, \delta_i}(z) = 1. \quad (5.3)
\]
Since, for $z \in \mathbb{R}$, we have
\[
\bigoplus_{i=1}^{n} \beta^{w_k U}_{i, \delta_i}(z) = \bigvee_{z = \sum_{i=1}^{n} x_i} \left\{ \bigwedge_{i=1}^{n} \beta^{w_k U}_{i, \delta_i}(x_i) \right\}.
\]
\[
= \bigvee_{z = \sum_{i=1}^{n} x_i} \left\{ \bigwedge_{i=1}^{n} \frac{r_i x_i}{x} \right\}.
\]
\[
= \frac{r}{x} \bigvee_{z = \sum_{i=1}^{n} x_i} \left\{ \bigwedge_{i=1}^{n} x_i \right\}.
\]
\[
= \frac{r}{x} \times \frac{x}{r} = 1.
\]

**Remark 5.12.** Because of (5.3), we call the fuzzy quantities $\beta^{w_k U}_{i, \delta_i}$ fuzzy weights, associated with the players’ participation in a fuzzy coalition. We define a Shapley mapping in $FG^\delta_B(N)$ as follows:

**Definition 5.13.** Let us take an arbitrary payment vector $x \in \mathbb{R}^n_+$, such that $x = (x_1, ..., x_n)$, $x_i$ being the amount to be paid to the $i$-th agent after the game is played. Also let $x = \sum_{i=1}^{n} x_i$, be the total amount to be paid to the participants of the game with respect to the payment vector $x$.

Define a function $\Phi^\delta : (FG^\delta_B(N))^\mathbb{R} \to [0, 1]^{L(N)}$ by

\[
\Phi^\delta_i(w)(U)(x_i) = \begin{cases} 
    w_\delta(U)(x) \wedge \beta^{w_k U}_{i, \delta_i}(x_i) & \text{if } i \in \text{Supp}(U) \\
    0 & \text{otherwise}
\end{cases}
\]

The following theorem follows immediately.

**Theorem 5.14.** The mapping given in Definition 5.13 is a Shapley mapping on $FG^\delta_B(N)$.

The alternative model of a class of games with fuzzy coalitions and fuzzy characteristic functions based on Hukuhara difference and given by Yu and Zhang [46] is described in the following. The underlying idea is that initially a cooperative game with fuzzy characteristic function over the set of crisp coalitions is considered according to Definition 4.4. The corresponding Shapley value for this class of games is obtained based on the Hukuhara
difference. This is followed by an extension of such games to incorporate fuzzy coalitions and a corresponding Shapley value is proposed for games with fuzzy coalitions and fuzzy characteristic functions.

**Definition 5.15.** For any two crisp coalitions \(S, T \in 2^N\) such that \(S \cap T = \emptyset\), if for \(\lambda \in (0, 1]\), the \(\lambda\)-level sets \(w_\lambda\) of the cooperative games with fuzzy characteristic function \(w \in FG_M(N)\) satisfy
\[
w_\lambda(S \cap T) \geq w_\lambda(S) + w_\lambda(T), \quad \lambda \in (0, 1].
\]
then \(w\) is called a superadditive game due to Yu and Zhang.

The following theorem follows from the existence of Hukuhara difference between any two fuzzy numbers [23].

**Theorem 5.16.** If for any crisp coalition \(T \subseteq S\) and any \(\lambda, \beta \in (0, 1]\) such that \(\beta > \lambda\), the superadditive game \((N, w)\) due to Yu and Zhang also satisfies:
\[
w^L_\lambda(S) - w^L_\lambda(T) \leq w^L_\beta(S) - w^L_\beta(T) \leq w^R_\beta(S) - w^R_\beta(T) \leq w^R_\lambda(S) - w^R_\lambda(T)
\]
then the Hukuhara difference \(w(S) \ominus_H w(T)\) exists.

We denote by \(FG_H(N)\) the set of superadditive games with fuzzy characteristic functions due to Yu and Zhang that satisfy the inequalities (5.5). In the following a Shapley value for the members of the class \(FG_H(N)\) is obtained.

**Theorem 5.17.** Let \(w \in FG_H(N)\). Then the mapping \(\Phi^H : G_H(N) \rightarrow (\mathbb{R}^n)^{2^N}\) given by
\[
\Phi^H_i(w)(S) = \begin{cases} 
\sum_{T \subseteq S \setminus \{i\}} \beta(|T|; |S|).\{w(T \cup \{i\}) \ominus_H w(T)\} & \text{if } i \in S \\
0 & \text{otherwise}
\end{cases}
\]
where \(\beta(|T|; |S|) = \frac{(|T|)(|S|-|T|-1)!}{(|S|)!}\), \(|T|\) and \(|S|\) being the cardinalities of \(S\) and \(T\) respectively, is a Shapley mapping on \(FG_H(N)\).
The Shapley mapping given by (5.6) is called the Hukuhara-Shapley mapping. In what follows, we define a special class of cooperative games with fuzzy coalitions and fuzzy characteristic functions. Given \( K \in L(N) \), let \( Q(K) = \{ K(i) | K(i) > 0, i \in N \} \) and let \( q(K) \) be the cardinality of \( Q(K) \). The elements of \( Q(K) \) in increasing order can be written as \( h_1 < ... < h_{q(K)} \).

**Definition 5.18.** Let \( K \in L(N) \). Then the function \( \tilde{w} : L(N) \to (\mathbb{R}^n)^L(N) \) with \( \tilde{w}(\emptyset) = <0> \) such that its restriction \( w \) to crisp coalitions belongs to \( FG_H(N) \) is said to be a “fuzzy game with indeterminate integral form” if and only if it satisfies:

\[
w(K) = \sum_{l=1}^{q(K)} w([K]_{h_l}).(h_l - h_{l-1}),
\]

for any \( K \in L(N) \), where \( w \) is the restriction of \( \tilde{w} \) to crisp coalitions and \( h_0 = 0 \). The set of all fuzzy games with indeterminate integral form is represented by \( FG_{YZ}^H(N) \).

Following theorem ensures the existence of a Shapley value for the class \( FG_{YZ}^H(N) \).

**Theorem 5.19.** Define a mapping \( \Phi_{YZ}^H : FG_{YZ}^H(N) \to (\mathbb{R}^n)^L(N) \) by

\[
\Phi_{YZ}^H(\tilde{w})(U) = \sum_{l=1}^{q(K)} \Phi_{i}^H(w)([U]_{h_l})(h_l - h_{l-1})
\]

where \( \Phi^H \) is the mapping given in (5.6). Then the mapping \( \Phi_{YZ}^H \) is the Shapley mapping on \( FG_{YZ}^H(N) \).

### 6 Conclusion

This paper surveys the recent developments in Cooperative Game Theory under fuzzy environment in general and the Shapley value in particular. The Shapley value as a single point solution concept for cooperative games in both crisp and fuzzy settings has received wide attention from the researchers in the field. However, the kind of full characterization it possesses in crisp environment has not yet been achieved for its fuzzy counterparts.
Therefore, the uniqueness of the Shapley value by means of a set of axioms as in crisp sense is yet to be accomplished for fuzzy case. Shapley value for games with fuzzy coalitions and/or vague expectations has sufficient scope for future research. There is no instance of using any triangular norms other than the minimum operator in the literature to explain the lattice structures of the fuzzy sets associated with the fuzzy games so far. However, it is possible to use other triangular norms generalizing Zadeh’s extension principle and obtain corresponding results in different contexts. Similarly, instead of dealing with more general fuzzy quantities, we can pursue with special fuzzy numbers such as triangular fuzzy numbers, LR-Fuzzy numbers etc. to highlight the special features of the game and its solution concepts.

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**References**


