Measuring Consensus in Group Decisions by Means of Qualitative Reasoning

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Abstract

This paper presents a mathematical framework to assess the consensus found among different evaluators who use ordinal scales in group decision-making and evaluation processes. This framework is developed on the basis of the absolute order-of-magnitude qualitative model through the use of quantitative entropy. As such, we study the algebraic structure induced in the set of qualitative descriptions given by evaluators. Our results demonstrate that it is a weak partial semi lattice structure that in some conditions takes the form of a distributive lattice. We then define the entropy of a qualitatively-described system. This enables us, on the one hand, to measure the amount of information provided by each evaluator and, on the other hand, to consider a degree of consensus among the evaluation committee. This new approach is capable of managing situations where the assessment given by experts involves different levels of precision. In addition, when there is no consensus regarding the group decision, an automatic process assesses

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the effort required to achieve said consensus.

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1. Introduction

Distributed decision-makers and decision-making systems need new efficient algorithms to assess consensus among evaluators. In many cases decision-making processes involve qualitative data. Based on the theories of Qualitative Reasoning (QR) [10] and, more specifically, on Absolute Order-of-Magnitude Qualitative Models (OM) [27], this paper proposes a theoretical framework to integrate evaluators’ opinions and their reasoning and measure their precision and degree of consensus. The use of the proposed measures within an automatic system for group decision-making will help detect and avoid any potential subjectivity arising from conflicts of interests among the evaluators in the group.

Qualitative Reasoning (QR) is a subarea of Artificial Intelligence that seeks to understand and explain human beings’ ability to reason without having precise information ([10], and [16]). The main objective of QR is
to develop systems that permit operating in conditions of insufficient or no numerical data.

Qualitative Reasoning also deals with problems in such a way that the principle of relevance is preserved, that is, each variable is valued with the level of precision required [9]. In group decision evaluation processes, it is not unusual for a situation to arise in which different levels of precision have to be worked with simultaneously depending on the information available to each evaluator.

To this end, the methodology we present tackles the problem of integrating the representation of existing uncertainty within the group and it defines the concept of entropy in a qualitative evaluation. This allows us to calculate each evaluator’s precision and the degree of consensus within the decision group. If there is no consensus within the group, an automatic process to achieve this consensus and compute the degree of consensus is activated.

Although the proposed methodology does not deal with the decision-making process itself, its main advantage is its ability to evaluate this process, managing situations where expert assessment involves different levels of precision.

This paper is structured as follows. Section 2 introduces the absolute orders-of-magnitude qualitative models and describes the formalism for the qualitative description of alternatives induced by an evaluator. Section 3 establishes the algebraic structure for the set of qualitative descriptions corresponding to the decision group. Section 4 defines the entropy of a qualitatively described system inspired on Shannon’s Theory of Information. We also present an index to measure the precision of the qualitative description.
of alternatives induced by each evaluator. Section 5 defines the degree of consensus among the evaluation committee and gives a simple example to demonstrate the application of this measure. Finally, in Section 6, we present our conclusions and possible future lines of research.

2. Theoretical Framework

Order of magnitude models are essential among the theoretical tools available for qualitative reasoning applied to physical systems ([13] and [25]). They aim to capture order of magnitude commonsense inferences [26], such as those used in the engineering world. Order-of-magnitude knowledge may be of two types: Absolute or relative. The absolute order-of-magnitudes are represented by a partition of $\mathbb{R}$, with each element of the partition standing for a basic qualitative label. A general algebraic structure, called Qualitative Algebra or $Q$-algebra, was defined on this framework ([28]), providing a mathematical structure to unify sign algebra and interval algebra through a continuum of qualitative structures built from the roughest to the finest partition of the real line. The most referenced order of magnitude $Q$-algebra divides the real line into 7 classes, labeled as follows: Negative Large(NL), Negative Medium(NM), Negative Small(NS), Zero(0), Positive Small(PS), Positive Medium(PM) and Positive Large(PL). $Q$-algebras and their algebraic properties have been extensively studied ([19] and [27]).

Order-of-magnitude knowledge may also be of relative type, in the sense that a quantity is qualified with respect to another quantity by means of a set of binary order-of-magnitude relations. The seminal relative orders of magnitude model was the formal FOG system ([21]), based on three basic relationships representing the intuitive concepts of "negligible with respect
to” (Ne), ”close to” (Vo) and ”comparable to” (Co). It was also described by 32 intuition-based inference rules. The relative orders of magnitude models that were proposed later improved on FOG not only in terms of incorporating the necessary rigorous formalization, but also by permitting the incorporation of quantitative information when available and the control of the inference process, in order to obtain valid results in the real world ([18], [4] and [5]).

In [27] the authors analyze the conditions under which an absolute orders-of-magnitude and a relative orders-of-magnitude model are consistent and they determine and interpret the constraints that consistency implies.

In [22] a generalization of qualitative absolute orders-of-magnitude was proposed, something which served as the theoretical basis to develop a Measure Theory in this context. The classic orders-of-magnitude qualitative spaces ([27]) verify the conditions of the generalized model introduced in [22]. These models are built from a set of ordered basic qualitative labels determined by a partition of the real line.

2.1. Absolute Order of Magnitude Qualitative Spaces

Let’s consider a finite set of basic labels, \( S^* = \{B_1, \ldots, B_n\} \), which is totally ordered as a chain: \( B_1 < \ldots < B_n \). Usually, each basic label corresponds to a linguistic term, for instance “extremely bad” < “very bad” < “bad” < “acceptable” < “good” < “very good” < “extremely good”. However, it is not unusual for basic labels to be defined by a discretization of a real interval or the real line, given by a set \( \{a_1, \ldots, a_{n+1}\} \) of real numbers as landmark such as \( B_i = [a_i, a_{i+1}] \), \( i = 1, \ldots, n \).

Nevertheless, we consider a more general case in this paper in which
knowledge of landmarks values is not required to introduce the basic labels.

The complete description universe for the Orders-of-Magnitude Space \( \text{OM}(n) \), with granularity \( n \), is the set \( S_n \):

\[
S_n = S_* \cup \{ [B_i, B_j] | B_i, B_j \in S_*, i < j \},
\]

where the label \([B_i, B_j]\) with \( i < j \) is defined as the set \( \{B_i, B_{i+1}, \ldots, B_j\}\).

Consistent with the former example of linguistic labels, the label “moderately good” can be represented by [“acceptable”, “good”], i.e. \([B_4, B_5]\), and the label “don’t know” is represented by [“extremely bad”, “extremely good”], i.e. \([B_1, B_7]\). This least precise label is denoted by the symbol \(?\), i.e. \([B_1, B_n] \equiv \?\).

There is a partial order relation \( \leq_P \) in \( S_n \), “to be more precise than”, given by:

\[
L_1 \leq_P L_2 \iff L_1 \subset L_2.
\] (1)

This structure permits working with all different levels of precision from
the basic labels to the 7 label (see Figure 2).

To introduce the classical concept of entropy by means of qualitative order of magnitude spaces, Measure Theory is required. This theory seeks to generalize the concept of “length”, “area” and “volume”, understanding that these quantities do not necessarily correspond to their physical counterparts, but that they may in fact represent others. The main use of this measure is to define the concept of integration for order-of-magnitude spaces.

2.2. Qualitative Description Induced by an Evaluator

Let \( \Lambda \) be the set that represents a magnitude or a feature that is qualitatively described by means of the labels of \( S_n \). Since \( \Lambda \) can represent both a continuous magnitude such as position and temperature and a discrete feature such as salary, \( \Lambda \) will be:

\[
\Lambda = \{ a(t) = a_t \mid t \in I \},
\]

where \( t \) is a continuous or discrete parameter, and \( I \) a set of indexes. An example would be \( I = [t_0, t_1] \) in the case where \( a(t) \) is a room temperature in an given instant \( t \) within a period of time or \( I = \{1, \ldots, n\} \) in the case of the salary of \( n \) people to be qualitatively described.

This qualitative description is carried out by each evaluator and is represented by the function:

\[
Q : \Lambda \rightarrow S_n,
\]

where \( a_t \mapsto Q(a_t) = E_t = \) is the minimum label (with respect to the inclusion \( \subset \)) which describes \( a_t \), i.e. the most precise qualitative label describing \( a_t \). All the elements of the set \( Q^{-1}(E_t) \) are “representatives” of the label \( E_t \) or
are qualitatively described” by \( \mathcal{E}_t \). From now on, this process of qualitative description will be referred as the *qualitativization* process.

Function \( Q \) induces a partition of \( \Lambda \) by means of the equivalence relation:

\[
a \sim_Q b \iff Q(a) = Q(b).
\]

This partition will be denoted by \( \Lambda/\sim_Q \), and its equivalence classes are the sets \( Q^{-1}(Q(a_j)) = Q^{-1}(\mathcal{E}_j), \forall j \in J \subset I \). Each of these classes contains all the elements of \( \Lambda \) which are described by the same qualitative label (see Figure 3).

**Example 1.** Suppose there is a kettle heating water on to heat and we want to qualitativize the water temperature during a period of five minutes: \( \Lambda = \{ T(t) \mid t \in [0, 5] \} \). Assuming that there are three evaluators and the space of qualitative description is \( S_5 \), with \( B_1 = \text{BOXVERY COLD}, B_2 = \text{COLD}, B_3 = \text{WARM}, B_4 = \text{HOT} \) and \( B_5 = \text{VERY HOT} \). Let us assume the three following qualitativizations:

\[
Q_1(T(t)) = \begin{cases} 
[B_1, B_2], & \text{if } t \in [0, 2); \\
[B_3, B_4], & \text{if } t \in [2, 4); \\
B_5, & \text{if } t \in [4, 5].
\end{cases}
\]
\[
Q_2(T(t)) = \begin{cases} 
B_1, & \text{if } t \in [0, 1); \\
B_2, & \text{if } t \in [1, 2); \\
B_3, & \text{if } t \in [2, 3); \\
B_4, & \text{if } t \in [3, 4); \\
B_5, & \text{if } t \in [4, 5]. 
\end{cases}
\]

and:

\[
Q_3(T(t)) = [B_1, B_5] \text{ if } t \in [0, 5]
\]

Note that the qualitative description given by \(Q_2\) is the most precise and that the description corresponding to \(Q_3\) is the least. In addition, the intersection of the \(Q_1\), \(Q_2\) and \(Q_3\) labels corresponding to each period of time is not empty. Thus, we can assume that there is some degree of consensus among the three evaluators.

The concepts of qualitative description precision and degree of consensus among a set of qualitativizations are formally introduced in the following sections.

3. The Algebraic Structure of a Set of Qualitative Descriptions

Let \(Q = \{Q \mid Q : \Lambda \rightarrow \mathbb{S}_n\}\) be the set of qualitativizations of \(\Lambda\) over \(\mathbb{S}_n\) given by a group of evaluators.

Given \(Q, Q' \in \mathcal{Q}\), two different operations are defined between them. Intuitively speaking, one is the result of mixing the two evaluations in a new evaluation that includes both opinions about each element of \(\Lambda\), and the other one is the result of taking what is common between the two evaluations.

3.1. The Mix \(\sqcup\) Operation

**Definition 1.** Given two qualitativizations \(Q, Q' \in \mathcal{Q}\), the operation \(Q \sqcup Q'\) leads to a new qualitativization function \(Q \sqcup Q' : \Lambda \rightarrow \mathbb{S}_n\) such that, for any \(a_t \in \Lambda\),

\[
(Q \sqcup Q')(a_t) = Q(a_t) \sqcup Q'(a_t),
\]

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where \( \sqcup \) is the connex union of labels, i.e. the minimum label that contains \( Q(a_t) \) and \( Q'(a_t) \):

\[
[B_i, B_j] \sqcup [B_h, B_k] = [B_{\min\{i,h\}}, B_{\max\{j,k\}}],
\]

using the convention \( [B_i, B_i] = B_i \).

Note that if \( Q(a_t) \cap Q'(a_t) \neq \emptyset \), then \( Q(a_t) \sqcup Q'(a_t) \) is the simple union \( Q(a_t) \cup Q'(a_t) \).

### 3.2. The Common \( \cap \) Operation

The concept of consensus between two qualitativizations \( Q \) and \( Q' \) is required in order to introduce the common operation:

**Definition 2.** Given a set \( \Lambda \) and a qualitative space \( \mathbb{S}_n \), two qualitativizations of \( \Lambda \), \( Q, Q' \) are in consensus, \( Q \rightleftarrows Q' \), iff

\[
Q(a_t) \cap Q'(a_t) \neq \emptyset, \quad \forall a_t \in \Lambda.
\]  

(2)

This last condition is equivalent to saying that \( Q(a_t) \approx Q'(a_t), \forall a_t \in \Lambda. \)  

It is clear that the relation \( \rightleftarrows \) is symmetric and reflexive.

In general, if \( \{Q_i\}_{i \in I} \subset Q \) of qualitativizations of \( \Lambda \) over \( \mathbb{S}_n \) is in consensus iff

\[
\cap_{i \in I} Q_i(a_t) \neq \emptyset \quad \forall a_t \in \Lambda.
\]

Note that, in this case, \( Q \rightleftarrows Q' \) for all \( Q, Q' \in \{Q_i\}_{i \in I} \).

**Definition 3.** Given two qualitativizations \( Q \) and \( Q' \) where \( Q \rightleftarrows Q' \), the common \( Q \cap Q' \) operation produces a new qualitativization function \( Q \cap Q' : \Lambda \to \mathbb{S}_n \) such that

\[
(Q \cap Q')(a_t) = Q(a_t) \cap Q'(a_t) \quad \forall a_t \in \Lambda.
\]

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\(^1\)In the absolute order-of-magnitude theory, two labels \( E, F \) are qualitatively equal, \( E \approx F \), iff \( E \cap F \neq \emptyset \).
In general, if \( \{Q_i\}_{i \in I} \subset Q \) is in consensus, the operation common \( \cap_{i \in I} Q_i \) produces a new qualitativization function: 
\[
(\cap_{i \in I} Q_i)(a_i) = \cap_{i \in I} Q_i(a_i) \quad \forall a_i \in \Lambda.
\]

3.3. The Algebraic Structure of the Set \( Q \)

The algebraic structure of the set \( Q \) and the \( \sqcup \) and \( \cap \) operations are given by the next proposition:

**Proposition 1** \((Q, \sqcup, \cap)\) is a weak partial lattice.

**Proof.** Demonstrating (see [11]) the following statements and their dual forms, changing \( \sqcup \) by \( \cap \) is sufficient; note that given \( Q, Q' \), the \( \sqcup \) operation always is defined and \( \cap \) exists iff \( Q \cong Q' \).

1. \( Q \sqcup Q = Q \).
2. \( Q \sqcup Q' = Q' \sqcup Q \).
3. \( (Q \sqcup Q') \sqcup Q'' = Q \sqcup (Q' \sqcup Q'') \).
4. \( Q \cap (Q \sqcup Q') \) exists and \( Q = Q \cap (Q \sqcup Q') \).

And the dual ones:

1’ \( Q \cap Q \) exists and \( Q \cap Q = Q \).
2’. If \( Q \cap Q' \) exists then \( Q' \cap Q \) exists and \( Q \cap Q' = Q' \cap Q \).
3’. If \( Q \cap Q' \), \( (Q \cap Q') \cap Q'' \) and \( Q' \cap Q'' \) exists, then \( Q \cap (Q' \cap Q'') \) exists and \( (Q \cap Q') \cap Q'' = Q \cap (Q' \cap Q'') \). If \( Q' \cap Q'' \), \( Q \cap (Q' \cap Q'') \) and \( Q \cap Q' \) exists, then \( (Q \cap Q') \cap Q'' \) exists and \( (Q \cap Q') \cap Q'' = Q \cap (Q' \cap Q'') \).
4’. If \( Q \cap Q' \) exists, then \( Q = Q \sqcup (Q \cap Q') \).
Statements 1, 2 and 3 are easily seen. Statement 4 is also true because for any $a \in \Lambda$ then $Q(a) \subset Q(a) \sqcup Q'(a)$, therefore $Q \rhd Q \sqcup Q'$ and $Q(a) \cap (Q(a) \sqcup Q'(a)) = Q(a)$. The dual statements are proved similarly proven.

From the general lattice theory (see [11] and [2]) applied to the weak partial lattice $(Q, \sqcup, \cap)$ we can stand the following statements can be made:

- $Q \leq Q'$ iff $Q \sqcup Q' = Q$ defines a a partial order relation.
- $Q \sqcup Q' = \inf\{Q, Q'\}$.
- If $Q \rhd Q'$, then $Q \cap Q' = \sup\{Q, Q'\}$.
- $Q \leq Q'$ iff $Q \cap Q' = Q'$.

Note that in the case of $Q \leq Q'$, $Q$ is less accurate than $Q'$, because $Q'(a) \subset Q(a) \ \forall a \in \Lambda$, i.e., each element of the set $\Lambda$ is more precisely described by $Q'$ than by $Q$:

$$Q \leq Q' \iff Q'(a) \leq_P Q(a) \ \forall a \in \Lambda.$$ 

**Proposition 2** Let $Q_L$ be a subset of $Q$ which is in consensus. Then $(Q_L, \sqcup, \cap)$ is a distributive lattice.

**Proof.** If the subset $Q_L$ of $Q$ is in consensus, then $(Q_L, \sqcup, \cap)$ is a lattice, because the operation $\cap$ is defined in all cases.

When there is consensus the $\sqcup$ and $\cap$ operations on $Q_L$ correspond exactly to union and intersection of $S_n$ labels respectively. As such the distributive axiom: if $Q, Q', Q'' \in Q_L$ then $Q \sqcup (Q' \cap Q'') = (Q \sqcup Q') \cap (Q \sqcup Q'')$, is satisfied.
Thus far, we have introduced the basics of the qualitativization mathematical structure: the definition of qualitativization, the aggregation information for mix and common operations, and the algebraic structure. In the next section these concepts are used to define the entropy of a given qualitativization and degree of consensus.

4. Entropy

The concept of entropy is introduced to measure the amount of information provided by the qualitativization given by an evaluator. The information measures’ concept has been the subject of study for a long time and the interest in this research has been considerably renewed by the development of Zadeh’s fuzzy sets ([29]), Possibility Theory and Shafer’s Theory ([23]) of Evidence. An excellent overview of information measures in these fields can be found in [6], [7] and [14]. Despite of this interest, however no work has been specifically dedicated to measuring the information in the order-of-magnitude qualitative reasoning framework. Entropy as defined in this paper is a measure of the information provided by an evaluator when he qualitativizes the set Λ. It can also be seen as the measure of the information the evaluator needs to assign a qualitative label to any element in Λ. The information used by the evaluator is the information that he has possesses or knows about the elements in the set. The first step is to measure how much information is needed in order to map an element of Λ to a specific label. The following definitions are an extension of Shannon’s Theory of Information ([24]).
4.1. The Information of a Label

The information of a label $\mathcal{E}$ is defined by a positive continuous real function within the measure of said label. It is denoted by $I(\mathcal{E})$. In [22] measures of the qualitative order-of-magnitude spaces are defined. It is assumed that if a label $\mathcal{E}$ is more precise than a label $\mathcal{E}'$, then more information is needed to assign an element to $\mathcal{E}$ than to $\mathcal{E}'$:

$$\mathcal{E} \leq_p \mathcal{E}' \Rightarrow I(\mathcal{E}) \geq I(\mathcal{E}').$$

Another assumption about $I$ function is that the information for the $\mathcal{E}$ label is zero (no information is needed to assign the $\mathcal{E}$ label to an element).

The following definition of $I$, inspired in the Shannon’s Theory of information, verifies these assumptions:

**Definition 4.** Let $\mu$ be a normalized measure (see [22]) defined in $\mathbb{S}_n$. The formula to define the information of a label $\mathcal{E} \in \mathbb{S}_n$ such that $\mu(\mathcal{E}) \neq 0$ is

$$I(\mathcal{E}) = \log \frac{1}{\mu(\mathcal{E})}.$$ 

Note that, for any $\mathcal{E}$, $\mu(\mathcal{E}) \leq 1$, and so $I(\mathcal{E}) \geq 0$.

Moreover, $I$ decreases with respect $\leq_p$:

$$\mathcal{E} \leq_p \mathcal{F} \Rightarrow \mathcal{E} \subset \mathcal{F} \Rightarrow \mu(\mathcal{E}) \leq \mu(\mathcal{F}) \Rightarrow \log \frac{1}{\mu(\mathcal{E})} \geq \log \frac{1}{\mu(\mathcal{F})}.$$ 

In addition, $I(?) = \log 1 = 0$.

**Example 2.** Given the classical $\mathbb{S}_n$ model, and by defining the measure $\mu(B_i) = \mu([a_i, a_{i+1}]) = (a_{i+1} - a_i)/(a_n - a_1)$, the information of a label is $I([a_i, a_{i+1}]) = \log \left(\frac{a_n - a_i}{a_{i+1} - a_i}\right)$.

**Proposition 3** For all labels $\mathcal{E}, \mathcal{F} \in \mathbb{S}_n$ it holds that $I(\mathcal{E} \sqcup \mathcal{F}) \leq I(\mathcal{E}) + I(\mathcal{F})$.

**Proof.** Since $\mathcal{E} \leq_p \mathcal{E} \sqcup \mathcal{F}$, $I(\mathcal{E}) \geq I(\mathcal{E} \sqcup \mathcal{F})$ then $I(\mathcal{E} \sqcup \mathcal{F}) \leq I(\mathcal{E}) + I(\mathcal{F})$. \qed
4.2. Definition of Entropy of a Qualitativization in $\mathbb{S}_n$

The entropy of a qualitativization, as mentioned above, is a measure of the information needed by the evaluator when qualitativizing the set $\Lambda$. The most natural way to express this concept mathematically is to define the entropy $H$ of a qualitativization $Q$ as a weighted average of the information regarding the elements within the set $\Lambda$ given by $Q$.

Let $\mu$ be a normalized measure defined in the set $\Lambda$.

**Definition 5.** Entropy $H$ of set $\Lambda$ given by $Q$ is:

$$H(Q) = \sum_{\mathcal{E} \in \mathbb{S}_n, \mu(\mathcal{E}) \neq 0} \mu(Q^{-1}(\mathcal{E})) I(\mathcal{E}).$$  \hspace{1cm} (3)

Note that $H(Q) \geq 0$ for all qualitativizations $Q$, and $H(Q) = 0$ when the whole set $\Lambda$ is described only by the label $\ast$.

If $\Lambda/\sim_Q = \{X_i, i \in J\}$, that is, the set of equivalence classes of $\sim_Q$, then (3) can be expressed as

$$H(Q) = \sum_{i \in J} \mu(X_i) I(Q(X_i)).$$  \hspace{1cm} (4)

Our next proposition shows the monotonicity of the entropy with respect to the accuracy relation between qualitativizations.

**Proposition 4** Given two qualitativizations $Q$ and $Q'$, then

$$Q \leq Q' \implies H(Q) \leq H(Q').$$

**Proof.** Let’s assume that $\Lambda/\sim_Q = \{X_i \mid i \in M\}$, $\Lambda/\sim_{Q'} = \{Y_j \mid j \in N\}$, and $(\Lambda/\sim_Q) \cap (\Lambda/\sim_{Q'}) = \{X_i \cap Y_j \neq \emptyset \mid i \in M, j \in N\}$.

Then, for all $X_i$ and $Y_j$,

$$X_i = \bigcup_{j \in N} (X_i \cap Y_j), \quad Y_j = \bigcup_{i \in M} (X_i \cap Y_j),$$
where only the non-empty intersections $X_i \cap Y_j$ are written, and the unions are disjoint unions, because \( \{Y_j\}_j \) and \( \{X_i\}_i \) are classes of equivalence.

Therefore, the entropy of $Q$ is:

\[
H(Q) = \sum_{i \in M} \pi(X_i) I(Q(X_i)) = \sum_{i \in M} \pi \left( \bigcup_{j \in N} (X_i \cap Y_j) \right) I(Q(X_i)) =
\]

\[
= \sum_{i \in M} \left( \sum_{j \in N} \pi(X_i \cap Y_j) \right) I(Q(X_i)) = \sum_{i \in M, j \in N} \pi(X_i \cap Y_j) I(Q(X_i)).
\]

Analogously, $H(Q') = \sum_{i \in M, j \in N} \pi(X_i \cap Y_j) I(Q'(Y_j))$.

Given that $X_i \cap Y_j \neq \emptyset$, the hypothesis $Q \leq Q'$ implies that, for an element $a_t \in X_i \cap Y_j$, $Q'(a_t) = Q'(Y_j) \leq_P Q(a_t) = Q(X_i)$, that is to say, $Q'(Y_j) \subset Q(X_i)$. As such, $I(Q(X_i)) \leq I(Q'(Y_j))$ for all the summands in $H(Q)$ and $H(Q')$, so the inequality $H(Q) \leq H(Q')$ is inferred.

As a corollary to the last proposition we obtain entropy’s subadditivity:

**Corollary 1** Let \( (Q_L, \cup, \cap) \) be a lattice of qualitativizations. Given $Q, Q' \in Q_L$ then

\[
H(Q \cup Q') \leq H(Q) + H(Q').
\]

**Proof.** Since \( (Q_L, \cup, \cap) \) is a lattice, $Q \cup Q' \leq Q, Q' \leq Q \cap Q'$. Then, from Proposition 4, $H(Q \cup Q') \leq H(Q), H(Q')$ and the inequality is concluded.

The property of monotonicity together with the subadditivity are two of the main properties of information measures ([6] and [15]).
4.3. Precision of a Qualitative Description

The entropy of a set $\Lambda$ when it is qualitativized by means of a space $S_n$ has a maximum value which allows us to define a measure of the precision of the qualitativizations given by the decision group.

**Proposition 5** Let $E^*_1, \ldots, E^*_k \in S_n$ be the labels (basic) with minimum measure $\mu$, $m^* = \mu(E^*_1) = \ldots = \mu(E^*_k) \neq 0$. Let us consider a qualitativization $\tilde{Q}$ such that $\tilde{Q}(\Lambda) \subset \{E^*_1, \ldots, E^*_k\}$, that is to say, $\tilde{Q}$ maps the entire set $\Lambda$ to the most precise labels. Then:

$$H(Q) \leq H(\tilde{Q}) = \log \frac{1}{m^*} \quad \forall Q.$$  

**Proof.** Since $m^* \leq \mu(E), \forall E \in S_n$ then $I(E) \leq \log(1/m^*)$. Thus, for any $Q$,

$$H(Q) = \sum_{E \in S_n} \bar{\mu}(Q^{-1}(E)) I(E) \leq \log \frac{1}{m^*} \sum_{E \in S_n} \bar{\mu}(Q^{-1}(E)) = \log \frac{1}{m^*},$$

because $\{Q^{-1}(E)\}_{E \in S_n}$ is a partition of $\Lambda$ and $\bar{\mu}$ is normalized. Moreover, since $\tilde{Q}(\Lambda) \subset \{E^*_1, \ldots, E^*_m\}$,

$$H(\tilde{Q}) = \sum_{E \in S_n} \bar{\mu}(\tilde{Q}^{-1}(E)) \log \frac{1}{m^*} = \log \frac{1}{m^*}.$$  

According to this proposition the precision of a qualitativization is defined in the following way:

**Definition 6.** The *precision* of a qualitativization $Q$ within set $\Lambda$, $h(Q)$, is the relative entropy respect to the maximum entropy $H(\tilde{Q})$ for the set $\Lambda$ in $S_n$:

$$h(Q) = \frac{H(Q)}{H(\tilde{Q})}. \quad (5)$$  

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This quantity is a real number between 0 and 1; the more accurate the
evaluator is, the closer \( h(Q) \) is to 1. When \( Q \) maps the whole \( \Lambda \) to the most
precise labels (basic labels with the smallest measure) then \( h(Q) = 1 \). In
the opposite case, \( h(Q) = 0 \) when \( Q \) maps the whole \( \Lambda \) to the least precise
label.

5. Consensus in the Group Decision

One of the main applications of the theory presented in this paper is
that it enables us to measure the precision of and consensus within a group
decision which rates or evaluates a given problem.

Measuring consensus has been tackled in the literature by several authors
in different ways. In [3] and [20] an average is consider to compute the degree
of consensus, in [8] it is related to a distance. The approach presented in this
paper is based on the entropy as defined in the previous section.

5.1. Degree of Consensus

In order to introduce a definition for the degree of consensus, let us sup-
pose that two evaluators qualitativize a set \( \Lambda \) by means of \( Q \) and \( Q' \). First
of all the degree of consensus can only be computed when consensus exists
among them, i.e. if \( Q \parallel Q' \).

If the two evaluators “think similarly”, then the operation \( \cap \) between
\( Q, Q' \) which extracts their coincidences will produce a qualitativization simi-
lar to the qualitativization obtained by mixing them. In this case \( H(Q \cap Q') \)
will be quite similar to \( H(Q \sqcup Q') \). Otherwise, \( Q \cap Q' \) will be a qualitativization
with a high degree of entropy, and \( Q \sqcup Q' \) will have a low degree.
On the other hand, $H(Q \cap Q') \geq H(Q \cup Q')$ because $Q \cap Q' \geq Q \cup Q'$; thus the quotient $H(Q \cup Q')/H(Q \cap Q')$ is a real number between 0 and 1.

In order to generalize the quotient above to the case of group decisions with $N$ evaluators, let us introduce the following notation:

Given a space $\mathbb{S}_n$, a finite non empty set $\Lambda = \{a_1, \ldots, a_N\}$ and a group of evaluators $\mathcal{E} = \{\alpha_1, \ldots, \alpha_M\}$, the group evaluation of $\Lambda$ is considered as the pair $(\Lambda, Q_\mathcal{E})$, where $Q_\mathcal{E} = \{Q_i : \Lambda \to \mathbb{S}_n \mid i \in \{1, \cdots, M\}\}$, and $Q_i$ is the evaluation of $\alpha_i$.

Let’s suppose that there is consensus among the group, i.e., $\cap_{i=1}^{M} Q_i(a_t) \neq \emptyset \ \forall a_t \in \Lambda$. The next definition regarding the degree of consensus measures the relation between the entropy of operations mix and common in the set of group qualitativizations:

**Definition 7.** Given a group evaluation $(\Lambda, Q_\mathcal{E})$ in consensus, the degree of consensus among the group, $\kappa(Q_\mathcal{E})$, is

$$\kappa(Q_\mathcal{E}) = \frac{H(\sqcup_{i=1}^{M} Q_i)}{H(\bigcap_{i=1}^{M} Q_i)}$$

(the only case in which $\kappa$ is not well defined corresponds to the case $H(\cap_{i=1}^{M} Q_i) = 0$, that is to say, when all evaluators describe the elements of the full set $\Lambda$ with the label ?).

This degree is a number between 0 and 1; the closer it is to 1, the closer the group is to being unanimous in its assessment. It is important to note that this degree does not depend on the number of the evaluators in the group, in the sense that if the number of evaluators that “think similarly” increases, the consensus degree does not increase. In addition, when many evaluators think similarly but one of them thinks differently, the degree of
consensus degree will be low for the group as a whole because of that single evaluator.

The next proposition shows that the degree of consensus within a group evaluation cannot be increased by adding a new evaluator to the group.

**Proposition 6** Consider a group evaluation \((\Lambda, Q_E)\) in consensus. Let be \(Q_{\text{new}}\) a new evaluator of \(\Lambda\) such that \(Q_E \cup \{Q_{\text{new}}\}\) is in consensus. Then

\[
\kappa(Q_E \cup \{Q_{\text{new}}\}) \leq \kappa(Q_E).
\]

**Proof.** Based on the fact that \(Q \sqcup Q' = \inf\{Q, Q'\}\) and \(Q \cap Q' = \sup\{Q, Q'\}\):

\[
(\bigcup_{i=1}^{M} Q_i) \sqcup Q_{\text{new}} \leq \bigcup_{i=1}^{M} Q_i \quad \text{and} \quad \bigcap_{i=1}^{M} Q_i \leq (\bigcap_{i=1}^{M} Q_i) \cap Q_{\text{new}}.
\]

Then, from Proposition 4:

\[
H((\bigcup_{i=1}^{M} Q_i) \sqcup Q_{\text{new}}) \leq H(\bigcup_{i=1}^{M} Q_i) \quad \text{and} \quad H(\bigcap_{i=1}^{M} Q_i) \leq H((\bigcap_{i=1}^{M} Q_i) \cap Q_{\text{new}}),
\]

and hence \(\kappa(Q_E \cup \{Q_{\text{new}}\}) \leq \kappa(Q_E)\). 

Therefore, the only way to increase the degree of consensus in a group is for an evaluator to reconsider the situation and his assessment.

### 5.2. Achieving Consensus

The necessary and sufficient condition for which there exists consensus is \(\bigcap_{i=1}^{M} Q_i(a_t) \neq \emptyset, \forall a_t \in \Lambda\). If this situation does not hold then a process has to be initiated to obtain consensus. In [3], [8], [12], [17] and [20], different approaches to this problem are found framed within fuzzy sets theory and aggregation operators. The algorithm presented here, which could be called an automatic negotiation, is based on the following idea: If two people disagree
on some fact and they want to reach an agreement, i.e. reach consensus, they have to reconsider their positions and find points in common. In this section this idea is formalized by using the concepts already given. It can be understood as a process of automatic negotiation.

**Definition 8.** Given a space $\mathcal{S}_n$ with basic labels $\mathcal{S} = \{B_1, \ldots, B_n\}$, and a space $\mathcal{S}_{n+1}$ with basic labels $\mathcal{S}' = \{B_1', \ldots, B_{n+1}'\}$, the **dive function** is the mapping $\phi_0 : \mathcal{S}_n \rightarrow \mathcal{S}_{n+1}$, defined as follows: For basic labels $B_i \in \mathcal{S}_n$, then

$$
\phi_0(B_i) = [B_i', B_{i+1}'],
$$

and, for non basic labels,

$$
\phi_0([B_i, B_j]) = \bigcup_{k=i}^{j} \phi_0(B_k) = [B_i', B_{j+1}'].
$$

The dive function $\phi_0$ is an injection of the $\mathcal{S}_n$ into $\mathcal{S}_{n+1}$. With this function, each basic label in $\mathcal{S}_n$ is “split” into two new basic labels in $\mathcal{S}_{n+1}$. And in general, for each label in $\mathcal{S}_n$ its image is obtained by adding a new basic label. In this same way, we can be defined it as $\phi_i : \mathcal{S}_{n+i} \rightarrow \mathcal{S}_{n+i+1}$, for $i \geq 1$, and the following chain can be assumed:

$$
\mathcal{S}_n \xrightarrow{\phi_0} \mathcal{S}_{n+1} \xrightarrow{\phi_1} \mathcal{S}_{n+2} \xrightarrow{\ldots} \mathcal{S}_{n+m} \xleftarrow{\phi_m} \mathcal{S}_{n+m+1}
$$
Then, given $E, F \in S_n$ such that $E \cap F = \emptyset$, we can see that there exists a natural number $1 \leq k \leq n$ (see figure 4) such that:

$$(\phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_0)(E) \cap (\phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_0)(F) \neq \emptyset.$$  

Similarly and given $E_1, \ldots, E_M \in S_n$, there exists $1 \leq k \leq n$ such that

$$\bigcap_{i=1}^M (\phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_0)(E_i) \neq \emptyset.$$  

The next proposition allows us to extend the measure defined in $S_n$ to the new space $S_{n+1}$.

**Proposition 7** Let $\mu$ be a normalized measure defined on $S_n$ and let us suppose that $S_n$ is divided in $S_{n+1}$. Then the measure $\mu$ can be extended to a normalized measure $\mu'$ in $S_{n+1}$ defined, taking weights $0 < \lambda_1 < \cdots < \lambda_n < 1$, in the following way (see Figure 5):

$$\begin{align*}
\mu'(B'_1) &= (1 - \lambda_1)\mu(B_1) \\
\mu'(B'_2) &= \lambda_1\mu(B_1) + (1 - \lambda_2)\mu(B_2) \\
&\vdots \\
\mu'(B'_i) &= \lambda_{i-1}\mu(B_{i-1}) + (1 - \lambda_i)\mu(B'_i) \\
&\vdots \\
\mu'(B'_{n+1}) &= \lambda_n\mu(B_n)
\end{align*}$$

where $\forall i \ 0 < \lambda_i < 1$. And for a non basic label $E' = [B'_i, B'_j] \in S_{n+1}$,

$$\mu'(E') = \sum_{k=i}^j \mu'(B'_k).$$

**Proof.** It is easy to check that $\mu'$ verifies the axioms of a measure and that it is normalized, i.e. $\mu'(B'_1) + \cdots + \mu'(B'_{n+1}) = \mu(B_1) + \cdots + \mu(B_n) = 1$. \qed
With the defined dive function and this extension of the measure, we can thus define a process to reach consensus in a group evaluation \((\Lambda, Q_E)\) can be performed.

Let’s suppose that the group is not in consensus, i.e. there exists a subset \(\Gamma \subset \Lambda\) such that \(\cap_{i=1}^{M} Q_i(a_t) = \emptyset \ \forall a_t \in \Gamma\). For each \(a_t \in \Gamma\), let \(n_{a_t}\) be the first natural number such that

\[
\cap_{i=1}^{n_{a_t}} (\phi_{n_{a_t}} \circ \cdots \circ \phi_0)(Q_i(a_t)) \neq \emptyset.
\]

Considering \(n^* = \max\{n_{a_t} | a_t \in \Gamma\}\), the group evaluation obtained, which is \(\{\phi_{n^*} \circ \cdots \circ \phi_1 \circ \phi_0 \circ Q_i | i = 1, \ldots, M\}\), is in consensus in the space \(S_{n+n^*}\).

Now, we can calculate the degree of consensus \(\kappa\) within the group evaluation which consensus which has been obtained.

**Example 3.** Let us suppose a committee consisting of two members \(\alpha_1\) and \(\alpha_2\) that evaluates an candidate for a grant. Let us also assume that this candidate is evaluated in terms of three attributes \(\Lambda = \{a_1, a_2, a_3\}\). Attribute \(a_1\) is the quality of his CV, attribute \(a_2\) is the quality of his publications, and \(a_3\) is the quality of his research projects. The evaluation of \(a_i\) is done over a space \(S_3\) where \(B_1 = \text{LOW}, B_2 = \text{NORMAL}\), and \(B_3 = \text{HIGH}\).

Suppose that evaluator \(\alpha_1\) gives the candidate the following appraisal:

\[Q_1(a_1) = B_1, Q_1(a_2) = B_1, Q_1(a_3) = [B_2, B_3],\]

and committee member \(\alpha_2\) the following:

\[Q_2(a_1) = B_2, Q_2(a_2) = [B_1, B_2], Q_2(a_3) = B_3.\]
The precision of both evaluators is
\[ h(Q_1) = h(Q_2) = 0.79. \]

There is no consensus within the group because \( Q_1(a_1) \cap Q_2(a_1) = \emptyset \). The automatic negotiation requires one step to achieve the consensus:

\[
\begin{align*}
(\phi_0 \circ Q_1)(a_1) &= [B'_1, B'_2], \quad (\phi_0 \circ Q_1)(a_2) = [B'_1, B'_2], \quad (\phi_0 \circ Q_1)(a_3) = [B'_2, B'_4], \\
(\phi_0 \circ Q_2)(a_1) &= [B'_2, B'_3], \quad (\phi_0 \circ Q_2)(a_2) = [B'_1, B'_3], \quad (\phi_0 \circ Q_2)(a_3) = [B'_3, B'_4],
\end{align*}
\]

and now \( \Phi_0 \circ Q_1 \rightleftharpoons \Phi_0 \circ Q_2 \) in \( S_4 \).

Let’s take \( \mu(B_1) = \mu(B_2) = \mu(B_3) = 1/3 \), and \( \lambda_1 = \lambda_2 = \lambda_3 = 1/2 \). As such \( \mu'(B'_1) = \mu'(B'_2) = 1/6 \) and \( \mu'(B'_3) = \mu'(B'_4) = 1/3 \).

To calculate the entropies, \( \mathcal{P} \) has been taken as the normalized counter measure, i.e.,
\[
\mathcal{P}(X_i) = \frac{\text{card}(X_i)}{\text{card}(\Lambda)}, \text{ for each } X_i \subset \Lambda.
\]
The degree of consensus is:
\[
\kappa(\{\Phi_0 \circ Q_1, \Phi_0 \circ Q_2\}) = \frac{H((\Phi_0 \circ Q_1) \cup (\Phi_0 \circ Q_2))}{H((\Phi_0 \circ Q_1) \cap (\Phi_0 \circ Q_2))}.
\]

Since
\[
((\Phi_0 \circ Q_1) \cup (\Phi_0 \circ Q_2))(\Lambda) = \{ [B'_1, B'_3], [B'_1, B'_3], [B'_2, B'_4] \}
\]
and
\[
((\Phi_0 \circ Q_1) \cap (\Phi_0 \circ Q_2))(\Lambda) = \{ B'_2, [B'_1, B'_2], [B'_3, B'_4] \},
\]
using the formula 3
\[
\kappa(\{\Phi_0 \circ Q_1, \Phi_0 \circ Q_2\}) = 0.22.
\]

The main feature of this automatic negotiation process is that it is done by considering data obtained from the evaluators without having to interact with them again. This is useful when working on problems where it is difficult to contact the evaluators later. This situation can be found, for example, when processing data from surveys or from databases. In addition, the degree of consensus and the automatic negotiation process allow us to compare the internal coherence of different decision groups. A simple example of this is given below.
Example 4. Let $E_1, E_2$ and $E_3$ be three committees from different areas of knowledge, selected to evaluate respective projects $\Lambda_1, \Lambda_2, \Lambda_3$ for an official announcement. Each committee consists of four evaluators, $E_i = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, and each project is characterized by two attributes $\Lambda_i = \{a_1, a_2\}$.

Qualitativization is done over the space $S_5$ where $B_1 = \text{VERY BAD}, B_2 = \text{BAD}, B_3 = \text{REGULAR}, B_4 = \text{GOOD}$ and $B_5 = \text{VERY GOOD}$. The measure $\mu$ of all these basic labels is $1/5$, and the measure $\Psi$ in $\Lambda$ is the normalized cardinal measure. In the next table we summarize the qualitativizations given by each committee (the qualitativization of member $\alpha_j^i$ is done by means of the function $Q_j^i$), together with the results of the common and mix operations for each case and the degree of consensus for the third committee.

<table>
<thead>
<tr>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1^i$</td>
<td>$B_3, B_4$</td>
<td>$B_1, B_2$</td>
</tr>
<tr>
<td>$Q_2^i$</td>
<td>$B_3, B_4$</td>
<td>$B_1, B_2$</td>
</tr>
<tr>
<td>$Q_3^i$</td>
<td>$B_3, B_4$</td>
<td>$B_1, B_2$</td>
</tr>
<tr>
<td>$\sqcup$</td>
<td>$B_4$</td>
<td>$B_2$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.32</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Committees’ evaluations

As there is no consensus in committees $E_1$ and $E_2$, the dive function must be applied to these two committees’ evaluations. In order to compare the degree of consensus among the three committees, the dive function is also applied to the third committee. The results are summarized in Table 2.

<table>
<thead>
<tr>
<th>$\Phi_1 \circ \Phi_0$</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1^i$</td>
<td>$B_3, B_5$</td>
<td>$B_4, B_6$</td>
<td>$B_1, B_3$</td>
</tr>
<tr>
<td>$Q_2^i$</td>
<td>$B_3, B_5$</td>
<td>$B_4, B_6$</td>
<td>$B_1, B_3$</td>
</tr>
<tr>
<td>$Q_3^i$</td>
<td>$B_3, B_5$</td>
<td>$B_4, B_6$</td>
<td>$B_1, B_3$</td>
</tr>
<tr>
<td>$\sqcup$</td>
<td>$B_4$</td>
<td>$B_5$</td>
<td></td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.58</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Committees’ evaluations

<table>
<thead>
<tr>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_3, B_4$</td>
<td>$B_5$</td>
<td></td>
</tr>
</tbody>
</table>

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Table 2. Final consensus degrees

Notice that, although committee three, $\Lambda_3$, was the only one achieving consensus at the beginning of the process, the final degree of consensus of $\Lambda_1$ is the greatest because, globally, the evaluations from its members were quite similar, and, therefore, the committee’s final consensus is more accurate.

6. Conclusions and Future Research

A mathematical framework and a methodology are presented in this study to measure precision and consensus in group decisions. The representation of the alternatives to be analyzed is based on an order-of-magnitude qualitative model. The operations considered to aggregate information provide this model a weak partial lattice structure. When there is consensus among the decision group, however, a distributive lattice structure is obtained.

The concept of entropy is then introduced in this framework to measure the amount of information within a system when using order-of-magnitude descriptions to represent it. On the other hand, entropy allows us to measure consensus in group decision-making problems. The degree of consensus is introduced in order to obtain an objective measure of the decision group’s reliability. If there is no consensus among the group, an automatic process is then initiated to achieve a global consensus. This process allows us to compare the internal coherence of different decision groups.

Future research could focus on one of two lines. From a theoretical point of view, different degrees of consensus could be considered and compared, for instance, by defining conditional entropy and the information distance [1] derived from it in this framework.

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From an application point of view, this work and the related methodology could be orientated in two directions: On the one hand, developing techniques to detect malfunctioning within an evaluation committee, i.e., finding incoherencies due to corruption or a lack of knowledge, and avoiding potential subjectivity caused by conflicts of interest regarding evaluators; on the other hand, it could also lead to the development of recommender systems based on the clustering process obtained through automatic negotiation.

\[ \mathcal{E} \]
\[ \mathcal{F} \]
\[ \mathcal{G} \]
\[ \mathcal{E} \cap \mathcal{F} \]
\[ \mathcal{E} \cup \mathcal{F} \]
\[ \mathcal{E} \wedge \mathcal{F} \]

References


