A TRANSMISSION PROBLEM FOR FLUID-STRUCTURE INTERACTION IN THE EXTERIOR OF A THIN DOMAIN*

HSIAO, G.C.† AND NIGAM, N.‡

May 7, 2002

1. Introduction. In this article, we study the interaction between a infinitely long cylinder coated with a sheath of an elastic material and a compressible, inviscid fluid, with *lubricated contact on the interface*. The problem is thus posed in $\mathbb{R}^2$. The thickness of this sheath is a small parameter in this problem, and we are particularly interested in the asymptotic behaviour of the solutions to the fluid-solid interaction problem as this parameter approaches zero. We attempt to resolve not only the fluid pressure in the exterior, but also the elastodynamic oscillations in the thin region. This is in contrast to earlier work, where the elastic region is replaced by an effective boundary condition on the fluid-solid interface for the acoustic problem in the exterior. The incident waves are time-harmonic in nature, enabling us to study a time-independant scattering problem.

It was shown in (26) that for certain frequencies, the inviscid fluid-structure interaction problem does not have unique solutions. Even in the absence of a pressure field in the exterior fluid region, the elastic domain may support so-called *traction-free oscillations*, whose normal components and tractions across the fluid-structure interface are continuous. This imposes constraints on our existence theory; we are only able to demonstrate solvability for the fluid-structure problem away from certain natural eigenfrequencies. Fortunately, such oscillations occur in highly specific situations and heavily depend upon the symmetry properties of the region. Therefore, our analysis holds for all but a highly specific class of fluid-structure problems. Moreover, these Jones frequencies form a discrete spectrum. If the structure under consideration supports Jones oscillations, the analysis is valid except for at most a discrete set of frequencies.

The existence of solutions is usually demonstrated by casting the problem into variational form, and then using standard techniques from Fredholm theory. We follow this procedure, but encounter difficulty in showing the invertibility of a certain operator. The use of Korn's inequality enables us to get around this problem.

We begin the article by describing the fluid-structure problem in Section 2, with a brief discussion on issues of uniqueness for solutions. The governing equations will be the Navier-Lamé system for the elastic displacements in the thin region, and the Helmholtz equation in the fluid, coupled via transmission conditions across the fluid-structure interface.

In order to present our key ideas, without getting lost in the technical details, in Section 3 we develop the analysis in the context of a very simple model scalar transmission problem. After precisely formulating this illustrative example in Section 3, we reduce the exterior problem by means of two integral equations. Using a variational formulation for the reduced problem leads to key estimates.

In Section 4, we return to the fluid-structure problem of interest, and formulate it carefully in a Hilbert space setting. We study the uniqueness and existence of solutions via variational techniques. Finally, a formal asymptotic scheme is described. The asymptotic procedure is justified by using a result (19) concerning the Korn's inequality in a thin region. The

*This work was conducted at the Dept. of Math Sciences, U. of Delaware.

†Department of Mathematical Sciences, Ewing Hall, U. of Delaware, Newark, DE 19716 (hsiao@math.udel.edu).

‡Department of Mathematics and Statistics, Burnside Hall, McGill U., 805 Sherbrooke St. W, Montréal, PQ, H3A 2K6 (nigam@math.mcgill.ca).

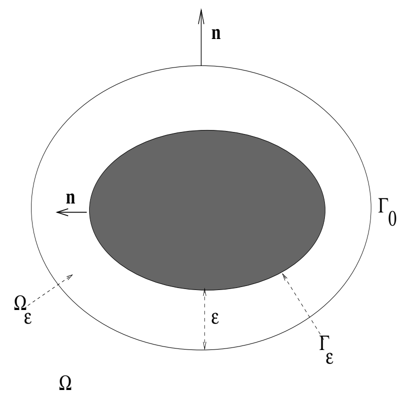

FIG. 2.1. *Geometry of problem. The traces of a function f approaching Γ_0 from inside (resp. outside) Ω_ϵ are denoted f^- (resp. f^+).*

proof follows a technique introduced earlier in the context of simpler problems, (18), (38).

In this paper, we are able to justify the formal asymptotics by making a fairly strong technical assumption. This assumption can be avoided by introducing an artificial layer, as described in (2) for a scalar Helmholtz transmission problem. We shall present this technique in the context of the fluid-structure interaction in future work, (20).

2. Problem description and the governing equations. In what follows Ω_ϵ is a bounded thin domain in $\mathbb{R}^2$, with smooth convex inner and outer boundaries Γ_ϵ and Γ_0. We assume that the thickness of Ω_ϵ is sufficiently small so that Γ_ϵ can be regarded as a scaled image of Γ_0. Specifically, if we denote by Ω' the bounded complement of Γ_0 in $\mathbb{R}^2$, then the thin domain and the inner boundary are defined precisely as

$$\Omega_\epsilon := \{x \in \Omega' |\ d(x, \Gamma_0) < \epsilon\}, \qquad \Gamma_\epsilon := \{x \in \Omega' |\ d(x, \Gamma_0) = \epsilon\},$$

where $d(x, \Gamma_0)$ denotes the Euclidean distance of the point x to the curve Γ_0.

The region Ω_∞ is the unbounded component of $\mathbb{R}^2 \setminus (\Omega_\epsilon \cup \Gamma_0)$, see Figure 2.1. A time-harmonic, monochromatic, plane acoustic wave is incident on an isotropic, elastic solid Ω_ϵ. We are interested in the resulting pressure in the fluid Ω_∞, and displacement in the region Ω_ϵ. The fluid is assumed to be homogenous, compressible and inviscid. Following (1), and after factoring out the time-dependance, the linearized, time-independent equations governing the displacement $\mathbf{u}_f$ and the pressure p_ϵ in the fluid are given by

$$\text{(2.1)} \qquad \left\{ \begin{array}{lcl} p_\epsilon - c^2 \rho_f & = & 0, \\ p_\epsilon + \rho_0 c^2 \nabla \cdot \mathbf{u}_f & = & 0, \\ \rho_0 c^2 \operatorname{grad} \nabla \cdot \mathbf{u}_f + \omega^2 \rho_0 \mathbf{u}_f & = & 0. \end{array} \right.$$

Here ρ_0 is the time-independent equilibrium density of the fluid, and c is the speed of sound in the fluid and where μ and λ are the (real) Lamé coefficients. In the physics and engineering community, it is more common to use the coefficients μ, K where μ is interpreted as the *modulus of rigidity*, and $K = \lambda + \mu$, the *bulk modulus*, or *modulus of hydrostatic compression* in $\mathbb{R}^2$. In a perfectly elastic medium, these moduli are strictly positive, that is,

$$\mu > 0, \qquad \lambda + \mu > 0.$$

Some simple manipulations now yield the equation for the time-independent (scattered wave) pressure in the fluid medium as

$$\text{(2.2a)} \qquad \triangle p_\epsilon + k^2 p_\epsilon = 0, \qquad x \in \Omega_\infty,$$

with $k^2 = \frac{\omega^2}{c^2}$. Since the fluid is inviscid, no shear waves are supported, and k is interpreted as *the wave number* of the longitudinal wave in the exterior region.

Using the ansatz the time dependant elastic displacement $\mathbf{U}_0$ is of the form $\mathbf{U}_0 = Re(\mathbf{u}_\epsilon e^{i\omega t})$ in the region Ω_ϵ, we get from the balance of forces in Ω_ϵ the reduced elasto-dynamics equation:

$$\triangle^* \mathbf{u}_\epsilon := \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) = \mu \triangle \mathbf{u}_\epsilon + (\lambda + \mu) \operatorname{grad} \nabla \cdot \mathbf{u}_\epsilon = -\rho\omega^2 \mathbf{u}_\epsilon, \quad x \in \Omega_\epsilon, \tag{2.2b}$$

where $\underline{\underline{\sigma}}(\mathbf{u})$ is the stress tensor. The connection between p_ϵ and $\mathbf{u}_\epsilon$ is given through the *kinematic transmission condition:*

$$\frac{\partial p^i}{\partial n} + \frac{\partial p_\epsilon}{\partial n} = \rho_f \omega^2 \mathbf{n} \cdot \mathbf{u}_\epsilon, \qquad x \in \Gamma_0, \tag{2.2c}$$

and the *dynamic transmission condition*

$$-(p^i + p_\epsilon)\mathbf{n} = \mathbf{n} \cdot \underline{\underline{\sigma}}(\mathbf{u}_\epsilon) = \mathbf{T}(\mathbf{u}_\epsilon), \qquad x \in \Gamma_0. \tag{2.2d}$$

The unit outer normal on Γ_0, $\mathbf{n} = n_1\hat{\mathbf{i}} + n_2\hat{\mathbf{j}}$ points towards Ω_∞. The given incident pressure p^i is a solution of the Helmholtz equation in Ω_∞, except perhaps at finitely many source points. The *traction* $\mathbf{T}(\mathbf{u})$ on the surface Γ_0 is defined by

$$\mathbf{T}(\mathbf{u}) := 2\mu \frac{\partial \mathbf{u}}{\partial n} + \lambda \mathbf{n} \nabla \cdot \mathbf{u} + \mu \mathbf{n} \times \nabla \times \mathbf{u}, \qquad x \in \Gamma_0. \tag{2.2e}$$

In two dimensions, the last term should be interpreted in Cartesian coordinates as

$$\mu \mathbf{n} \times \nabla \times \mathbf{u} = (\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x})(n_2\hat{\mathbf{i}} - n_2\hat{\mathbf{j}}),$$

where $\mathbf{u} = u_1\hat{\mathbf{i}} + u_2\hat{\mathbf{j}}$. The displacement $\mathbf{u}_\epsilon$ also satisfies the *homogenous Dirichlet condition* on the surface of the hard object, Γ_ϵ, given by

$$\mathbf{u}_\epsilon = \mathbf{0}, \qquad x \in \Gamma_\epsilon. \tag{2.2f}$$

In addition, the pressure p_ϵ must satisfy the radiation condition:

$$\frac{\partial p_\epsilon}{\partial r} - ikp_\epsilon = o(\frac{1}{\sqrt{r}}), \qquad r \to \infty, \tag{2.2g}$$

uniformly in all directions. At this juncture we recall that in a homogenous isotropic elastic material, *Hooke's law* gives

$$\underline{\underline{\sigma}}(\mathbf{u}) = \lambda \nabla \cdot \mathbf{u} \mathbf{I} + 2\mu\, \underline{\underline{\mathbf{e}}}(\mathbf{u}), \tag{2.3}$$

where $\underline{\underline{\mathbf{e}}}(\mathbf{u}) := \frac{1}{2}\left(\operatorname{grad}\mathbf{u} + \operatorname{grad}\mathbf{u}^T\right)$ is the linearized strain tensor.

The usual classical solution space for this problem is $\mathcal{U} := \left(C^2(\Omega_\epsilon) \cap C^1(\overline{\Omega_\epsilon})\right)^2 \times \left(C^2(\Omega_\infty) \cap C^1(\overline{\Omega_\infty})\right)$. The classical fluid-solid interaction problem can now be precisely formulated as follows: *For a given incident field p^i, find $(\mathbf{u}_\epsilon, p_\epsilon) \in \mathcal{U}$ such that*

$$\left\{ \begin{array}{rcll} (\triangle^* + \rho\omega^2)\mathbf{u}_\epsilon & = & \mathbf{0} & \text{in } \Omega_\epsilon, \\ \mathbf{u}_\epsilon^- & = & \mathbf{0} & \text{on } \Gamma_\epsilon, \\ \mathbf{u}_\epsilon^- \cdot \mathbf{n} & = & \frac{1}{\rho_f \omega^2}(\frac{\partial p_\epsilon}{\partial n} + \frac{\partial p^i}{\partial n})^+ & \text{on } \Gamma_\epsilon, \\ T(\mathbf{u}_\epsilon)^- & = & -(p_\epsilon^+ + p^i)^+ \cdot \mathbf{n} & \text{on } \Gamma_0, \\ \triangle p_\epsilon + k^2 p_\epsilon & = & 0 & \text{in } \Omega_\infty, \\ (\frac{\partial p_\epsilon}{\partial r} - ikp_\epsilon) & = & o(\frac{1}{\sqrt{r}}) & |x| = r \to \infty. \end{array} \right. \tag{2.4}$$

We assume throughout the article that k, ρ, ω and ρ_f are real parameters.

2.1. A word on classical uniqueness results. We will study the asymptotic behaviour of weak solutions to the transmission problem (2.4) as the thickness $\epsilon \to 0$. However, before we derive a weak formulation, we must establish the uniqueness of classical solutions. Such a uniqueness result would then enable us to invoke the Fredholm alternative (or its variants) at a later stage. Unfortunately such a uniqueness result does not hold for all geometries Ω_ϵ and frequencies ω. We must exclude a certain (discrete) set of eigenfrequencies. As such, the following classical result concerning uniqueness is the most general result that one can hope for.

THEOREM 2.1. *The homogenous fluid-structure interaction problem,* (2.4) *with* $p^i = 0$, *has a solution in the form* $(\mathbf{u}_0, 0) \in \mathcal{U}$ *where* $\mathbf{u}_0$ *is a solution of the traction-free problem*

$$\begin{aligned} \triangle^* \mathbf{u}_0 + \rho\omega^2 \mathbf{u}_0 = \mathbf{0},\ \ x \in \Omega_\epsilon; \\ \mathbf{u}_0 = \mathbf{0},\ \ x \in \Gamma_\epsilon; \\ \mathbf{u}_0 \cdot \mathbf{n} = 0, \qquad \mathbf{T}(\mathbf{u}_0) = 0,\ \ x \in \Gamma_0. \end{aligned}$$

In other words, the problem (2.4) *has unique solutions up to eigenmodes* $\mathbf{u}_0$.
When $p^i = 0$, the exterior pressure field p_ϵ vanishes because of the radiation condition. The non-uniqueness, if any, occurs due to displacements in the bounded elastic region, which arise from the absence of constraints on the tangential components of the displacement on the fluid-structure interface, Γ_0. These components cannot be controlled since the *no-slip boundary conditions* are not physical for an inviscid fluid. The non-zero solutions of the reduced elastodynamics problem in a bounded region D with smooth boundary ∂D,

$$\triangle^* \mathbf{u}_0 + \rho\omega^2 \mathbf{u}_0 = \mathbf{0} \text{ in } D, \qquad \mathbf{u}_0 \cdot \mathbf{n} = 0, \mathbf{T}(\mathbf{u}_0) = \mathbf{0} \text{ on } \partial D, \tag{2.5}$$

are called *Jones modes*, or *traction free oscillations*, and the associated frequencies ω the *Jones frequencies*. They were discussed in depth by (26); however, (33), (24) showed their existence for spheres much earlier, and (40) demonstrated their existence for prolate spheroids. All these examples are for three-dimensional structures; we are not aware of any literature concerning these modes in $\mathbb{R}^2$. It is also known that the frequencies ω at which these *traction-free oscillations* occur form a discrete spectrum in $\mathbb{R}^3$ (26), and we conjecture that this is also the case in $\mathbb{R}^2$. For this reason, in the engineering literature, the displacement fields for such frequencies are studied at slightly perturbed, non-Jones modes, and then a continuity argument with respect to frequency is used.

Fortunately, these traction-free oscillations occur only in highly specific situations. It is known, for example, that such modes do not occur in the absence of axial symmetry in the structure (12); or in visco-elastic materials (see e.g. (36)). In addition, if the incident acoustic wave has ω, k with positive imaginary parts (c.f. (36)) these modes are not supported.

The proof of Theorem (2.1) is standard (see, e.g., (37)), and is not reproduced here.

3. A model problem. In order to gain insight about the fluid-structure interaction problem (2.4), we first study a simple scalar example. This will illustrate the key steps in the analysis of such problems. A different treatment of this example are available in (18).

We begin by precisely describing the model transmission problem. We then scale the problem in the thickness coordinate, and examine the reduced problem.This leads in turn to a formal asymptotic procedure. To justify the procedure, we cast the problem into variational form and derive appropriate invertibility and approximation results.

3.1. Statement of model problem. As in the previous section, let Ω_ϵ denote a bounded thin domain in $\mathbb{R}^2$ with smooth (at least C^2) inner and outer boundaries Γ_ϵ and Γ_0 respectively, as shown in Figure (2.1). The unbounded component of $\mathbb{R}^2 \setminus (\Omega_\epsilon \cup \Gamma_0)$ is denoted by

Ω_∞. Let $\mathcal{U}$ denote the classical function space $C^2(\Omega_\epsilon \cup \Omega_\infty) \cap C^1\left(\overline{\Omega_\epsilon} \cup \overline{\Omega}_\infty\right)$, and consider the following transmission problem:

Find $u_\epsilon \in \mathcal{U}$ satisfying the Laplace equation

$$\triangle u_\epsilon = 0, \qquad x \in \Omega_\epsilon \cup \Omega_\infty, \tag{3.1a}$$

with the *homogenous Dirichlet boundary condition,*

$$u_\epsilon^- = 0 \qquad \text{on } \Gamma_\epsilon, \tag{3.1b}$$

the *transmission conditions*

$$\left\{ \begin{array}{rcll} u_\epsilon^- & = & u_\epsilon^+ + f & \text{on } \Gamma_0, \\ \frac{\partial u_\epsilon^-}{\partial n} & = & \frac{\partial u_\epsilon^+}{\partial n} + g & \text{on } \Gamma_0, \end{array} \right. \tag{3.1c}$$

for given smooth functions f and g, as well as the *growth conditions*

$$\left\{ \begin{array}{rcll} u_\epsilon & = & O(1) & \text{as } |x| \to \infty, \\ \frac{\partial u_\epsilon}{\partial r} & = & O(\frac{1}{|x|^2}) & \text{as } r = |x| \to \infty. \end{array} \right. \tag{3.1d}$$

Here, as in the sequel, $\frac{\partial}{\partial n}$ always denotes the normal derivative with respect to the outer normal, which points towards Ω_∞. We denote by v^+ (respectively v^-) the limit of the function v on Γ_0 from the exterior (respectively the interior) of Ω_ϵ. The problem (3.1) is singular in the sense that when $\epsilon = 0$, the reduced problem to find $u_\epsilon \in \left(C^2(\Omega_\infty) \cap C^1(\overline{\Omega_\infty})\right)$, such that

$$\left\{ \begin{array}{rcll} \triangle u_0 & = & 0 & \text{in } \Omega_\infty, \\ u_0^- & = & f & \text{on } \Gamma_0, \\ \frac{\partial u_0^-}{\partial n} & = & g & \text{on } \Gamma_0, \\ u_0(x) & = & O(1) & \text{as } |x| \to \infty, \\ \frac{\partial u_0}{\partial r} & = & O(\frac{1}{|x|^2}) & \text{as } r = |x| \to \infty, \end{array} \right. \tag{3.2}$$

has no solution, since one cannot prescribe the Cauchy data f and g independently. Hence one may expect some sort of boundary layer in the neighborhood of Γ_0. However, as will be shown after a careful scaling, there is no boundary layer, and the problem can be treated as a regular perturbation problem.

3.2. Scaling. In what follows, we assume that the thickness $\epsilon > 0$ is fixed and the outer boundary is a fixed, smooth and rectifiable curve Γ_0. In particular, if we consider distances measured along the direction of the outer normal as positive, we can scale $d(x, \Gamma_0)$ by

$$t = \frac{d(x, \Gamma_0) + \epsilon}{\epsilon}.$$

When $d(x, \Gamma_0) = 0$, $t = 1$ and $x \in \Gamma_0$. When $d(x, \Gamma_0) = -\epsilon$, $t = 0$, and the point under consideration is on Γ_ϵ. We now denote the length of Γ_0 by L, and use a parametrization of Ω_ϵ by the manifold $\Omega := (0, L) \times (0, 1)$ through the mapping

$$\left\{ \begin{array}{rcl} \Omega & \longrightarrow & \Omega_\epsilon, \\ (s,t) & \to & x = \mathbf{X}(s) + \epsilon(t-1)\mathbf{n}(s) =: \mathbf{r}(s,t), \end{array} \right. \tag{3.3}$$

Here s is the arclength parameter on Γ_0, and $\mathbf{X}(s) = X\hat{i} + Y\hat{j}$ is the projection of the point x onto Γ_0. As usual, $\mathbf{n}(s)$ is the unit outward normal to Γ_0, and t is the scaled length defined previously.

In order to precisely determine various differential operators in the scaled coordinates s and t, we need to find an orthonormal basis in the new, curvilinear system, and find the appropriate scale factors. Since the curve Γ_0 is smooth, its local curvature κ is well-defined. The orthogonal unit vectors in the curvilinear coordinate system are thus given by

$$\hat{e}_s := \frac{1}{1+\epsilon(t-1)\kappa}\frac{\partial \mathbf{r}}{\partial s}, \qquad \hat{e}_t := \frac{1}{\epsilon}\frac{\partial \mathbf{r}}{\partial t}.$$

We notice that $\epsilon_t = \mathbf{n}(s)$.

Once we have these unit vectors, we can use standard tensor analysis to obtain important differential and integral operators. To whit, *the gradient* of a scalar ϕ becomes:

$$\operatorname{grad}\phi = \frac{1}{\epsilon m}(m\frac{\partial \phi}{\partial t}\hat{e}_t + \epsilon\frac{\phi}{\partial t}e_s) = \frac{1}{\epsilon}\frac{\partial \phi}{\partial t}\mathbf{n} + \frac{1}{m}\frac{\partial \phi}{\partial s}\hat{e}_s, \tag{3.4}$$

the *divergence* of a vector $\mathbf{U} = U_1\hat{e}_t + U_2\hat{e}_s$ is

$$\begin{aligned} \nabla\cdot\mathbf{U} &= \tfrac{1}{\epsilon m}\left\{\tfrac{\partial}{\partial t}(mU_1) + \tfrac{\partial}{\partial s}(\epsilon U_2)\right\} \\ &= \tfrac{1}{\epsilon}\tfrac{\partial}{\partial t}U_1 + \tfrac{\kappa}{m}U_1 + \tfrac{1}{m}\tfrac{\partial}{\partial s}U_2, \end{aligned} \tag{3.5}$$

and the *Laplacian* is given as

$$\triangle\phi = \frac{1}{\epsilon^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\kappa}{\epsilon m}\frac{\partial \phi}{\partial t} + \frac{1}{m^2}\frac{\partial^2\phi}{\partial s^2}, \tag{3.6}$$

where we have used the notation $m = 1+\epsilon(t-1)\kappa$. Moreover,

$$\int_{\Omega_\epsilon} dx = \int_0^L\int_0^1 \epsilon\ m\ dt\,ds, \qquad \int_{\Gamma_0} d\Gamma_0 = \int_0^L ds, \tag{3.7}$$

where the last relation follows from the fact that Γ_0 corresponds to the coordinate curve $t=1$, and $m=1$ on this curve,

Scalar functions $u = u(x)$, $x \in \Omega_\epsilon$ can now be written in terms of the curvilinear coordinates as $u = u(x) = \tilde{u}(s,t)$, where $(s,t)\in\Omega$. The function $\tilde{u}$ is L-periodic in its first argument.

The problem (3.1) in the annular region therefore becomes: *find* $\tilde{u} \in C^{\,2}(\Omega)\cap C^1\left(\overline{\Omega}\right)$, such that $\tilde{u}(s,t) = \tilde{u}(s+L,t)$ and

$$\frac{1}{\epsilon^2}\frac{\partial^2\tilde{u}}{\partial t^2} + \frac{\kappa}{\epsilon m}\frac{\partial \tilde{u}}{\partial t} + \frac{1}{m^2}\frac{\partial^2\phi}{\partial s^2} = 0, \tag{3.8}$$

along with the boundary conditions

$$\tilde{u}(s,0)\equiv 0, \qquad \frac{1}{\epsilon}\frac{\partial\tilde{u}}{\partial t} = \frac{\partial\tilde{u}}{\partial n} = \frac{\partial u_\epsilon^+}{\partial n} + g \quad \text{on } \Gamma_0,$$

where u_ϵ solves the exterior problem with appropriate radiation conditions. We drop the tilde $\tilde{}$, making note of the space $\tilde{u}$ is defined in.

To leading order, the transmission problem looks like

$$\left\{\begin{array}{rcll} \frac{\partial^2 u_0}{\partial t^2} &=& 0, & \text{in}\,\Omega, \\ u_0(s,0) &\equiv& 0, & \\ \frac{\partial u_0}{\partial t}(s,1) &=& 0, & \\ u_0 &=& u_e + f, & \text{on t=1} \\ \triangle u_e &=& 0, & \text{in}\,\Omega_\infty \\ u_e &=& O(1), & |x|\to\infty. \end{array}\right.$$

The solution of the BVP for u_0 is $u_0 \equiv 0$. Therefore, the exterior problem becomes

$$\begin{cases} \triangle u_e &= 0, & \text{in } \Omega_\infty \\ u_e &= -f, & \text{on t=1} \\ u_e &= O(1), & |x| \to \infty. \end{cases}$$

Let us denote by σ_0 the trace of the normal derivative of u_ϵ from Ω_∞ on Γ_0, ie,$\sigma_0 = \frac{\partial u_e}{\partial n}|_{\Gamma_0}$.

At the next order, the problem in the annular region is

$$\begin{cases} \frac{\partial^2 u_1}{\partial t^2} &= 0, & \text{in } \Omega, \\ u_1(s,0) &\equiv 0, \\ \frac{\partial u_1}{\partial t}(s,1) &= \sigma_0 + g \end{cases}$$

The solution of this BVP enables us to solve the correction for the exterior problem. We can proceed in this fashion to obtain higher order terms in the asymptotic sequence.

3.3. Reduction to bounded domain. The technique for formulating the problem (3.1) as a non-local boundary value problem is to reduce the boundary value problem in Ω_∞ to an integral equation on the boundary Γ_0. More precisely, we seek a solution of the exterior problem in Ω_∞ in the form

$$u_\epsilon(x) := \int_{\Gamma_0} \left[\frac{\partial \gamma}{\partial n_y}(x,y) u_\epsilon^+(y) - \gamma(x,y) \frac{\partial u_\epsilon^+}{\partial n_y} \right] ds_y + \omega, \qquad \forall x \in \Omega_\infty \tag{3.9}$$

where ω is an unknown constant. Here, u_ϵ^+ and $\frac{\partial u_\epsilon^+}{\partial n}$, which are the Cauchy data for the solution of the Laplace equation, are the traces of u_ϵ and its dual $\frac{\partial u_\epsilon}{\partial n}$ on Γ_0 respectively, and

$$\gamma(x,y) = \frac{-1}{2\pi} \ln(|x-y|), \qquad x \neq y, \tag{3.10}$$

is the fundamental solution for this equation in two dimensions. By taking the limits in the above *representation formula* (3.9) for u_ϵ and its gradient in the direction of the normal as $\mathbf{x} \to \Gamma_0$ from inside Ω_∞, we arrive at the following two boundary integral equations on Γ_0:

$$u_\epsilon^+ = \left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{I} + \mathbf{K} \right) u_\epsilon^+ - \mathbf{V} \frac{\partial u_\epsilon^+}{\partial n} + \omega, \qquad x \in \Gamma_0, \tag{3.11a}$$

$$\frac{\partial u_\epsilon^+}{\partial n} = \left(\frac{\mathbf{1}}{\mathbf{2}}\mathbf{I} - \mathbf{K}' \right) \frac{\partial u_\epsilon^+}{\partial n} - \mathbf{W} u_\epsilon^+, \qquad x \in \Gamma_0. \tag{3.11b}$$

The operators $\mathbf{V}, \mathbf{K}, \mathbf{K}', \mathbf{W}$ are the *boundary integral operators* defined below: $\mathbf{V}$ is the *single layer operator,*

$$(\mathbf{V}\sigma)(x) := \int_{\Gamma_0} \gamma(x,y)\sigma(y) ds_y, \qquad x \in \Gamma_0, \tag{3.12a}$$

$\mathbf{K}$, *the double-layer operator,*

$$(\mathbf{K}\mu)(x) := \int_{\Gamma_\epsilon} \frac{\partial \gamma(x,y)}{\partial n_y} \mu(y) ds_y, \qquad x \in \Gamma_0, \tag{3.12b}$$

$\mathbf{K}'$, *the adjoint of the double-layer operator,*

$$(\mathbf{K}'\sigma)(x) := \int_{\Gamma_0} \frac{\partial \gamma(x,y)}{\partial n_x} \sigma(\mathbf{y}) ds_y, \qquad x \in \Gamma_0, \tag{3.12c}$$

and finally $\mathbf{W}$, *the hypersingular operator*,

$$(\mathbf{W}\mu)(x) := -\frac{\partial}{\partial n_x}\int_{\Gamma_0}\frac{\partial\gamma(x,y)}{\partial n_y}\mu(y)ds_y, \qquad x\in\Gamma_0. \tag{3.12d}$$

These boundary integral operators are defined for smooth boundary functions σ and μ, related to the Cauchy data, and the mapping properties of these operators are discussed in detail elsewhere ((13)). We note that from the representation formula (3.9), the solution u_ϵ will be completely known in Ω_∞ if the Cauchy data on Γ_0, namely u_ϵ^+ and $\frac{\partial u_\epsilon^+}{\partial n}$, are known. On the other hand, from the transmission condition, u_ϵ^+ equals $u_\epsilon^- - f$ on Γ_0, where u_ϵ is completely determined by the solution u_ϵ in the bounded region Ω_ϵ. This allows us to convert (3.1) to an equivalent boundary value problem in the domain Ω_ϵ: *Given* f, g, *find* $(u_\epsilon, \sigma_\epsilon, \omega_\epsilon)$ *such that*

$$\begin{cases} -\triangle u_\epsilon &= 0 & \text{in } \Omega_\epsilon,\\ u_\epsilon^- &= 0 & \text{on } \Gamma_\epsilon,\\ \frac{\partial u_\epsilon}{\partial n}^- &= g+\sigma_\epsilon & \text{on } \Gamma_0,\\ (\frac{1}{2}\mathbf{I}+\mathbf{K}')\sigma_\epsilon + \mathbf{W}(u_\epsilon^-) &= \mathbf{W}f & \text{on } \Gamma_0,\\ \mathbf{V}\sigma_\epsilon - \omega_\epsilon + \left(\frac{1}{2}\mathbf{I}-\mathbf{K}\right)u_\epsilon^- &= \left(\frac{1}{2}\mathbf{I}-\mathbf{K}\right)f & \text{on } \Gamma_0, \end{cases} \tag{3.13}$$

together with the compatibility condition

$$\int_{\Gamma_0}\sigma_\epsilon ds = 0. \tag{3.14}$$

In the formulation (3.13), we employed both the boundary integral equations (3.11a) and (3.11b), with the Cauchy data u_ϵ^+ replaced by the transmission condition (3.1c). Here, $\sigma_\epsilon := \frac{\partial u_\epsilon^+}{\partial n}$ on Γ_0, and the compatibility condition is a consequence of the growth condition (3.1d).

We remark that the weak solution u_ϵ in Ω_ϵ will be completely determined by the boundary value problem

$$\begin{cases} -\triangle u_\epsilon &= 0 & \text{in } \Omega_\epsilon,\\ u_\epsilon^- &= 0 & \text{on } \Gamma_\epsilon,\\ \frac{\partial u_\epsilon}{\partial n}^- &= g+\sigma_\epsilon & \text{on } \Gamma_0, \end{cases}$$

provided σ_ϵ is known. The reduced boundary value problem (3.13) in Ω_ϵ is called a *non-local boundary value problems*. The condition (3.11a) is called a *non-local boundary condition* in the sense that in order to find σ_ϵ at a single point on Γ_0, one needs the values of u_ϵ^- on the whole boundary. We also note that one may have used only one integral equation to truncate the original problem. However, using both proves advantageous in establishing certain estimates later on. This is also in contrast to previous work, (18).

This idea of reduction of a transmission problem to a non-local boundary value problem consisting of a coupled partial differential equation and boundary integral equation was first presented by (3) and (25). Such a formulation lends itself very well to a combined boundary-element and finite-element method, and is especially useful when dealing with infinite (exterior) domains.

The reduction of an exterior problem to an integral equation is by no means a unique procedure. Here we have employed what is called the *direct method* based on the Green's formula in terms of the Cauchy data, while *indirect methods* in terms of the layer potentials may also be used. In the next subsection, we will consider the appropriate solution spaces of these non-local boundary value problems and their variational formulation.

3.4. Variational Formulation. There are two versions of the non-local boundary value problems. We shall confine ourselves only to the form of (3.13). As is usual for partial differential equations, we multiply the PDE in (3.13) by a test function $v \in H^1_{\Gamma_\epsilon}(\Omega_\epsilon)$, and integrate the resulting equation by parts over the domain Ω_ϵ. We similarly multiply the boundary integral equations by appropriate test functions and integrate over Γ_0. The test function for (3.12a) is chosen as $\chi \in H_0^{-1/2}(\Gamma_0) := \{\chi \in H_0^{-1/2}(\Gamma_0) | \ \langle \chi, 1\rangle = 0, \}$. This allows us to drop ω_ϵ as an unknown, since we note that the compatibility condition (3.14) has been built into the test space $H_0^{-1/2}(\Gamma_0)$. This yields the following *weak formulation: Given* $f \in H^{1/2}(\Gamma_0)$, $g \in H_0^{-1/2}(\Gamma_0)$, *find* $(u_\epsilon, \sigma_\epsilon) \in H_0^1(\Omega_\epsilon) \times H_0^{1/2}(\Gamma_0)$ *such that*

$$
\left\{
\begin{array}{rcl}
a_\epsilon(u_\epsilon, v) - \langle \sigma_\epsilon, v\rangle &=& \langle g, v\rangle, \\
\langle \chi, \mathbf{V}\sigma_\epsilon\rangle + \left\langle \chi, (\frac{1}{2}\mathbf{I} - \mathbf{K})u_\epsilon\right\rangle &=& \left\langle \chi, (\frac{1}{2}\mathbf{I} - \mathbf{K})f\right\rangle, \\
\left\langle (\frac{1}{2}\mathbf{I} + \mathbf{K}')\sigma_\epsilon, v\right\rangle + \langle \mathbf{W}u_\epsilon, v\rangle &=& \langle \mathbf{W}f, v\rangle
\end{array}
\right.
\tag{3.15}
$$

for all $(v, \chi) \in H_0^1(\Omega_\epsilon) \times H_0^{1/2}(\Gamma_0)$. Here $a_\epsilon(u, v)$ is the bilinear form defined by

$$a_\epsilon(u, v) := \int_{\Omega_\epsilon} \operatorname{grad} u \cdot \operatorname{grad} v dx.$$

For ease of notation, we denote $\mathcal{H}_\epsilon = H_0^1(\Omega_\epsilon) \times H_0^{1/2}(\Gamma_0)$, equipped with the inner product $[\cdot, \cdot]_\epsilon$. The equivalent product space in the scaled region is $\mathcal{H}_0 = H_0^1(\Omega) \times H_0^{1/2}(\Gamma_0)$, with the norm

$$\|(u, \sigma)\|^2_{\mathcal{H}_0)} := \left\{ \int_0^L \int_0^1 \left(|u|^2 + |\frac{\partial u}{\partial t}|^2 + |\frac{\partial u}{\partial s}|^2 \right) dt\, ds + \|\sigma\|^2_{H^{-1/2}(\Gamma_0)} \right\}.$$

We use the orthogonality of the unit vectors in the curvilinear coordinate system (s, t) to rewrite (3.15) as: *Find* $(u_\epsilon, \sigma_\epsilon) \in \mathcal{H}_\epsilon$ *such that for all* $(v, \sigma) \in \mathcal{H}_0$,

$$
\left\{
\begin{array}{rcl}
\frac{1}{\epsilon} a_o(u_\epsilon, v) + a_1(u_\epsilon, v) + \langle \sigma_\epsilon, v\rangle + \epsilon a_2(u_\epsilon, v) && \\
+ .... + \epsilon^{n-1} a_n(u_\epsilon, v) + \mathcal{Q}_n(u_\epsilon, v) &=& \langle g, v\rangle, \\
\langle \chi, 2\mathbf{V}\sigma_\epsilon\rangle + \langle \chi, (\mathbf{I} - 2\mathbf{K})u_\epsilon\rangle &=& \langle \chi, (\mathbf{I} - 2\mathbf{K}_2)f\rangle, \\
\langle (\mathbf{I} + 2\mathbf{K}')\sigma_\epsilon, v\rangle + 2\langle \mathbf{W}u_\epsilon, v\rangle &=& 2\langle \mathbf{W}f, v\rangle
\end{array}
\right.
\tag{3.16}
$$

where

$$a_0(u, v) := \int_0^L \int_0^1 \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} dt\, ds, \tag{3.17a}$$

$$a_1(u, v) := \int_0^L \int_0^1 (t - 1)\kappa \frac{\partial u}{\partial t} \frac{\partial v}{\partial t} dt\, ds, \tag{3.17b}$$

$$a_k := \int_0^L \int_0^1 ((1 - t)\kappa)^{(k-2)} \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} dt\ ds, \quad 2 \le k \le n, \tag{3.17c}$$

and $\mathcal{Q}_n(u, v)$ is the remainder term from the bilinear form defined by

$$
\begin{aligned}
\mathcal{Q}_n(u, v) &:= a_\epsilon(u, v) \sum_{k=0}^{n} \epsilon^{k-1} a_k(u, v) \\
&= \epsilon^n \int_0^L \int_0^1 \frac{[(1 - t)\kappa]^{n-1}}{1 + \epsilon(t - 1)\kappa} \frac{\partial u}{\partial s} \frac{\partial v}{\partial s} dt\, ds.
\end{aligned}
$$

We see that $\mathcal{Q}_n(u,v) = O(\epsilon^n)$ for fixed $u, v \in H^1(\Omega)$. This permits us to adopt the notation

$$\epsilon^n Q(u,v) := \mathcal{Q}_n(u,v), \qquad \forall u,v \in H^1(\Omega). \tag{3.18}$$

The boundary integral equations remains unchanged. This is because the curve Γ_0 does not move or change with varying ϵ, and always corresponds to $t = 1$. Also note that the solution pairs u_ϵ and σ_ϵ depend on ϵ.

3.5. Sequences of Uncoupled Problems. We now use the formal series expansions

$$u_\epsilon(s,t,\epsilon) = \sum_{j=0}^{n} \epsilon^j u_j(s,t) + R_n(s,t), \tag{3.19}$$

$$\sigma_\epsilon(s,\epsilon) = \sum_{j=0}^{n} \epsilon^j \sigma_j(s) + S_n(s), \tag{3.20}$$

in (3.16) and collect like powers of ϵ to obtain the following sequence of problems: *Find* $(u_i, \sigma_i) \in \mathcal{H}_0$, $i = 0, 1, 2, ..., n$ *such that for all* $(v, \chi) \in \mathcal{H}_0$,

$$\left\{ \begin{array}{rcl} a_0(u_o, v) & = & 0, \\ \langle \chi, 2\mathbf{V}_0\sigma_o \rangle & = & \langle \chi, (\mathbf{I} - 2\mathbf{K}_2)f \rangle - \langle \chi, (\mathbf{I} - 2\mathbf{K})u_o \rangle ; \\ \langle (\mathbf{I} + 2\mathbf{K}')\sigma_0, v \rangle & = & 2 \langle \mathbf{W}(f - u_0), v \rangle \end{array} \right.$$

$$\left\{ \begin{array}{rcl} a_0(u_1, v) & = & -a_1(u_o, v) + \langle g, v \rangle + \langle \sigma_o, v \rangle, \\ \langle \chi, 2\mathbf{V}\sigma_1 \rangle & = & - \langle \chi, (\mathbf{I} - 2\mathbf{K})u_1 \rangle ; \\ \langle (\mathbf{I} + 2\mathbf{K}')\sigma_1, v \rangle & = & -2 \langle \mathbf{W}(u_1), v \rangle \end{array} \right.$$

and

$$\left\{ \begin{array}{rcl} a_0(u_k, v) & = & -\sum\limits_{l=0}^{k-1} a_{k-l}(u_l, v) + \langle \sigma_{n-1}, v \rangle \\ \langle \chi, 2\mathbf{V}_0\sigma_k \rangle & = & - \langle \chi, (\mathbf{I} - 2\mathbf{K})u_k \rangle ; \\ \langle (\mathbf{I} + 2\mathbf{K}')\sigma_k, v \rangle & = & -2 \langle \mathbf{W}(u_k), v \rangle \end{array} \right.$$

for $2 \le k \le n$. From the definitions of the sequence of problems, we see that the general scheme is as follows: For each fixed k, first we solve a variational problem of the form

$$a_0(u_k, v) = F_k(u_0, u_1, ..., u_{k-1}; \sigma_0, \sigma_1 ... \sigma_{k-1}; v)$$

for u_k, and then use the trace of this function to solve the associated variational problem for σ_k. Either of the formulations

$$\langle \chi, 2\mathbf{V}_0\sigma_k \rangle = - \langle \chi, (\mathbf{I} - 2\mathbf{K})u_k \rangle ; \qquad \langle (\mathbf{I} + 2\mathbf{K}')\sigma_k, v \rangle = -2 \langle \mathbf{W}(u_k), v \rangle$$

may be used to obtain σ_k; we pick the first. Thus, after obtaining u_k, we solve a variational problem of the form

$$\langle \chi, 2\mathbf{V}_0\sigma_k \rangle + \omega_k \langle \chi, 1 \rangle = G_k\left(u_0, u_1, u_2, ...u_k; \sigma_0, \sigma_1, \sigma_2, ...\sigma_{k-1}; \chi\right)$$

for in σ_k. Here, the linear functionals F_k, G_k are defined by

$$\begin{aligned} F_0(v) &:= 0, \\ G_0(u_0; \chi) &:= - \langle \chi, (\mathbf{I} - 2\mathbf{K})u_0 \rangle + \langle \chi, (\mathbf{I} - 2\mathbf{K})f \rangle , \\ F_1(u_0; \sigma_0; v) &:= -a_1(u_0, v) + \langle g, v \rangle + \langle \sigma_0, v \rangle, \\ G_1(u_0, u_1; \sigma_0; \chi) &:= - \langle \chi, (\mathbf{I} - 2\mathbf{K})u_1 \rangle , \end{aligned}$$

and for $k \geq 2$,

$$\begin{aligned} F_k(u_0, u_1, ..., u_{k-1}; \sigma_0, \sigma_1, ..., \sigma_{k-1}; v) &:= -\sum_{l=0}^{k-1} a_{k-l}(u_l, v) + \langle \sigma_{k-1}, v \rangle, \\ G_k\,(u_0, u_1, u_2, ..., u_k; \sigma_0, \sigma_1, \sigma_2, ..., \sigma_{k-1}; \chi) &:= -\langle \chi, (\mathbf{I} - 2\mathbf{K}) u_k \rangle\,. \end{aligned}$$

In principle, for each solution pair (u_j, σ_j) in the formal series expansion (3.19), we are solving two similar uncoupled equations. The existence and uniqueness for solutions to these problems was discussed in (18). Schematically, this can be represented via the following algorithm:

$u_0^- = 0$ in Ω $\longrightarrow$ $u_0^+ = u_0^- - f$ on Γ_0

$\downarrow$

$\sigma_0 = \frac{\partial u_0^+}{\partial n}$ on Γ_0

$\swarrow$

$\frac{\partial u_1^-}{\partial n} = \sigma_0 + g$ on Γ_0

$\downarrow$

u_1 in Ω $\longrightarrow$ $u_1^+ = u_1^-$ on Γ_0

$\downarrow$

$\sigma_1 = \frac{\partial u_1^+}{\partial n}$ on Γ_0

$\swarrow$

$\frac{\partial u_2^-}{\partial n} = \sigma_1$ on Γ_0

$\downarrow$

u_2 in Ω $\longrightarrow$ $u_2^+ = u_2^-$ on Γ_0

$\downarrow$

$\sigma_2 = \frac{\partial u_2^+}{\partial n}$ on Γ_0

$\swarrow$

...

At this stage, we have a purely formal asymptotic procedure for solving the transmission problem in the exterior of a thin domain. The justification of this procedure requires further investigation.

3.6. A crucial estimate. In order to justify the asymptotics, we return to the weak formulation (3.15): *Find* $(u_\epsilon, \sigma_\epsilon) \in \mathcal{H}_\epsilon$ *such that* $\forall (v, \lambda) \in \mathcal{H}_\epsilon$

$$\left\{ \begin{aligned} a_\epsilon(u_\epsilon, v) - \langle \sigma_\epsilon, v \rangle &= \langle g, v \rangle \\ 2\langle \lambda, \mathbf{V}\sigma_\epsilon \rangle + \langle \lambda, (\mathbf{I} - 2\mathbf{K}) u_\epsilon \rangle &= \langle \chi, (\mathbf{I} - 2\mathbf{K}) f \rangle \\ \langle (\mathbf{I} + 2\mathbf{K}')\sigma_\epsilon, v \rangle + 2\,\langle \mathbf{W} u_\epsilon, v \rangle &= 2\,\langle \mathbf{W} f, v \rangle \end{aligned} \right. \tag{3.21}$$

Let us introduce the bilinear operator $\mathcal{A}$ acting on $\mathcal{H}_\epsilon \times \mathcal{H}_\epsilon$:

$$\begin{aligned}[(v,\lambda);\mathcal{A}_\epsilon(u,\xi)]_\epsilon := a_\epsilon(u_\epsilon, v) - \langle \sigma_\epsilon, v\rangle \\ + \langle \lambda, \mathbf{V}\sigma_\epsilon\rangle + \langle \lambda, (\frac{1}{2}\mathbf{I} - \mathbf{K})u_\epsilon\rangle \\ + \langle \mathbf{W}u_\epsilon, v\rangle + \langle (\frac{1}{2}\mathbf{I} + \mathbf{K}')\sigma_\epsilon, v\rangle.\end{aligned}$$

We now present a crucial estimate for the operator $\mathcal{A}_\epsilon$. This estimate, following certain norm equivalence relationships, will allow us to justify the formal asymptotic procedure presented in the previous Subsection.

THEOREM 3.1. *Let Γ_0 be a smooth (C^2) rectifiable curve, and k not be an exceptional value for the transmission problem. Then, for all $0 < \epsilon < \epsilon_0$, there exists constants $c_1, c_2 > 0$ depending only on k, α_v and α_w, such that*

$$c_1\|(v,\lambda)\|^2_{\mathcal{H}_\epsilon} \le |[(v,\lambda);\mathcal{A}_\epsilon(v,\lambda)]_\epsilon| \le c_2\|(v,\lambda)\|^2_{\mathcal{H}_\epsilon}, \qquad \forall\,(v,\lambda)\in\mathcal{H}_\epsilon. \tag{3.22}$$

The thickness ϵ_0 depends on k.

Proof. The right hand estimate follows immediately from the boundedness of the operators involved. We omit the details here.

The left hand estimate requires more investigation. We begin by examining $\langle (v,\lambda); \mathcal{A}_\epsilon(v,\lambda)\rangle$. From the definition of $\mathcal{A}_\epsilon$, we get

$$\begin{aligned}[(v,\lambda);\mathcal{A}_\epsilon(v,\lambda)]_\epsilon &= a_\epsilon(v,v) + \langle\lambda, V\lambda\rangle + \langle \mathbf{W}v, v\rangle \\ &\quad - \langle \lambda, Kv\rangle + \langle K'\lambda, v\rangle \\ &= a_\epsilon(v,v) + \langle\lambda, V\lambda\rangle + \langle \mathbf{W}v, v\rangle.\end{aligned}$$

We have used the fact that $\mathbf{K}$ and $\mathbf{K}'$ are adjoint operators. Proceeding,

$$\begin{aligned}[(v,\lambda);\mathcal{A}_\epsilon(v,\lambda)]_\epsilon &\ge \|\nabla v\|^2_{L^2(\Omega_\epsilon)} + \alpha_v\|\lambda\|^2_{H_0^{-1/2}(\Gamma_0)} + \langle \mathbf{W}v, v\rangle \\ &\ge \|\nabla v\|^2_{L^2(\Omega_\epsilon)} + \alpha_v\|\lambda\|^2_{H_0^{-1/2}(\Gamma_0)}\end{aligned} \tag{3.23}$$

We have used the positivity of the boundary energy $\langle \mathbf{W}v, v\rangle \ge 0$ and the coerciveness of $\mathbf{V}$ on $H_0^{-1/2}(\Gamma_0)$ in the final inequality.

The next part of the estimate crucially depends on the Poincaré inequality for a thin domain, which (37) states that

$$\|v\|^2_{L^2(\Omega_\epsilon)} \le c_o\epsilon^2\|\nabla v\|^2_{L^2(\Omega_\epsilon)}, \qquad \forall v\in H_0^1(\Omega_\epsilon). \tag{3.24}$$

Using this inequality, we are able to estimate

$$\|v\|^2_{H_0^1(\Gamma_0)} = \|\operatorname{grad} v\|^2_{L^2(\Omega_\epsilon)} + \|v\|^2_{L^2(\Omega_\epsilon)} \le (1+c_0\epsilon^2)\|\operatorname{grad} v\|^2_{L^2(\Omega_\epsilon)} \le 2\|\operatorname{grad} v\|^2_{L^2(\Omega_\epsilon)}$$

provided $0 < c\epsilon^2 \le 1$. Thus, for all $\epsilon \in (0,\epsilon_0)$ where $c\epsilon_0^2 = 1$, we have

$$\begin{aligned}[(v,\lambda);\mathcal{A}_\epsilon(v,\lambda)]_\epsilon &\ge \|\nabla v\|^2_{L^2(\Omega_\epsilon)} + \alpha_v\|\lambda\|^2_{H_0^{-1/2}(\Gamma_0)} \\ &\ge \frac{1}{2}\|v\|^2_{H_0^1(\Omega_\epsilon)} + \alpha_v\|\lambda\|^2_{H_0^{-1/2}(\Gamma_0)} \\ &\ge c_1\|(v,\lambda)\|^2_{\mathcal{H}_\epsilon},\end{aligned}$$

with $c_1 = \min\{\frac{1}{2}, c_v\}$. This is the required lower bound. □

3.7. Justification of the asymptotic procedure.. We now return to the justification of the asymptotic procedure. We derive the variational equations for the remainder terms $(R_n, S_n) \in \mathcal{H}_\epsilon$. By definition of these remainders, we have

$$a_\epsilon(R_n, v) - \langle S_n, v\rangle = a_\epsilon(u_\epsilon, v) - \langle \sigma_\epsilon, v\rangle - a_\epsilon\left(\sum_{j=0}^{n} \epsilon^j u_j, v\right) + \left\langle \sum_{j=0}^{n} \epsilon^j \sigma_j, v\right\rangle,$$

and

$$\langle \chi, 2\mathbf{V}S_n\rangle + \langle \chi, (\mathbf{I} - 2\mathbf{K})R_n\rangle = 0$$
$$\langle 2\mathbf{W}R_n, v\rangle + \langle(\mathbf{I} + 2\mathbf{K}')S_n, v\rangle = 0.$$

A simple, but messy computation allows us to write this problem as

$$[(v,\chi); \mathcal{A}_\epsilon(R_n, S_n)]_\epsilon = [(v,\chi); \mathcal{F}_n]_\epsilon, \qquad \forall (v,\chi) \in \mathcal{H}_\epsilon, \tag{3.25}$$

where

$$\|\mathcal{F}_n\|_{\mathcal{H}_\epsilon} \le \epsilon^{n-1}(\mathcal{M})\|(v,\chi)\|_{\mathcal{H}_\epsilon}$$

as $\epsilon \to 0^+$. The constant $\mathcal{M}$ depends on $_0, u_1, ..u_{n-1}$. In deriving the estimate for $\mathcal{F}$, we used the following relationship between the energy norms in Ω and Ω_ϵ:

$$\epsilon c_3 \|u\|_{H_0^1(\Omega)} \le \||u\|_{H_0^1(\Omega_\epsilon)} \le \frac{c_4}{\epsilon}\|u\|_{H_0^1(\Omega)}, \tag{3.26}$$

where c_3, c_4 are positive constants independant of ϵ.

Using Theorem 3.1 and this estimate, we get

$$c_1\|(R_n, S_n)\|^2_{\mathcal{H}_\epsilon} \le |[(R_n, S_n); \mathcal{A}_\epsilon(R_n, S_n)]_\epsilon| \le \epsilon^{n-1}(\mathcal{M})\|(R_n, S_n)\|_{\mathcal{H}_\epsilon}. \tag{3.27}$$

which leads to the main theorem.

THEOREM 3.2. *For sufficiently smooth data f, g and a smooth boundary Γ_0, the following asymptotic estimate holds:*

$$\left\|(u_\epsilon, \sigma_\epsilon) - \left(\sum_{k=0}^{n} \epsilon^n u_n, \sum_{k=0}^{n} \epsilon^n \sigma_n\right)\right\|_{\mathcal{H}_0} = O(\epsilon^{n-2}), \qquad \textit{as } \epsilon \to 0^+, \tag{3.28}$$

where $\{(u_j, \sigma_j)\}$ is the asymptotic sequence constructed by the formal procedure. For details, we refer the reader to (18).

Before we conclude this section, we draw attention to the key ingredients in the analysis.

- The coercivity of the single layer operator $\mathbf{V}$ over a suitable subspace of $H^{-1/2}(\Gamma_0)$,
- The positivity of the boundary energy $\langle \mathbf{W}v, v\rangle$,
- Poincaré's inequality for thin domains,
- The norm equivalence relationship (3.26).

In analysing the fluid-structure interaction problem, we will follow the same procedure. However, the analysis is complicated by the need for a suitable Korn's inequality, and the presence of an L^2 term which needs to be balanced.

4. Non-local Boundary Value Problem. We return now to the fluid-structure interaction problem (2.4). This is posed on an infinite computational region, $\Omega_\epsilon \cup \Omega_\infty$. As in Subsection (3.3), we truncate this domain by reducing the exterior boundary value problem for p_ϵ, to an integral equation for the Cauchy data of the problem on Γ_0. Once this integral equation is solved, we can use a suitable analytic representation formula to obtain p_ϵ at any point in the exterior region Ω_∞. AAs in the previous problem, any solution $p_\epsilon \in C^2(\Omega_\infty) \cap C^1(\overline{\Omega}_\infty)$ of the exterior Helmholtz problem (2.2a,2.2c, 2.2d), which satisfies the radiation condition (2.2f), has a representation for p_ϵ for all $x \in \Omega_\infty$:

$$p_\epsilon(x) = \int_{\Gamma_0} \frac{\partial p_\epsilon}{\partial n_y} \gamma_k(y) - \frac{\partial \gamma_k}{\partial n_y} p_\epsilon(y)\, ds_y. \tag{4.1}$$

Here $\gamma_k(x,y) := \imath k H_0^1(k|x-y|)$ is the fundamental solution for the Helmholtz equation in two dimensions. By allowing $x \to \Gamma_0$ in this representation formula, we arrive at

$$(\frac{1}{2}\mathbf{I} - \mathbf{K}_k)p_\epsilon + \mathbf{V}_k \frac{\partial p_\epsilon}{\partial n} = 0, \qquad x \in \Gamma_0, \tag{4.2a}$$

whilst taking the normal derivative first and then allowing $\mathbf{x} \to \Gamma_0$ yields

$$(\frac{1}{2}\mathbf{I} + \mathbf{K}'_k)\frac{\partial p_\epsilon}{\partial n} + \mathbf{W}_k p_\epsilon = 0, \qquad x \in \Gamma_0. \tag{4.2b}$$

We note the use of the jump conditions in these derivations. Here, as in Subsection 3.3, the boundary integral operators $\mathbf{K}_k, \mathbf{K}'_k, \mathbf{V}_k$ and $\mathbf{W}_k$ are defined as their analogues in Subsection 3.3, with the fundamental solution for the Laplacian, γ, replaced by that of the Helmholtz equation, γ_k. We can use the integral equations, combined with the transmission conditions across Γ_0 to obtain the equivalent non-local boundary value problems below: *Find* $(\mathbf{u}_\epsilon, p)$ *such that*

$$\left\{ \begin{array}{rcll} (\triangle^* + \rho\omega^2)\mathbf{u}_\epsilon &=& \mathbf{0}, & x \in \Omega_\epsilon, \\ \mathbf{u}_\epsilon &=& \mathbf{0}, & x \in \Gamma_\epsilon, \\ \mathbf{T}\mathbf{u}_\epsilon \cdot \mathbf{n} + p\mathbf{n} &=& p^i\mathbf{n} & x \in \Gamma_0, \\ (\frac{1}{2}\mathbf{I} - \mathbf{K}_k)p_\epsilon + \rho\omega^2\mathbf{V}_k(\mathbf{u}\cdot\hat{\mathbf{n}}) &=& \mathbf{V}_k\frac{\partial p^i}{\partial n}, & x \in \Gamma_0, \end{array} \right. \tag{4.3}$$

or *Find* $(\mathbf{u}_\epsilon, p)$ *such that*

$$\left\{ \begin{array}{rcll} (\triangle^* + \rho\omega^2)\mathbf{u}_\epsilon &=& \mathbf{0}, & x \in \Omega_\epsilon, \\ \mathbf{u}_\epsilon &=& \mathbf{0}, & x \in \Gamma_\epsilon, \\ \mathbf{T}\mathbf{u}_\epsilon \cdot \mathbf{n} + p\mathbf{n} &=& p^i\mathbf{n}, & x \in \Gamma_0, \\ \mathbf{W}_k p + (\frac{1}{2}\mathbf{I} + \mathbf{K}'_k)(\omega^2\rho\mathbf{u}\cdot\mathbf{n}) &=& (\frac{1}{2}\mathbf{I} + \mathbf{K}'_k)(\frac{\partial p^i}{\partial n}) & x \in \Gamma_0. \end{array} \right. \tag{4.4}$$

The reader may observe that we have not yet specified the function spaces in which we seek solutions. This is deliberate at this stage. Following the discussion in (23), we see that the weak formulation of the problem (4.3) is problematic. This is because the transmission conditions lead to inconsistencies in the spaces used; the trace of p lies "naturally" in $H^{1/2}(\Gamma_0)$, but is associated with the traction, which belongs to $H^{-1/2}(\Gamma_0)$. This can be resolved by requiring that the trace of p lie in the "unnatural" space $L^2(\Gamma_0)$, however then the crucial Gårding inequality will not hold for Lipschitz boundaries. To avoid these problems, we will use the second formulation (4.4), since it has a variational equivalent which is posed in the "natural" spaces for both $\mathbf{u}_\epsilon$ and p_ϵ. However, unlike the simple example previously discussed, we are no longer able to use the two integral equations to establish the analog of estimate (3.1). Instead, we need a technical assumption on the "boundary energy". In other work]citehn2002 this assumption has been removed, but at the expense of introducing an extra aritifical boundary *a la* (2).

5. Weak Formulation. This section forms the core of our analysis. We shall define the function spaces in which the variational formulation of (4.4) is posed, carefully describing the norms we use. We also collect some properties about the integral operators introduced in the previous section. These properties, combined with Korn's inequality and a uniqueness result, yield an existence theorem for the solutions. Finally, we derive some estimates for the operators, which will be used to justify our asymptotic technique later.

5.1. Some function spaces and notation. We are interested in the elastic displacement in the region Ω_ϵ, and therefore employ the following (natural) energy space for these vector-valued functions:

$$\vec{H}^1(\Omega_\epsilon) := \left\{ \mathbf{u} = (u_1, u_2) | u_1, u_2 \in H^1(\Omega_\epsilon) \right\}, \tag{5.1}$$

with the associated norm

$$\|\mathbf{u}\|_{\vec{H}^1(\Omega_\epsilon)} := \left\{ \int_{\Omega_\epsilon} \operatorname{grad} \mathbf{u} : \operatorname{grad} \bar{\mathbf{u}} + |\mathbf{u}|^2 dx \right\}^{1/2}. \tag{5.2}$$

Here, $\operatorname{grad} \mathbf{u}$ is the 2×2 tensor with components $\frac{\partial u_i}{\partial x_j}$, $1 \leq i, j \leq 2$, and this tensor is not, in general, symmetric. This lack of symmetric derivatives in the definition of the norm is made explicit if we notice that

$$\operatorname{grad} \mathbf{u} : \operatorname{grad} \bar{\mathbf{u}} = \underline{\underline{\mathbf{e}}}(\mathbf{u}) : \underline{\underline{\mathbf{e}}}(\bar{\mathbf{u}}) + \frac{1}{2}(\nabla \times \mathbf{u}) \cdot (\nabla \times \bar{\mathbf{u}}).$$

(see, e.g., (7)). The subspace $\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon) \subset \vec{H}^1(\Omega_\epsilon)$ has the obvious interpretation. The correct function space for p_ϵ is the trace space $H^{1/2}(\Gamma_0)$. We shall seek solutions to a problem posed in the product space

$$\vec{\mathcal{H}}_\epsilon := \left\{ (\mathbf{u}, p) | \mathbf{u} \in \vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon), p \in H^{1/2}(\Gamma_0) \right\}, \tag{5.3}$$

equipped with the norm

$$\|(\mathbf{u}, \mu)\|_{\vec{\mathcal{H}}_\epsilon} = \left(\|\mathbf{u}\|^2_{\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)} + \|\mu\|^2_{H^{1/2}(\Gamma_0)} \right)^{1/2}.$$

We note that this product space depends on ϵ. Since we are interested ultimately in an asymptotic analysis as $\epsilon \to 0$, it will become necessary to identify a solution space in scaled coordinates. A mapping between the given thin region Ω_ϵ and the scaled region $\Omega := (0,1) \times [0, L]$ is possible, since Γ_0 is smooth. A detailed discussion on how to achieve such a scaling was presented in Subsection 3.2. Here, we only accumulate notation about various function spaces defined over Ω.

The energy space $\vec{H}^1(\Omega)$ for vector-valued functions $\mathbf{u} := u_1 \hat{\mathbf{e}}_t + u_2 \hat{\mathbf{e}}_s$ in the scaled t, s variables is defined by

$$\vec{H}^1(\Omega) := \left\{ \mathbf{u}(t,s) \in L^2(\Omega) \mid \frac{\partial u_i}{\partial t}, \frac{\partial u_i}{\partial s} \in L^2(\Omega) \right\},$$

while $\mathcal{H}_0$ denotes the subspace of $\vec{H}^1(\Omega)$ which are L-periodic in the variable s and have vanishing trace on the curve $t = 0$. The norm on $\mathcal{H}_0$ is given by

$$\|\mathbf{u}\|_{\mathcal{H}_0} := \left\{ \int_0^L \int_0^1 \operatorname{grad} \mathbf{u} : \overline{\operatorname{grad} \mathbf{u}} + |\mathbf{u}|^2 \, dt \, ds \right\}^{1/2},$$

where

$$\operatorname{grad}\mathbf{u} := \begin{pmatrix} \frac{\partial u_1}{\partial t} & \frac{\partial u_1}{\partial s} \\ \frac{\partial u_2}{\partial t} & \frac{\partial u_2}{\partial s} \end{pmatrix}.$$

The product space $\mathcal{H}_0 \times H^{-1/2}(\Gamma_0)$ is denoted by $\vec{\mathcal{H}}_0$, and is defined in the natural way.

We shall also make use of several duality pairings and inner products. In what follows, we denote by $\langle \cdot, \cdot \rangle_0$, the L^2-duality pairing on $H^{-1/2}(\Gamma_0) \times H^{1/2}(\Gamma_0)$. The L^2-inner product over Ω_ϵ is

$$(\mathbf{u}, \mathbf{v})_{0,\Omega_\epsilon} := \int_{\Omega_\epsilon} \mathbf{u} \cdot \bar{\mathbf{v}}\, dx, \qquad \forall \mathbf{u}, \mathbf{v} \in \vec{L}^2(\Omega_\epsilon).$$

The inner product on $H^{1/2}(\Gamma_0)$ is denoted by $(\cdot, \cdot)_{1/2,\Gamma_0}$. Likewise, the inner product on the product space $\vec{\mathcal{H}}_\epsilon$ is

$$\begin{aligned}[(\mathbf{u}, p, \sigma), (\mathbf{v}, q, \chi)]_\epsilon := \int_{\Omega_\epsilon} \underline{\underline{\sigma}}(\mathbf{u}) : \overline{\underline{\underline{\mathbf{e}}}(\mathbf{v})} dx + (\mathbf{u}, \bar{\mathbf{v}})_{0,\Omega_\epsilon} + (\operatorname{grad} p, \operatorname{grad} \bar{q})_{0,\Omega_R} \\ +(p, \bar{q})_{0,\Omega_R} + (\sigma, \chi)_{-1/2,\Sigma}.\end{aligned}$$

We claim that $[\cdot, \cdot]_\epsilon$ actually defines a norm on $\vec{\mathcal{H}}_\epsilon$. To see this, it suffices to show that $\int_{\Omega_\epsilon} \underline{\underline{\sigma}}(\mathbf{u}) : \overline{\underline{\underline{\mathbf{e}}}(\mathbf{u})}\, dx + \|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}$ is a norm equivalent to $\|\mathbf{u}\|_{\vec{H}^1(\Omega_\epsilon)}$. This is not immediately obvious, since $\underline{\underline{\sigma}}(\mathbf{u}) : \overline{\underline{\underline{\mathbf{e}}}(\mathbf{u})}$ contains only the certain combinations of derivatives of $\mathbf{u}$, while $\|\mathbf{u}\|_{\vec{H}^1(\Omega_\epsilon)}$ contains all the combinations. From constitutive law, (see, e.g.,(10), (34)),

$$\sigma_{ij} = a_{ijkl} e_{kl},$$

with $a_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where Einstein's summation convention has been employed. Thus, we can write

$$\int_{\Omega_\epsilon} \underline{\underline{\sigma}}(\mathbf{u}) : \overline{\underline{\underline{\mathbf{e}}}(\mathbf{v})}\, dx = \int_{\Omega_\epsilon} a_{ijkl} e_{kl}(\mathbf{u}) \overline{e_{ij}(\mathbf{v})} dx.$$

From the ellipticity properties of the coefficients of elasticity, it follows that

$$\int_{\Omega_\epsilon} a_{ijkl} e_{kl}(\mathbf{u}) \overline{e_{ij}(\mathbf{u})}\, dx \geq \alpha_1 \int_{\Omega_\epsilon} e_{ij}(\mathbf{u}) \overline{e_{ij}(\mathbf{u})}\, dx,$$

where α_1 is a positive constant depending only on λ and μ for an isotropic, homogenous medium. We now use *Korn's inequality* which states that

$$\int_{\Omega_\epsilon} e_{ij}(\mathbf{u}) \overline{e_{ij}(\mathbf{u})}\, dx + \int_{\Omega_\epsilon} |\mathbf{u}|^2\, dx \geq c_k \|\mathbf{u}\|^2_{\vec{H}^1(\Omega_\epsilon)}$$

for some constant $c_k > 0$ which depends on Ω_ϵ. In other words, this inequality asserts that if the "symmetric" part of grad $\mathbf{u}$ is controlled, this automatically controls the skew-symmetric part. As a consequence, we see that there holds the inequality

$$\int_{\Omega_\epsilon} \underline{\underline{\sigma}}(\mathbf{u}) : \overline{\underline{\underline{\mathbf{e}}}(\mathbf{u})} + |\mathbf{u}|^2\, dx \geq c\|\mathbf{u}\|^2_{\vec{H}^1(\Omega_\epsilon)}$$

for some constant $c > 0$. We also have the equivalence between the semi-norm $\left(\int_{\Omega_\epsilon} \underline{\underline{\sigma}}(\mathbf{u}) : \overline{\underline{\underline{\mathbf{e}}}(\mathbf{u})}\, dx\right)^{1/2}$ and the $\vec{H}^1(\Omega_\epsilon)$-norm for $\mathbf{u} \in \vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)$. For smooth Γ_0, mapping properties of the integral operators are given below:

1. The operator $\mathbf{W}_k$ is continuous as a mapping from $H^{s+1}(\Gamma_0)$ to $H^s(\Gamma_0)$, for all $s \in \mathbb{R}$. Further, $\mathbf{W}_k$ satisfies a Gårding inequality in the form

$$Re\langle \mathbf{W}_k \chi, \overline{\chi}\rangle \geq \alpha \|\chi\|^2_{H^{1/2}(\Sigma)} - \beta_v \|\chi\|^2_{H^{1/2-\delta}(\Sigma)}, \tag{5.4}$$

where $\alpha_v > 0, \beta_v \geq 0$ are independent of $\chi \in H^{1/2}(\Sigma_0)$. This implies that $\mathbf{W}_k$ permits a decomposition of the form

$$\mathbf{W}_k = \mathbf{W}_0 + C_v, \tag{5.5}$$

where $\mathbf{W}_0$ is $H^{1/2}(\Gamma_0)$-elliptic and C_v is compact on $H^{1/2}(\Gamma_0)$.
2. The operator $\mathbf{K}'_k$ is continuous as a mapping from $H^s(\Gamma_0) \longrightarrow H^{s+1}(\Sigma)$ for all $s \in \mathbb{R}$, and in particular, is compact as a mapping from $H^{1/2}(\Gamma_0) \longrightarrow H^{1/2}(\Gamma_0)$.
3. When k is not an exceptional value, (that is, when k^2 is not an eigenvalue for the Neumann problem for $-\triangle$ in the interior of Γ_0) , $\mathbf{W}_k$ is an isomorphism from $H^{s+1}(\Gamma_0) \to H^s(\Gamma_0)$). See, e.g., (17).

There are thus two sets of frequencies we must exclude: those which are *physical* eigenmodes of the problem, i.e. the Jones modes, and those that are *mathematical*, the Neumann eigenvalues. The latter are a consequence of the method we have employed to truncate the region; one may get around this problem by either modifying the kernels (as in (28)), or by combining integral equations, ((4)).

5.2. Weak formulation of the transmission problem. In the remainder of this article, we assume that k is not an exceptional value, and therefore $\mathbf{W}_k$ is an isomorphism, and the equation $\mathbf{W}_k\mu = 0$ on Γ_0 has only the trivial solution. Making use of the transmission conditions on Γ_0 and Σ, we now have the variational formulation for the fluid-structure interaction problem. This is precisely stated as: *For given $p^i \in H^{1/2}(\Gamma_0)$, find $(\mathbf{u}_\epsilon, p_\epsilon) \in \vec{\mathcal{H}}_\epsilon$ such that for all $(\mathbf{v}, q) \in \vec{\mathcal{H}}_\epsilon$,*

$$\left\{ \begin{array}{rcl} a_\epsilon(\mathbf{u}_\epsilon, \mathbf{v}) - \rho\omega^2(\mathbf{u}, \mathbf{v})_{0,\Omega_\epsilon} + \langle p, \bar{\mathbf{v}} \cdot \mathbf{n}\rangle_{\Gamma_0} & = & -\langle p^i, \bar{\mathbf{v}}, \cdot \mathbf{n}\rangle_{\Gamma_0}, \\ \langle \mathbf{W}_k p, \bar{q}\rangle_{\Gamma_0} + \rho\omega^2 \left\langle (\frac{1}{2}\mathbf{I} + \mathbf{K}'_k)(\mathbf{u}\cdot\mathbf{n}), \bar{q}\right\rangle_{\Gamma_0} & = & \left\langle (\frac{1}{2}\mathbf{I} + \mathbf{K}'_k)\frac{\partial p^i}{\partial n}, \bar{q}\right\rangle_{\Gamma_0}. \end{array}\right. \tag{5.6}$$

In order to demonstrate existence and uniqueness results for this formulation, we first convert this system into an operator equation. We then proceed to demonstrate uniqueness for the operator equation, and prove existence via the Fredholm theory. This latter proof shall follow by showing compactness for one operator, and a Gårding inequality for the other.

We begin by converting the weak formulation (5.6) into an operator equation,

$$(\mathcal{A}_\epsilon + \mathcal{K}_0)(\mathbf{u}_\epsilon, p_\epsilon) = \mathcal{F}_\epsilon, \tag{5.7}$$

by making use of the Riesz representation theorem to define linear operators $\mathcal{A}_\epsilon, \mathcal{K}_0 : \vec{\mathcal{H}}_\epsilon \to \vec{\mathcal{H}}_\epsilon$. These operators are given by

$$\left\{ \begin{array}{rcl} [\mathcal{A}_\epsilon(\mathbf{u}_\epsilon, p_\epsilon), (\mathbf{v}, q)]_\epsilon & := & a_\epsilon(\mathbf{u}, \mathbf{v}) - \rho\omega^2(\mathbf{u}, \mathbf{v})_{0,\Omega_\epsilon} + \langle \mathbf{W}_o p, \bar{q}\rangle_{\Gamma_0}, \\ {[\mathcal{K}_0(\mathbf{u}_\epsilon, p_\epsilon), (\mathbf{v}, q)]_\epsilon} & := & \langle p, \bar{\mathbf{v}}\cdot\mathbf{n}\rangle_{\Gamma_0} + \rho\omega^2 \left\langle (\frac{1}{2}\mathbf{I} + \mathbf{K}'_k)(\mathbf{u}\cdot\mathbf{n}), \bar{q}\right\rangle_{\Gamma_0} \\ & & + \langle C_v p, q\rangle. \end{array}\right. \tag{5.8}$$

The operator $\mathcal{F} = \mathcal{F}(p^i)$ is defined by

$$[\mathcal{F}, (\mathbf{v}, q)]_\epsilon := -\langle p^i, \bar{\mathbf{v}}\cdot\mathbf{n}\rangle_{\Gamma_0} + \left\langle (\frac{1}{2}\mathbf{I} + \mathbf{K}'_k)\frac{\partial p^i}{\partial n}, \bar{q}\right\rangle_{\Gamma_0}.$$

We now state and prove the following uniqueness theorem for the variational problem (5.6), (equivalently (5.7)).

THEOREM 5.1. *Let the material properties* λ, μ *and the curves* Γ_0 *and* Γ_ϵ *be such that there are no traction-free solutions of (2.5). Suppose in addition that* k *is not an exceptional value for the integral equation. Then the nonlocal variational problem has at most one solution* $(\mathbf{u}_\epsilon, p_\epsilon) \in \vec{\mathcal{H}}_\epsilon$.

Proof. It suffices to show that if $(\mathbf{u}_0, p_0)$ is a solution of the homogenous problem for (5.7) (equivalently 5.6), then $(\mathbf{u}_0, p_0) \equiv (\mathbf{0}, 0)$. Indeed, $\mathbf{u}_0, p_0$ are the weak solutions of (IE) and (IR), where

$$(IE) \begin{cases} \triangle^* \mathbf{u}_0 + \rho\omega^2 \mathbf{u}_0 &= \mathbf{0}, \quad x \in \Omega_\epsilon, \\ \mathbf{u}_0 &= 0, \quad x \in \Gamma_\epsilon, \\ \mathbf{T}(\mathbf{u}) &= -p_0^+ \mathbf{n}, \quad x \in \Gamma_0, \end{cases}$$

$$(IR) \qquad \langle \mathbf{W} p_0, \overline{q} \rangle + \langle (\frac{1}{2}\mathbf{I} + \mathbf{K}_k')(\mathbf{u}_0 \cdot \mathbf{n}), \overline{q} \rangle = 0, \qquad \forall q \in H^{1/2}(\Gamma_0).$$

Now let $p_e \in H^1_{loc}(\Omega_\infty)$ be a solution of

$$(5.9) \qquad (EH) \begin{cases} \triangle p_e + k^2 p_e &= \mathbf{0}, \quad x \in \Omega_\infty, \\ p_e^+ &= \mathbf{u}_0 \cdot \mathbf{n}, \quad x \in \Gamma_0, \\ \frac{\partial p_e^-}{\partial n} - ikp_e &= o(\frac{1}{\sqrt{r}}), \quad r = |x| \to \infty. \end{cases}$$

It can easily be shown that p_e exists, and is unique. Moreover, it has the representation formula

$$p_e(x) = \int_{\Gamma_0} \left[\frac{\partial \gamma(x,y)}{\partial n} p_0 - \gamma(x,y) \mathbf{u}_0 \cdot \mathbf{n} \right] ds_y, \qquad x \in \Omega'_\infty,$$

where $\gamma(x, y)$ is the fundamental solution for the Helmholtz equation. By taking normal derivative of p_ϵ in this formula, and then the limit as $x \to \Gamma_0$ from inside Ω_∞, we get the following integral equation (in weak form):

$$(IR) \qquad \langle \mathbf{W}_k p_e, \overline{q} \rangle + \langle (\frac{1}{2}\mathbf{I} + \mathbf{K}_k')(\mathbf{u}_0 \cdot \mathbf{n}), \overline{q} \rangle = 0, \qquad \forall q \in H^{1/2}(\Gamma_0).$$

Subtracting this integral equation from (IR), we obtain

$$\langle \mathbf{W}_k (p_e - p_0), \overline{q} \rangle_{\Gamma_0} = 0, \qquad \forall q \in H^{1/2}(\Gamma_0).$$

This implies that $\mathbf{W}_k(p_e - p_0) = 0$ on Γ_0. Since k is not an exceptional value for the Helmholtz problem, we use the isomorphism properties of $\mathbf{W}_k$ to conclude $p_e = p_0$ on Γ_0. That is, $\frac{\partial p_e^+}{\partial n} = \mathbf{u}_0 \cdot \mathbf{n}$, and $p_e^+ = p_0^+$ (more particularly, $p_e^+ \mathbf{n} = p_0 \mathbf{n}$) on Γ_0. Thus, $(\mathbf{u}_0, p_e)$ satisfies the classical transmission problem (2.4) with homogenous data. By assumption, this problem has only a unique solution, and thus $\mathbf{u}_0 \equiv \mathbf{0}$. Further, $p_e \equiv 0$, and therefore $pe^+ = p_0 = 0$ on Γ_0, completing the proof. □

We would like to be able to show that $\mathcal{A}_\epsilon$ is invertible, and that the norm of its inverse does not depend on ϵ.

THEOREM 5.2. *Assuming that* k *is not an exceptional value, the sesquilinear form* $\mathbf{A}_\epsilon$ *satisfies an inequality of the form*

$$(5.10) \qquad Re(\mathbf{A}_\epsilon(\mathbf{u}, p), (\mathbf{u}, p, \sigma)) \geq \alpha \|(\mathbf{u}, p)\|^2_{\vec{\mathcal{H}}_\epsilon}$$

for all $(\mathbf{u}, p) \in \vec{\mathcal{H}}_\epsilon$*. Here,* $\alpha > 0$ *is a constant independent of* $(\mathbf{u}, p) \in \vec{\mathcal{H}}_\epsilon$.

Proof. We begin by setting $(\mathbf{u}, p) = (\mathbf{v}, q)$ in (5.8) and then estimating term-by-term. Therefore, using the properties of the Lamé constants and Korn's inequality, we get that for any $(\mathbf{u}, p) \in \vec{\mathcal{H}}_\epsilon$,

$$\begin{aligned} Re\,[\mathcal{A}_\epsilon(\mathbf{u}, p), (\mathbf{u}, p)]_\epsilon &= a_\epsilon(\mathbf{u}, \mathbf{u}) - \rho\omega^2(\mathbf{u}, \bar{\mathbf{u}})_{0,\Omega_\epsilon} + \langle \mathbf{W}_0 p, p\rangle_{\Gamma_0} \\ &\geq a_\epsilon(\mathbf{u}, \mathbf{u}) - \rho\omega^2(\mathbf{u}, \bar{\mathbf{u}})_{0,\Omega_\epsilon} + \alpha \|p\|^2_{H^{1/2}(\Gamma_0)}. \end{aligned}$$

We have made use of the $H^{1/2}(\Gamma_0)$-ellipticity of $\mathbf{W}_0$. The L^2-norm of $\mathbf{u}$ can be estimated as

$$\|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)} \leq c_o \epsilon^2 \sum_{i=1}^{2} \|\operatorname{grad} u_i\|^2_{L^2(\Omega_\epsilon)} \leq c_o \epsilon^2 \operatorname{grad} \mathbf{u} : \operatorname{grad} \mathbf{u},$$

by using Poincaré's inequality on each component. In addition, we easily see that

$$\|\mathbf{u}\|^2_{\vec{H}_{\Gamma_\epsilon}(\Omega_\epsilon)} \leq (1 + c_0\epsilon^2) \operatorname{grad} \mathbf{u} : \operatorname{grad} \mathbf{u}.$$

Moreover, from the discussion on Korn's inequality in thin domains presented in other work, (19), we have that

$$a_\epsilon(\mathbf{u}, \mathbf{u}) = \int_{\Omega_\epsilon} \underline{\underline{\sigma}} : \underline{\underline{\mathbf{e}}}\; dV \geq c = \int_{\Omega} \underline{\underline{\mathbf{e}}} : \underline{\underline{\mathbf{e}}}\; dV \geq c_k \|\mathbf{u}\|^2_{\vec{H}^1_{\Gamma_\epsilon}(\Omega)},$$

where the constant $c_k = O(1)$.

Thus, the real part of $[\mathcal{A}_\epsilon(\mathbf{u}, p), (\mathbf{u}, p)]_\epsilon$ satisfies the inequality

$$c_k \|\mathbf{u}\|^2_{\vec{H}^1(\Omega_\epsilon)} - \epsilon^2(\rho\omega^2) \|\operatorname{grad} \mathbf{u}\|^2_{L^2(\Omega_\epsilon)} + c_w \|p\|^2 \leq Re\,[\mathcal{A}_\epsilon(\mathbf{u}, p), (\mathbf{u}, p)]_\epsilon\,. \tag{5.11}$$

We can now assert that

$$Re\,[\mathcal{A}_\epsilon(\mathbf{u}, p), (\mathbf{u}, p)]_\epsilon \geq C \|(u, p)\|^2_{\mathcal{H}_\epsilon},$$

for all suitably small ϵ.□

The next step in the existence proof is to show the compactness of $\mathcal{K}_0$. This is the subject of the next result, which follows from the compactness of $\mathbf{K}'_k$ and C_v on $H^{-1/2}(\Gamma_0)$ to $H^{1/2}(\Gamma_0)$.

THEOREM 5.3. *The sesquilinear operator* $\mathcal{K}_0$ *is a compact sesquilinear form on* $\vec{\mathcal{H}}_\epsilon \times \vec{\mathcal{H}}_\epsilon$.

The uniqueness result for solutions to the variational equation of the fluid-structure interaction problem, and Theorems 5.2 and 5.3 guarantee the existence of the solution through the Fredholm theory.

We are now able to state one of the central results of this paper, which will later help us justify the asymptotics. We need a strong technical assumption (5.13) for the result to hold; in future work we show that this assumption can be removed. However, the resultant analysis is more complicated.

THEOREM 5.4. *Suppose* k *is not an exceptional value, and that there are no Jones modes for the problem. Then there exists an* $\tilde{\epsilon}_0$, $0 < \tilde{\epsilon}_0 \ll 1$, *such that for all* $\epsilon \in (0, \tilde{\epsilon}_0)$, *the inequalities*

$$c\|(u, \sigma)\|^2_{\vec{\mathcal{H}}_\epsilon} \leq \|\,[(\mathcal{A}_\epsilon + \mathcal{K}_0)(u, \sigma), (u, \sigma)]_\epsilon\,\|_{\vec{\mathcal{H}}_\epsilon} \leq C\|(u, \sigma)\|^2_{\vec{\mathcal{H}}_\epsilon} \tag{5.12}$$

hold for all $(u,\sigma)\in\vec{\mathcal{H}}_\epsilon$. *Further, the constants* c, C *are independent of* $\epsilon\in(0,\tilde{\epsilon}_0)$ *under the assumption that the boundary energy, defined by*

$$\rho\omega^2\left(Re\,\langle \mathbf{S_k}\mathbf{u}\cdot\mathbf{n},\bar{\mathbf{u}}\cdot\mathbf{n},\bar{u}\rangle+\frac{k_0^2\beta}{\epsilon}\|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}\right),\quad \forall u\in\mathcal{H}_\epsilon, \tag{5.13}$$

is non-negative, for some finite constant $\beta>0$ *independant of* ϵ. *Here,* $\mathbf{S_k}: H^{-1/2}(\Gamma_0)\to H^{1/2}(\Gamma_0)$ *is the Poincaré-Stekhlov mapping defined by*

$$\mathbf{S_k}\mu := -\mathbf{W}_k^{-1}(\frac{1}{2}\mathbf{I}+\mathbf{K}_k')\mu,\qquad \forall\mu\in H^{-1/2}(\Gamma_0). \tag{5.14}$$

The mapping $\mathbf{S_k}$ is also called a Neumann-Dirichlet map, and is well-defined since k is not an exceptional value. We note the analogy between the non-negativity assumption on the boundary energy for this problem and the model problem.

Proof. The definition of the Poincaré -Stekhlov map allows us to decouple the variational formulations for the displacement $\mathbf{u}_\epsilon$ and the trace of the pressure p_ϵ. Indeed, by noting that

$$\mathbf{W}_k p_\epsilon = -\left(\frac{1}{2}\mathbf{I}+\mathbf{K}_k'\right)\left(\omega^2\rho\mathbf{u}\cdot\mathbf{n}-\frac{\partial p^i}{\partial n}\right)\Rightarrow p=\mathbf{S_k}\left(\omega^2\rho\mathbf{u}\cdot\mathbf{n}-\frac{\partial p^i}{\partial n}\right),$$

the solution $\mathbf{u}$ in Ω_ϵ is completely determined by the variational problem: *For given* ℓ_1, *find* $\mathbf{u}_\epsilon\in\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)$ *such that for all* $\mathbf{v}\in\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)$,

$$\begin{aligned}a_\epsilon(\mathbf{u},\mathbf{v})-\rho\omega^2(\mathbf{u},\bar{\mathbf{v}})_{0,\Omega_\epsilon}+\rho\omega^2\langle\mathbf{S_k}\mathbf{u}\cdot\mathbf{n},\bar{\mathbf{v}}\cdot\mathbf{n}\rangle &= -\langle p^i,\bar{\mathbf{v}}\cdot\mathbf{n}\rangle+\langle\mathbf{S_k}\frac{\partial p^i}{\partial n},\bar{\mathbf{v}}\cdot\mathbf{n}\rangle\\ &=\ell_1(\mathbf{v}).\end{aligned}$$

By assumption, since $0\le\rho\omega^2(Re\rho\omega^2\langle\mathbf{S_k}(\mathbf{u}\cdot\mathbf{n}),\bar{\mathbf{v}}\cdot\mathbf{n}\rangle+\frac{k_0^2\beta}{\epsilon}\|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)})$,

$$a_\epsilon(\mathbf{u},\mathbf{u})-\rho\omega^2(\epsilon+\frac{k_0^2\beta}{\epsilon})\|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}\le|\ell_1(\mathbf{u})|.$$

Poincaré's inequality over the thin region now plays a critical role again, since $\|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}\le c_o\epsilon^2\|\nabla\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}$. This allows us to estimate

$$\rho\omega^2\left(1+\frac{k_0^2\beta}{\epsilon}\right)\|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}\le c_0\epsilon\rho\omega^2(\epsilon+\beta k_0^2)\|\operatorname{grad}\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}.$$

Putting these inequalities together, we obtain that

$$\begin{aligned}|\ell_1(\mathbf{u})| &\ge a_\epsilon(\mathbf{u},\mathbf{u})-\rho\omega^2(\mathbf{u},\bar{\mathbf{v}})_{0,\Omega_\epsilon}+\rho\omega^2 Re\langle\mathbf{S_k}\mathbf{u}\cdot\mathbf{n},\bar{\mathbf{v}}\cdot\mathbf{n}\rangle\\ &\ge a_\epsilon(\mathbf{u},\mathbf{u})-(\rho\omega^2+\frac{k_0^2\beta}{\epsilon})\|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}\\ &\ge a_\epsilon(\mathbf{u},\mathbf{u})-c_0\epsilon(\epsilon+\beta k_0^2)\|\operatorname{grad}\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}\\ &\ge a_\epsilon(\mathbf{u},\mathbf{u})-c_0\epsilon(\epsilon+\beta k_0^2)(\|\operatorname{grad}\mathbf{u}\|^2_{L^2(\Omega_\epsilon)}+\|\mathbf{u}\|^2_{L^2(\Omega_\epsilon)})\\ &\ge a_\epsilon(\mathbf{u},\mathbf{u})-c'\epsilon^2\|\mathbf{u}\|^2_{\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)},\end{aligned}$$

where $c'>0$ is a positive constant independant of $\epsilon>0$. Using Korn's inequality next, we see that

$$a_\epsilon(\mathbf{u},\mathbf{u})\ge\alpha c_k\|\mathbf{u}\|^2_{\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)},$$

where c_k is $O(1)$ and depends on the geometry. Therefore

$$|\ell_1(\mathbf{u})| \geq \left(c_k - c'\epsilon^2\right) \|\mathbf{u}\|^2_{\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)} \geq \frac{c_k}{2}\|\mathbf{u}\|^2_{\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)},$$

for all $0 \leq \epsilon \leq \tilde{\epsilon}$, $\tilde{\epsilon}$ small enough. In particular, in the context of our fluid-structure problem this means that for ϵ small enough, there is a constant c_1, depending on c_k, c' and $\tilde{\epsilon}$, *but independant of* ϵ, such that

$$c_1\|\mathbf{u}\|^2_{\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)} \leq \|\left[(\mathcal{A}_\epsilon + \mathcal{K}_0)(\mathbf{u},p),(\mathbf{u},p)\right]\|_{\vec{\mathcal{H}}_\epsilon}, \qquad \forall (\mathbf{u},p) \in \vec{\mathcal{H}}_\epsilon. \tag{5.15}$$

To estimate p, we see that the isomorphism property of $\mathbf{W}_k$ implies

$$c_w\|p\|_{H^{1/2}(\Gamma_0)} \leq \|\mathbf{W}_k p\|_{H^{-1/2}(\Gamma_0)}, \qquad \forall p \in H^{1/2}(\Gamma_0). \tag{5.16}$$

The constant c_w is independant of ϵ, since the integral operator is defined only over a non-moving boundary. Further,

$$\|\mathbf{W}_k p\|_{H^{-1/2}(\Gamma_0)} \leq \|\mathbf{W}_k p + (\frac{1}{2}\mathbf{I} + \mathbf{K}'_k)(\mathbf{u}\cdot\mathbf{n})\|_{H^{-1/2}(\Gamma_0)} + c_K\|\mathbf{u}\|_{\vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon)}.$$

The constant c_K is again independant of ϵ. These two inequalities show that there is some constant, $c_3 > 0$, such that

$$c_3\|p\|_{H^{1/2}(\Gamma_0)} \leq \|(\mathcal{A}_\epsilon + \mathcal{K}_0)(\mathbf{u},p)\|_{\vec{\mathcal{H}}_\epsilon}, \qquad \forall (\mathbf{u},p) \in \vec{\mathcal{H}}_\epsilon. \tag{5.17}$$

As a consequence of (5.15) and (5.17), we have the desired estimate

$$c\|(\mathbf{u},p)\|^2_{\vec{\mathcal{H}}_\epsilon} \leq \|\left[(\mathcal{A}_\epsilon + \mathcal{K}_0)(\mathbf{u},p),(\mathbf{u},p)\right]_\epsilon\|_{\vec{\mathcal{H}}_\epsilon}, \qquad \forall (\mathbf{u},p) \in \vec{\mathcal{H}}_\epsilon.$$

□

6. Formal Asymptotic Scheme. In order to make the dependence on ϵ explicit, we first consider the weak formulation (5.6) in polar coordinates (r,θ), and then rewrite the corresponding variational formulation in the scaled coordinates (t,θ). Finally, we will introduce the formal asymptotic scheme.

Since we are dealing with elastodynamic displacements in the interior domain, the formulation of the problem in the scaled coordinates becomes rather unwieldy. However, the essential features of the formulation are still akin to those which were studied in Section 3. The leading-order behaviour is given by derivatives in the radial direction, and the zeroeth order solution in the interior is zero. The sequence of problems involves solving for the displacement $\mathbf{u}_i$ in the scaled region Ω, and then determining the associated p_i by solving an integral equation.

6.1. Weak formulation for the asymptotic expansion. Recall that the variational equations for the fluid-structure interaction problem (5.6) are of the form:

$$\left\{\begin{array}{rcl} a_\epsilon(\mathbf{u}_\epsilon,\mathbf{v}) - \rho\omega^2(\mathbf{u}_\epsilon,\mathbf{v})_\epsilon + \langle p_\epsilon, \bar{v}\cdot\mathbf{n}\rangle & = & -\langle p^i, \bar{v}\cdot\mathbf{n}\rangle, \\ 2\langle \mathbf{W}_k p_\epsilon, \overline{q}\rangle_{\Gamma_0} + \rho\omega^2\langle(\mathbf{I}+2\mathbf{K}'_k)\mathbf{u}_\epsilon\cdot\mathbf{n}, \overline{q}\rangle_{\Gamma_0} & = & \langle(\mathbf{I}+2\mathbf{K}'_k)\frac{\partial p^i}{\partial n}, \overline{q}\rangle_{\Gamma_0}. \end{array}\right. \tag{6.1}$$

In polar coordinates, we express $\mathbf{u} = \mathbf{u}(r,\theta) = u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta$. Here $\hat{\mathbf{e}}_r$ is the unit tangent to the coordinate curves r=constant, and $\hat{\mathbf{e}}_\theta$ is the unit tangent to the coordinate curve θ=constant. In terms of u_r and u_θ, the strain tensor is defined by

$$\underset{=}{\mathbf{e}}(\mathbf{u}) := \begin{pmatrix} e_{rr} & e_{r\theta} \\ e_{r\theta} & e_{\theta\theta} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2\frac{\partial u_r}{\partial r} & \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\ \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} & 2(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}) \end{pmatrix}. \tag{6.2}$$

In the scaled variables, the radial derivative $\frac{\partial}{\partial r} = \frac{1}{\epsilon}\frac{\partial}{\partial t}$, but the derivatives with respect to θ remain unchanged. Thus, we can rewrite $\underline{\underline{\mathbf{e}}}(\mathbf{u})$ as

$$\underline{\underline{\mathbf{e}}}(\mathbf{u}) = \frac{1}{2}\begin{pmatrix} \frac{2}{\epsilon}\frac{\partial u_t}{\partial t} & \frac{1}{\epsilon}\frac{\partial u_\theta}{\partial t} - \frac{u_\theta}{(R+\epsilon(t-1))} + \frac{1}{R+\epsilon(t-1)}\frac{\partial u_t}{\partial \theta} \\ \frac{1}{\epsilon}\frac{\partial u_\theta}{\partial t} - \frac{u_\theta}{R+\epsilon(t-1)} + \frac{1}{R+\epsilon(t-1)}\frac{\partial u_t}{\partial \theta} & \frac{2}{R+\epsilon(t-1)}\frac{\partial u_\theta}{\partial \theta} + \frac{2u_t}{R+\epsilon(t-1)} \end{pmatrix}. \tag{6.3}$$

With this expression for the strain tensor, we can rewrite the weak formulation as,

$$\begin{cases} \frac{1}{\epsilon}a_0(\mathbf{u}_\epsilon,\mathbf{v}) + a_1(\mathbf{u}_\epsilon,\mathbf{v}) + \langle p_\epsilon, \bar{\mathbf{v}}\cdot n\rangle + \epsilon a_2(\mathbf{u}_\epsilon,\mathbf{v}) + & \\ \quad +\ldots + \epsilon^{n-1}a_n(\mathbf{u}_\epsilon,\mathbf{v}) + \mathcal{Q}_n(\mathbf{u}_\epsilon,\mathbf{v}) = \langle p^i, \bar{\mathbf{v}}\cdot\mathbf{n}\rangle, & \\ 2\langle \mathbf{W}p_\epsilon, \overline{q}\rangle_{\Gamma_0} + \rho\omega^2\langle(\mathbf{I}+2\mathbf{K}')\mathbf{u}_\epsilon\cdot\mathbf{n}, \overline{q}\rangle_{\Gamma_0} = \langle(\mathbf{I}+2\mathbf{K}')\frac{\partial p^i}{\partial n}, \overline{q}\rangle_{\Gamma_0}. & \end{cases} \tag{6.4}$$

Here

$$a_0(\mathbf{u},\mathbf{v}) := \int_0^{2\pi}\int_0^1 \left\{(\lambda+2\mu)\frac{\partial u_t}{\partial t}\frac{\partial \bar{v}_t}{\partial t} + \mu\frac{\partial u_\theta}{\partial t}\frac{\partial \bar{v}_\theta}{\partial t}\right\} R\,dt\,d\theta, \tag{6.5}$$

$$\begin{aligned} a_1(\mathbf{u},\mathbf{v}) := &\int_0^{2\pi}\int_0^1 \left\{(\lambda+2\mu)\frac{\partial u_t}{\partial t}\frac{\partial \bar{v}_t}{\partial t} + \mu\frac{\partial u_\theta}{\partial t}\frac{\partial \bar{v}_\theta}{\partial t}\right\}(t-1)\,dt\,d\theta \\ &+ \int_0^{2\pi}\int_0^1 \lambda\left(\bar{v}_t\frac{\partial u_t}{\partial t} + u_t\frac{\partial \bar{v}_t}{\partial t} + \frac{\partial u_t}{\partial t}\frac{\partial \bar{v}_\theta}{\partial \theta} + \frac{\partial \bar{v}_t}{\partial t}\frac{\partial u_\theta}{\partial \theta}\right) dt\,d\theta \\ &+ \int_0^{2\pi}\int_0^1 \mu\left(\frac{\partial u_t}{\partial \theta}\frac{\partial \bar{v}_\theta}{\partial t} + \frac{\partial \bar{v}_t}{\partial \theta}\frac{\partial u_\theta}{\partial t} - u_\theta\frac{\partial \bar{v}_\theta}{\partial t} - \bar{v}_\theta\frac{\partial u_\theta}{\partial t}\right) dt\,d\theta, \end{aligned} \tag{6.6}$$

$$a_2(\mathbf{u},\mathbf{v}) := -\rho\omega^2\int_0^{2\pi}\int_0^1 \mathbf{u}\cdot\bar{\mathbf{v}}R dt d\theta + \int_0^{2\pi}\int_0^1 \frac{1}{R}Z(\mathbf{u},\mathbf{v})\,dt\,d\theta, \tag{6.7}$$

$$\begin{aligned} a_3(\mathbf{u},\mathbf{v}) := &-\rho\omega^2\int_0^{2\pi}\int_0^1 \mathbf{u}\cdot\bar{\mathbf{v}}(t-1)\,dt\,d\theta \\ &+ \int_0^{2\pi}\int_0^1 \frac{1}{R}\frac{\epsilon(1-t)}{R}Z(\mathbf{u},\mathbf{v})\,dt\,d\theta, \end{aligned} \tag{6.8}$$

$$a_n(\mathbf{u},\mathbf{v}) := \int_0^{2\pi}\int_0^1 \frac{1}{R}\left(\frac{(1-t)}{R}\right)^{n-2} Z(\mathbf{u},\mathbf{v})\,dt\,d\theta, \qquad n\geq 3.. \tag{6.9}$$

In the definitions above, the bilinear form Z contains combinations of radial and angular derivatives, and is defined by

$$\begin{aligned} Z(\mathbf{u},\mathbf{v}) := &(\lambda+2\mu)\frac{u_\theta}{\partial\theta}\frac{\partial \bar{v}_\theta}{\partial \theta} + \lambda\left(u_r\bar{v}_r + u_r\frac{\partial \bar{v}_\theta}{\partial \theta} + \bar{v}_t\frac{\partial u_\theta}{\partial \theta}\right) \\ &+\mu\left(u_\theta\bar{v}_\theta + \frac{\partial u_t}{\partial \theta}\frac{\partial \bar{v}_t}{\partial t} - u_\theta\frac{\partial \bar{v}_t}{\partial \theta} - \bar{v}_\theta\frac{\partial u_t}{\partial \theta}\right). \end{aligned}$$

The remainder term from the expansion of the bilinear form is given by

$$\begin{aligned} \mathcal{Q}_n(\mathbf{u},\mathbf{v}) &:= a_\epsilon(\mathbf{u},\mathbf{v}) - \rho\omega^2(\mathbf{u},\mathbf{v})_\epsilon - \sum_{k=0}^{n}\epsilon^{k-1}a_k(\mathbf{u},\mathbf{v}) \\ &= \int_0^{2\pi}\int_0^1 \left[\frac{\epsilon}{R+\epsilon(t-1)} - \sum_{k=0}^{n-2}\frac{\epsilon^{k+1}}{R}\left(\frac{(1-t)}{R}\right)^k\right] Z(\mathbf{u},\mathbf{v})\,dt\,d\theta \\ &= \epsilon^n\int_0^{2\pi}\int_0^1 \frac{(1-t)^{n-1}}{R+\epsilon(t-1)}Z(\mathbf{u},\mathbf{v})\,dt\,d\theta. \end{aligned} \tag{6.10}$$

from which we see that $\mathcal{Q}_n(\mathbf{u},\mathbf{v}) = O(\epsilon^n)$ for fixed $(\mathbf{u},\mathbf{v}) \in \vec{\mathcal{H}}_0$. We therefore write $\mathcal{Q}_n(\mathbf{u},\mathbf{v}) = \epsilon^n Q_n(\mathbf{u},\mathbf{v})$, where Q_n is the bilinear term defined by the integral in (6.10).

The variational equations over Γ_0 remain unchanged, as this boundary is not affected by varying ϵ.

The leading order term, $a_0(\mathbf{u},\mathbf{v})$, contains only the t derivatives of $\mathbf{u},\mathbf{v}$, and we expect the behaviour of the solutions in Ω_ϵ to resemble, at least to leading order, the solution of the Laplacian. We do not expect a boundary layer in the problem for this reason, and shall now replace $\mathbf{u}_\epsilon$ in (6.4) by a regular asymptotic series to obtain a sequence of coupled problems, as before.

6.2. Sequence of coupled problems. As mentioned previously, even though the scaled variational equations for the fluid-structure interaction is much more complicated in appearance, the leading order behaviour of the elastic displacements in Ω_ϵ is still governed by the radial derivatives, as long as the incident pressure field is sufficiently smooth. We therefore use the formal expansions

$$\mathbf{u}_\epsilon(t,\theta;\epsilon) = \sum_{j=0}^{n} \epsilon^j \mathbf{u}_j(t,\theta) + \mathbf{R}_n(t,\theta), \qquad x \in \Omega, \tag{6.11}$$

$$p_\epsilon(\theta;\epsilon) = \sum_{j=0}^{n} \epsilon^j p_j(\theta) + S_n(\theta), \qquad x \in \Gamma_0, \tag{6.12}$$

where $\mathbf{u}_i, p_i$ are 2π-periodic in the θ variable, and $\Omega := \{(t,\theta) | t \in (0,1),\ \theta \in [0,2\pi]\}$. Substituting (6.11), (6.12) in (6.4), and collecting like powers of ϵ, we get the following sequence of problems: *Find* $(\mathbf{u}_j, p_j) \in \vec{\mathcal{H}}_0$ *such that for all* $(\mathbf{v}, q, \chi) \in \vec{\mathcal{H}}_0$,

$$a_0(\mathbf{u}_j,\mathbf{v}) = F_j(\mathbf{u}_0,\mathbf{u}_1,\ldots\mathbf{u}_{j-1};p_0,p_1,\ldots p_{j-1};\mathbf{v}), \qquad \forall \mathbf{v} \in \mathcal{H}_0. \tag{6.13a}$$

and

$$2\langle \mathbf{W}p_j, \overline{q}\rangle_{\Gamma_0} = G_j(\mathbf{u}_0,\mathbf{u}_1,\ldots\mathbf{u}_{j-1},\mathbf{u}_j;p_0,p_1,\ldots p_{j-1};q). \tag{6.13b}$$

where we use the trace of $\mathbf{u}_j$ in the normal direction from (6.13a) as the natural boundary data for the variational problem (6.13b). Here, the linear functionals F_j, G_j are given by

$$\begin{aligned} F_0(\mathbf{v}) &:= 0 \\ G_0(\mathbf{u}_0, q) &:= \left\langle (\mathbf{I}+2\mathbf{K}')(\tfrac{\partial p^i}{\partial n} - \rho\omega^2 \mathbf{u}_0 \cdot \mathbf{n}), \bar{q} \right\rangle_{\Gamma_0}, \\ F_1(\mathbf{u}_0, p_0; \mathbf{v}) &:= -a_1(\mathbf{u}_0,\mathbf{v}) - \langle (p^i + p_0), \bar{\mathbf{v}} \cdot \mathbf{n}\rangle_{\Gamma_0}, \\ G_1(\mathbf{u}_0,\mathbf{u}_1;p_0;q) &:= -\rho\omega^2 \langle \mathbf{u}_1 \cdot \mathbf{n}, \overline{q}\rangle_{\Gamma_0}, \end{aligned}$$

and for general $j \geq 2$,

$$\begin{aligned} F_j(\mathbf{u}_0,\mathbf{u}_1,\ldots\mathbf{u}_{j-1};p_0,p_1,\ldots p_{j-1};\mathbf{v}) &:= -\sum_{l=1}^{j-1} a_l(\mathbf{u}_{j-l},\mathbf{v}) - \langle p_j, \bar{\mathbf{v}}\cdot\mathbf{n}\rangle_{\Gamma_0}, \\ G_j(\mathbf{u}_0,\mathbf{u}_1,\ldots\mathbf{u}_j;p_0,p_1,\ldots p_{j-1};q) &:= -\rho_f\omega^2 \langle (\mathbf{I}+2\mathbf{K}')\mathbf{u}_j\cdot\mathbf{n}, \bar{q}\rangle_{\Gamma_0}. \end{aligned}$$

6.3. Uniqueness results. In order to demonstrate the unique solvability of our formal asymptotic scheme, we shall now present a uniqueness theorem, and then derive a representation formula for the solution in the thin domain. At each order in ϵ, our scheme requires us to solve the problem: *Find* $(\mathbf{u}_j, p_j) \in \vec{\mathcal{H}}_0$ *such that*

$$\left\{ \begin{aligned} a_0(\mathbf{u}_j,\mathbf{v}) &= F_j(\mathbf{u}_0,\mathbf{u}_1,\ldots\mathbf{u}_{j-1};p_0,p_1,\ldots p_{j-1};\mathbf{v}) \\ 2\langle \mathbf{W}p_j,\overline{q}\rangle_{\Gamma_0} &= G_j(\mathbf{u}_0,\mathbf{u}_1,\ldots\mathbf{u}_{j-1},\mathbf{u}_j;p_0,p_1,\ldots p_{j-1};q) \end{aligned} \right. \tag{6.14}$$

for $(\mathbf{v}, q) \in \vec{\mathcal{H}}_0$. We need to establish existence and uniqueness results for these problems. We have the following two theorems in this regard:

THEOREM 6.1. *Under the assumptions of Theorem* (5.1)*, there exists at most one solution* $(\mathbf{u}_j, p_j) \in \vec{\mathcal{H}}_0$ *of* (6.14).

Proof. Suppose $(\mathbf{u}_j^1, p_j^1)$ and $(\mathbf{u}_j^2, p_j^2)$ both satisfy (6.14). Denote the difference $\mathbf{u}_j^1 - \mathbf{u}_j^2$ by $\mathbf{u}$. Then if $\mathbf{u}$ solves the homogenous problem

$$a_0(\mathbf{u}, \mathbf{v}) = \int_0^{2\pi} \int_0^1 \left[(\lambda + 2\mu) \frac{\partial u_t}{\partial t} \frac{\partial \bar{v}_t}{\partial t} + \mu \frac{\partial u_\theta}{\partial t} \frac{\partial \bar{v}_\theta}{\partial t} \right] \, dt \, d\theta = 0, \qquad \forall \mathbf{v} \in \mathcal{H}_0,$$

then $\frac{\partial u_t}{\partial t} \equiv 0$, $\frac{\partial u_\theta}{\partial t} \equiv 0$ in Ω. This implies that $u_t = \Psi_1(\theta), u_\theta = \Psi_2(\theta)$ where Ψ_1, Ψ_2 are functions of θ alone. However, since we are seeking solutions in $\mathcal{H}_0$, $\mathbf{u}$ must have zero trace on the curve $t = 0$, which implies that $\Psi_1, \Psi_2 = 0$ and hence $\mathbf{u} \equiv 0$ in Ω. Similarly, the difference $p := p_j^1 - p_j^2$ solves the variational problem

$$\begin{aligned} 2\langle \mathbf{W} p, \overline{q} \rangle = {} & G_j(\mathbf{u}_0, \mathbf{u}_1, \ldots \mathbf{u}_j^1; p_0, p_1, \ldots p_{j-1}; q) \\ & - G_j(\mathbf{u}_0, \mathbf{u}_1, \ldots \mathbf{u}_j^2; p_0, p_1, \ldots p_{j-1}; q) \end{aligned} \tag{6.15}$$

for all $(q, \chi) \in H^1(\Omega_R) \times H^{-1/2}(\Sigma)$. We have just shown, however, that $\mathbf{u}_j^1 = \mathbf{u}_j^2$ in Ω_ϵ, and consequently have the same trace on Γ_0. Hence the right hand side of (6.15) vanishes. Thus, $\langle \mathbf{W} p, \overline{q} \rangle = 0, \quad \forall q \in H^{1/2}(\Gamma_0)$. Using the isomorphism result for $\mathbf{W}$, we obtain that $p \equiv 0$. This proves the assertion.□

We also have the following representation theorem, analogous to those presented in the context of previous work ((38),(18)).

THEOREM 6.2. *Let* $F_1, F_2, P_1, P_2, Q_1, Q_2, R$ *be scalar functions such that for* $i = 1, 2$,

1. $F_i \in H^1(\Omega)$*, and for fixed* $t \in (0,1)$, $F_i(\cdot, t)$, $\frac{\partial F_i}{\partial \theta}$, $\frac{\partial^2 F_i}{\partial \theta^2} \in L^2(\Gamma_0)$.
2. $P_i \in H^1(\Omega)$ *and* $\frac{\partial^2 P_i}{\partial t^2} \in L^2(0,1)$ *for all fixed* s.
3. $R \in H^{1/2}(\Gamma_0)$.
4. F_i, P_i, R *satisfy the periodicity requirement in the* θ *variable.*

Then the variational equation

$$\begin{aligned} & \int_0^{2\pi} \int_0^1 \left[(\lambda + 2\mu) \frac{\partial u_1}{\partial t} \frac{\bar{v}_1}{\partial t} + \mu \frac{\partial u_2}{\partial t} \frac{\partial \bar{v}_2}{\partial t} \right] \, dt \, d\theta \\ & = - \sum_{i=1}^{2} \int_0^{2\pi} \int_0^1 \left[\frac{\partial F_i}{\partial \theta} \frac{\bar{v}_i}{\partial \theta} + (t-1) \frac{\partial P_i}{\partial t} \frac{\partial \bar{v}_i}{\partial t} + Q_i \bar{v}_i \right] \, dt \, d\theta + \int_0^{2\pi} R v_2 \, d\theta, \end{aligned} \tag{6.16}$$

for all $\mathbf{v} = v_1 \hat{\mathbf{e}}_t + v_2 \hat{\mathbf{e}}_\theta \in \mathcal{H}_0$ *determines a unique* $\mathbf{u} = u_1 \hat{\mathbf{e}}_t + u_2 \hat{\mathbf{e}}_\theta \in \mathcal{H}_0$*, given explicitly by*

$$\begin{aligned} u_t(\theta, t) = {} & t \, R(\theta) \\ & - \int_0^1 \left\{ (\tau - 1) \frac{\partial P_1}{\partial \tau}(\theta, \tau) + \int_\tau^1 \left[\frac{\partial^2 F_1}{\partial \theta^2} - Q_1(\theta, y) \right] \, dy \right\} \, d\tau \end{aligned} \tag{6.17a}$$

and

$$u_\theta(\theta, t) = \int_0^t \int_\tau^1 \left[\frac{\partial^2 F_2}{\partial \theta^2} - Q_2(\theta, y) \right] dy \, d\tau - \int_0^1 (\tau - 1) \frac{\partial P_2}{\partial \tau}(\theta, \tau) \, d\tau. \tag{6.17b}$$

Proof. By choosing test functions $v_2 \equiv 0$, the variational equation (6.16) reduces to a variational equation for u_t, which is the same as in the representation theorem for the acoustic scattering problem presented in (38), with F, p, q, r being replaced by F_1, P_1, Q_1 and R. Using the representation derived there gives (6.17a).

Similarly, by choosing $v_1 \equiv 0$ we get a variational equation for u_θ, and using the same technique as above with $r = 0$ gives us the resultant expression (6.17b). □

Theorem 6.2 provides the existence result for the solution $\mathbf{u}_j$ in our formal asymptotic expansion in Ω_ϵ. In this case, since $\mathbf{u}_j$ is a vector-valued function, it amounts to solving two non-homogenous ordinary differential equations, whose solutions are found explicitly from the representation formula. Following the uniqueness result of Theorem 6.1, this will be the solution of the variational problem in Ω_ϵ. We note again the extra regularity required for F_i and P_i in the θ and t vairables, respectively. Once $\mathbf{u}_j$ is obtained, the existence of the solution p_j is guaranteed by standard existence results for boundary integral equations.

7. Justification of formal asymptotic procedure. In this section, we shall justify the use of the asymptotics developed in the previous section. We begin by determining the variational problem satisfied by the remainder terms $(\mathbf{R}_n, S_n)$, and then use the estimates derived in Theorem (5.4) to show that

$$\|(\mathbf{R}_n, S_n)\| \leq O(\epsilon^m)$$

in some appropriate norm, where m increases monotonically with n, and $m > 0$ for all $n > N_0$. For this problem, the scaled differential operator contains leading order terms of $O(\epsilon^{-2})$, and therefore $m = n - 2,\ n \geq 2$. This means that we need two terms in the asymptotic expansion of $\mathbf{u}, p$ to capture the angular behaviour of the solutions. This "lag' in the asymptotics does not present any problems since we already have a precise representation formula for the $\mathbf{u}_j$.

7.1. Variational formulation for remainder terms. Using the weak formulation (5.6) for the non-local boundary value problem (4.4), and the scaled problems developed in the previous section, we are able to derive a system for the remainder terms $(\mathbf{R}_n, S_n)$:

$$\begin{aligned}
&a_\epsilon(\mathbf{R}_n, \mathbf{v}) - \rho\omega^2(\mathbf{R}_n, \bar{\mathbf{v}})_{0,\Omega_\epsilon} + \langle S_n, \bar{\mathbf{v}}\cdot\mathbf{n}\rangle_{\Gamma_0} \\
&= a_\epsilon\left(\mathbf{u}_\epsilon - \sum_{j=0}^{n}\epsilon^j\mathbf{u}_j, \mathbf{v}\right) - \rho\omega^2\left(\mathbf{u}_\epsilon - \sum_{j=0}^{n}\epsilon^j\mathbf{u}_j, \mathbf{v}\right)_{0,\Omega_\epsilon} + \left\langle p_\epsilon - \sum_{j=0}^{n}\epsilon^j p_j\right\rangle \\
&= -\epsilon^n\left[\sum_{j=1}^{n} a_j(\mathbf{u}_{n+1-j}, bv) + \epsilon\sum_{j=2}^{n} a_j(\mathbf{u}_{n+2-j}, bv) + \ldots + \epsilon^{n-1}a_n(\mathbf{u}_n, \mathbf{v})\right] \\
&\quad -\epsilon^n Q_n(\sum_{j=0}^{n}\epsilon^j\mathbf{u}_j, bv) + \epsilon^n\langle p_n, \bar{\mathbf{v}}\cdot\mathbf{n}\rangle_{\Gamma_0}.
\end{aligned}$$

The other equation is much simpler:

$$\langle \mathbf{W}S_n, \bar{q}\rangle_{\Gamma_0} + \langle(\frac{1}{2}\mathbf{I} + \mathbf{K}')(\mathbf{R}_n\cdot\mathbf{n}, \bar{q}\rangle_{\Gamma_0} = 0.$$

Combining these results, we see that the remainder $(\mathbf{R}_n, S_n)$ satisfies the operator equation

$$\begin{aligned}
[(\mathcal{A}_\epsilon + \mathcal{K}_0)(\mathbf{R}_n, S_n), (\mathbf{v}, q)]_\epsilon &= [\mathcal{F}_n, (\mathbf{v}, q)]_\epsilon \\
&:= -\epsilon^n\left(Q_n(\sum_{j=0}^{n}\epsilon^j\mathbf{u}_j, bv) - \langle p_n, \bar{\mathbf{v}}\cdot\mathbf{n}\rangle_{\Gamma_0}\right).
\end{aligned}$$

It is not difficult to see that

$$\|\mathcal{F}_n\|_{\vec{\mathcal{H}}_\epsilon} = O(\epsilon^{n-1}), \tag{7.1}$$

while Theorem (5.4) says that

$$c_1\|(\mathbf{R}_n, S_n)\|_{\vec{\mathcal{H}}_\epsilon} \leq \|(\mathcal{A}_\epsilon + \mathcal{K}_0)(\mathbf{R}_n, S_n)\|_{\vec{\mathcal{H}}_\epsilon}.$$

The norm equivalence relation between $\mathcal{H}_\epsilon$ and $\vec{\mathcal{H}}_\epsilon$ is

$$c\epsilon\|(\mathbf{R}_n, S_n)\|_{\mathcal{H}_\epsilon} \leq \|(bR_n, S_n)\|_{\vec{\mathcal{H}}_\epsilon} \leq \frac{c}{\epsilon}\|(\mathbf{R}_n, S_n)\|_{\mathcal{H}_\epsilon}$$

for positive ϵ, and therefore

$$\|(\mathbf{R}_n, S_n)\|_{\mathcal{H}_\epsilon} \leq \frac{1}{c\epsilon}\|(bR_n, S_n)\|_{\vec{\mathcal{H}}_\epsilon} \leq \frac{1}{c_1 c\epsilon}\|(\mathcal{A}_\epsilon + \mathcal{K}_0)(\mathbf{R}_n, S_n)\|_{\vec{\mathcal{H}}_\epsilon} \leq C'_n(\epsilon^{n-2})$$

where C'_n depends on the $(\mathbf{u}_0, \mathbf{u}_1, ...\mathbf{u}_{n-1}), (p_0, p_1, ...p_{n-1})$. We have thus proved

THEOREM 7.1. *Under the hypotheses of theorem (5.4), and for small enough $\epsilon > 0$, the following estimates hold:*

$$\left\|(\mathbf{u}_\epsilon, p_\epsilon) - \left(\sum_{j=0}^{n} \epsilon^j \mathbf{u}_j, \sum_{j=0}^{n} \epsilon^j p_j\right)\right\|_{\mathcal{H}_\epsilon} = O(\epsilon^{n-2}), \qquad as\ \epsilon \to 0^+ \tag{7.2}$$

where the terms $\{(\mathbf{u}_j, p_j)\}$ are constructed by the formal asymptotic procedure.

References.

[1] G. BATCHELOR, *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967.

[2] A. BENDALI AND K. LEMRABET, *The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation*, SIAM. J. Appl. Math., 56 (1996), pp. 1664–1693.

[3] F. BREZZI AND C. JOHNSON, *On the coupling of boundary integral and finite element methods*, Calcolo, 16 (1979), pp. 189–201.

[4] A. J. BURTON AND G. F. MILLER, *The application of integral equation methods to the numerical solution of some exterior boundary-value problems*, Proc. Roy. Soc. London. Ser. A, 323 (1971), pp. 201–210. A discussion on numerical analysis of partial differential equations (1970).

[5] G. CHEN AND J. ZHOU, *Boundary Element Methods*, Computational Mathemtics and Applications, Academic Press, 1992.

[6] M. COSTABEL, *Boundary integral operators on lipschitz domains: Elementary results*, SIAM J. Math. Anal., 19 (1988), pp. 613–626.

[7] R. DAUTRAY AND J. LIONS, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 4, Springer-Verlag,Berlin, 1990.

[8] ——, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 1, Springer-Verlag,Berlin, 1990.

[9] ——, *Mathematical Analysis and Numerical Methods for Science and Technology*, vol. 2, Springer-Verlag,Berlin, 1990.

[10] G. DUVAUT AND J.-L. LIONS, *Inequalities in Mechanics and Physics*, vol. 219 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Heidelberg, 1 ed., 1976.

[11] G. N. GATICA AND G. C. HSIAO, *Boundary-field Equation methods for a Class of Non-linear Problems*, Pitman Research Notes in Mathematics Series, 1995.

[12] T. HARGÉ, *Valeurs propres d'un corps élastique*, C.R.Acad.Sci. Paris, (1990), pp. 857–859.

[13] G. HSIAO AND W. WENDLAND, *A finite element method for some integral equations of the first kind*, vol. 58, J. Math. Anal. Appl. (197), pp. 533–547.

[14] G. HSIAO, *Integral representations of solutions for two-dimensional viscous flow problems*, Integral Equations and Operator Theory, 5 (1982), pp. 533–547.

[15] G. HSIAO, *The coupling of bem and fem – a brief review*, in Mathematical and Computational Aspects, A. Brebbia, ed., vol. 1 of Boundary Elements X, Springer-Verlag, 1988, pp. 431–441.

[16] ——, *Coupling of be and fe methods*, in Plenary Lectures, vol. 70 of ZAMM, Akademie-Verlag Berlin, 1990, pp. 493–503.

[17] G. HSIAO, *On the boundary-field equation methods for fluid-structure interactions*, in Problems and Methods in Mathematical Physics, L. Jentsch and F. Troltzsch, eds., vol. 34 of Teubner-Text zur Mathematik, B.G.Teubner, Stuttgart, 1994.

[18] G. HSIAO AND N. NIGAM, *A transmission problem in the exterior of a thin domain*, Quaderni di Mathematica, (2000).

[19] G. HSIAO, N. NIGAM, AND A. -M. SANDIG, *A transmission problem in elasticity*. in preparation.

[20] G. HSIAO AND N. NIGAM, *A fluid-structure interaction problem exterior to a thin domain using an artificial boundary.* in preparation.

[21] G. HSIAO AND W. WENDLAND, *Variational methods for boundary integral equations*. (in preparation).

[22] G. HSIAO AND S. ZHANG, *Optimal order multigrid methods for solving exterior boundary value problems*, SIAM J.Numer.Anal., 31 (1994), pp. 680–694.

[23] G. C. HSIAO, R. E. KLEINMAN, AND G. F. ROACH, *Weak solutions of fluid-solid interaction problems*, Math. Nachr., 218 (2000), pp. 139–163.

[24] P. JAERISCH, *Vibrations in a sphere*, J.F.Math (Crelle), 88 (1880), pp. 131–145.

[25] C. JOHNSON AND J. NEDELEC, *On the coupling of boundary integral and finite element methods*, Mathematics of Computation, 35 (1980), pp. 1063–1079.

[26] D. JONES, *Low-frequency scattering by a body in lubricated contact*, Quart.J.Mech.Appl.Math, 36 (1983), pp. 111–138.

[27] ——, *A uniqueness theorem in elastodynamics*, J. Mech. Appl. Math., 37 (1984), pp. 121–142.

[28] R. E. KLEINMAN AND G. F. ROACH, *On modified Green functions in exterior problems for the Helmholtz equation*, Proc. Roy. Soc. London Ser. A, 383 (1982), pp. 313–332.

[29] R. KRESS AND G. ROACH, *Transmission problems for the helmholtz equation*, J. Math.Phy., 19 (1978), pp. 1433–1437.

[30] V. KUPRADZE, *Dynamical Problems in Elasticity*, vol. 3 of Progress in Solid Mechanics, North-Holland Publishing Company, 1 ed., 1963.

[31] ——, *Potential Methods in the Theory of Elasticity*, Israel Program for Scientific Translations, 1 ed., 1965.

[32] ——, *Three Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity.*, North-Holland Publishing Com., 1979.

[33] H. LAMB, *On the vibrations of an elastic sphere*, London Math.Soc.Proc, 13 (1882), pp. 189–212.

[34] L. LANDAU AND E. LIFSHITZ, *Theory of Elasticity*, vol. 7 of Course of Theoretical

Physics, Pergamon Press, 3rd ed., 1986.
[35] R. LEIS, *Inital Boundary Value Problems in Mathematical Physics*, B.G.Teubner, Stuttgart; John Wiley and Sons, 1986.
[36] C. LUKE AND P. MARTIN, *Fluid-solid interaction: acoustic scattering by a smooth elastic object*, SIAM J. Appl. Math., 55 (1995), pp. 904–922.
[37] N. NIGAM, *Variational methods for a class of boundary value problems exterior to a thin domain*, PhD thesis, U. Delaware, May 1999.
[38] N. NIGAM, *A variational method in acoustics related to an impenetrable scatterer coated by a thin penetrable shell*, in Mathematical and numerical aspects of wave propagation (Santiago de Compostela, 2000), SIAM, Philadelphia, PA, 2000, pp. 90–95.
[39] A. QUARTERONI AND A. VALLI, *Numerical Appproximation of Partial Differential Equations*, no. 23 in Springer Series in Computational Mathematics, Springer, 2 ed., 1997.
[40] R. RAND, *Torsional vibrations of elastic prolate spheroids*, J. Acoust.Soc.Amer., 44 (1968), pp. 749–751.

Appendix: Norm equivalences. We collect some well-known, but necessary, results concerning the various norms we have employed in this article. In the variational formulation of the non-local boundary value problems, we employed the function space

$$\vec{\mathcal{H}}_\epsilon := \left\{(\mathbf{u},\sigma)|\mathbf{u}\in \vec{H}^1_{\Gamma_\epsilon}(\Omega_\epsilon),\, \sigma\in H^{-1/2}(\Gamma_0)\right\}.$$

The norm on this space depends on ϵ, and we would like to establish the relationship between this norm, and the norm defined by $\vec{\mathcal{H}}_0$. The following theorem establishes the equivalence relationship between the energy norms in the scaled and unscaled regions for scalar valued functions, ie, for

$$\mathcal{H}_\epsilon := \left\{(u,\sigma)|u\in H^1_{\Gamma_\epsilon}(\Omega_\epsilon),\, \sigma\in H^{-1/2}(\Gamma_0)\right\}.$$

from which the desired equivalence result follows in a straightforward fashion.

THEOREM 7.2. *There exists C_1, C_2 independent of ϵ such that for all $0<\epsilon\le \frac{1}{2min(\kappa)}, \epsilon << 1$,*

$$C_1\epsilon\|u\|_{H^1(\Omega)} \le \|u\|_{H^1(\Omega_\epsilon)} \le C_2\frac{1}{\epsilon}\|u\|_{H^1(\Omega)}. \tag{7.3}$$

Proof. We begin with the definition of the energy norm in $H^1(\Omega_\epsilon)$, and use the scaling results derived in the previous section:

$$\begin{aligned} \|u\|_{H^1(\Omega_\epsilon)} &= \left\{\int_{\Omega_\epsilon}|u|^2+|\nabla u|^2dx\right\}^{1/2} \\ &= \left\{\int_0^L\int_0^1\left[|u|^2+\tfrac{1}{\epsilon}|u_t|^2+\tfrac{1}{(R+\epsilon t-\epsilon)^2}|u_s|^2\right]\epsilon(1+\epsilon(t-1)\kappa)dtds\right\}^{1/2} \\ &\ge \left\{\tfrac{1}{\epsilon}\int_0^L\int_0^1\left[\epsilon^2|u|^2+\epsilon^2|u_t|^2+\tfrac{\epsilon^2}{(1+\epsilon(t-1)\kappa)^2}|u_\theta|^2\right](1+\epsilon(t-1)\kappa)dtd\theta\right\}. \end{aligned}$$

Now, using the fact that $0<\epsilon<<1$, we get the estimate

$$\|u\|_{H^1(\Omega_\epsilon)} \ge \left\{\epsilon^2\int_0^L\int_0^1\left[|u|^2+\frac{1}{(R+1)^2}|u_\theta|^2+|u_t|^2\right](1-\epsilon\kappa)dtds\right\}^{1/2},$$

and finally

$$\|u\|_{H^1(\Omega_\epsilon)} \geq \left(\sqrt{\frac{C}{2}}\right) \epsilon \|u\|_{H^1(\Omega)}$$

where $C(R) = \min\{1, \frac{1}{(R+1)^2}\}$. For the other inequality in the equivalence relation, we again begin with the definition, and

$$\|u\|_{H^1(\Omega_\epsilon)} = \left\{ \frac{1}{\epsilon} \int_0^L \int_0^1 \epsilon^2 (1+\epsilon(t-1)\kappa)^2 |u|^2 + (1+\epsilon(t-1)\kappa)|u_t|^2 \right.$$
$$\left. + \frac{\epsilon^2}{(1+\epsilon(t-1)\kappa))} |u_s|^2 dtds \right\}^{1/2}.$$

Clearly, $\frac{1}{\epsilon} < \frac{1}{\epsilon^2}$ if $0 < \epsilon < 1$, whence

$$\|u\|_{H^1(\Omega_\epsilon)} \leq \frac{1}{\epsilon} \left\{ \int_0^L \int_0^1 (2)|u|^2 + (2)|u_t|^2 + \frac{2}{(1)} |u_s|^2 dtds \right\}^{1/2}$$
$$\leq \frac{1}{\epsilon} \sqrt{2} \|u\|_{H^1(\Omega)}.$$

So,

$$\|u\|_{H^1(\Omega_\epsilon)} \leq \left(\sqrt{2}\right) \frac{1}{\epsilon} \|u\|_{H^1_{t=o}(\Omega)}.$$

Putting everything together, we have

$$\left(\sqrt{\frac{C}{2}}\right)^2 \epsilon \|u\|_{H^1_{t=0}(((0,1)\times(0,L))} \leq \|u\|_{H^1(\Omega_\epsilon)} \leq \left(\sqrt{2}\right) \frac{1}{\epsilon} \|u\|_{H^1_{t=o}((0,1)\times(0,L))}.$$

□

THEOREM 7.3. *The seminorm* $\|(u,0)\|_{\mathcal{H}_\epsilon}$ *and* $\|u\|_{H^1_0(\Omega)}$ *are equivalent.*

Proof. This follows directly from Poincaré's inequality. Therefore, we get the result: *there exists constants* C_1, C_2*, independent of* $\epsilon > 0$*, depending only on* R*, such that*

$$C1\epsilon \|(u,\sigma)\|_{\mathcal{H}_0} \leq \|(u,\sigma\|_{H^1_{\Gamma_\epsilon}(\mathcal{H}_\epsilon} \leq \frac{C2}{\epsilon} \|(u,\sigma)\|_{\mathcal{H}_0}. \quad (7.4)$$

□