Solitons and the Korteweg de Vries Equation: Starting with Shallow-Water Waves

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Abstract

Soliton, or solitary wave, solutions to nonlinear equations arise across several areas of physics and applied mathematics. They are distinguished physically as travelling wave solutions that maintain their shape, and mathematically by their symmetries, amenability to closed-form solutions (integrability), and their stability to perturbations. In this paper, we derive the Korteweg de Vries equation for shallow gravity waves in water, demonstrate a few of the soliton solutions it admits and their properties, and discuss interactions between these nonlinear waves and others of their characteristics.

1 Introduction

The Korteweg de Vries equation, which we derive below, describes the propagation of nonlinear, shallow-water waves in a dispersive medium. A special class of solutions, called “solitons,” or solitary waves, for their particle-like persistence and localization, have remarkable mathematical and stability properties. These solutions, which arose in the context of hydrodynamics, have been extended to a huge variety of other situations, even those involving other nonlinear dispersive equations that admit traveling wave solutions. In this paper, we review the basic mathematical properties of solitary waves and then discuss their unreasonable effectiveness as the basis for a host of nonlinear models across the physical and biological sciences and engineering.

2 Derivation of the KdV equation in Hydrodynamics

Many references on the Korteweg de Vries (KdV) equations present the equations a priori as PDEs to be analyzed mathematically. The rich mathematical structure of these equations encourages this approach, which we will touch on later. But to show the physical use of these equations and the approximations they arise from, we take the approach of Dauxois and develop the equations from an analysis of nonlinear shallow surface water waves, including gravity and surface tension.

We consider the flow of an ideal irrotational fluid with mean depth $h$, and take the flow to be two-dimensional, so that $u_y = 0$ and all fluid properties are independent of the $y$ direction: this two dimensional formulation is, for example, a good approximation for relatively small-amplitude water waves with large spatial extent, which most deep water waves are. The free surface is given by $z = h + \eta(x,t)$, at the air-water interface where $p = p_{atm}$, with a gravitational body force. We must thus solve the Euler equations with body force, subject to the irrotational condition and the kinematic condition at the free surface that $D\eta/Dt = w$, where $\vec{v} = (u, 0, w)$ is the velocity field of the fluid. In components, these equations become:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$
$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g$$
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
with kinematic condition
\[ w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x}, z = h + \eta(x,t) \]

and boundary conditions \( w = 0 \) at the bottom of the bed \( z = 0 \), \( p = p_{atm} \) at the free surface \( z = h + \eta \).

We must impose initial conditions as well, and the further integral condition that the average depth of the fluid is \( h \), meaning the \( \eta = 0 \) surface is at \( z = h \) on average: \( \lim_{r \to \infty} \int_{-r}^{r} \eta(x,t) \text{d}x = 0 \).

To simplify these equations, we can separate out the static component (\( \bar{\nu} = 0 \) solution of the Euler equations) of the pressure and nondimensionalize the equations.

If we take out the static pressure \( p_0 = -g\delta(z-h) + p_{atm} \) and rewrite the Euler equations for just the “dynamic pressure,” the \(-g\) term in the \( z \) component equation drops out, and we’re left with the boundary condition \( p' = \rho g \eta(x,t) \) at \( z = h + \eta(x,t) \).

We take the standard step of non-dimensionalizing the equations. As usual, this at once makes them mathematically more tractable, physically interpretable, and makes asymptotic and scaling analysis to develop the appropriate approximations work out in terms of fundamental length and time scales that we can easily tune to the appropriate regime of interest. We nondimensionalize \( t \to t/\ell_0, x, z \to x/L, z/L \), and \( \eta \to \eta/A \). Note that there are different characteristic parameters for the spatial coordinates and the coordinate measuring the oscillation of the free surface: though both are distances, for spatial coordinates we’re interested in some fundamental size parameter for the overall domain, and for the oscillations it’s the amplitude \( A \) that’s of interest.

With these transformed parameters, we can rescale the velocities and pressure—note that since it’s the free surface motion that’s of interest, we use the \( A \) length scale to do this:

\[ u \to u/(A/\ell_0), w \to w/(A/\ell_0), p \to p/(\rho AL/t_0^2) \]

(The dynamic pressure \( p = p' \), allowing abuse of notation, should involve both the oscillations and the system size). If we write the equations in the new coordinates using the Froude number \( F = \frac{\ell_0 \sqrt{\rho g}}{L} \) and dimensionless parameters that represent the relative amplitude magnitude and depth of the water, \( \epsilon = \frac{A}{L} \) and \( \delta = \frac{h}{L} \), then the equations are:

\[ \frac{\partial u}{\partial t} + \epsilon \left( \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} \]

\[ \frac{\partial w}{\partial t} + \epsilon \left( u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} \]

for the Euler equations, the same condition for incompressibility, \( w = 0 \) at \( z = 0 \), and, at \( z = \delta + \epsilon \eta \), the kinematic and dynamic conditions

\[ w = \frac{\partial \eta}{\partial t} + \epsilon u \frac{\partial \eta}{\partial x} \]

\[ F \eta = p \]

If we assume that the flow is irrotational too, so that \( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 \), we can define a velocity potential \( \phi \) (this is standard). These two conditions give an alternate expression for the kinematic boundary condition, incorporating the dynamic one too:

\[ \frac{\partial \nu}{\partial t} + \epsilon \left( \frac{\partial \nu}{\partial x} + \frac{\partial w}{\partial z} \right) + F \frac{\partial \eta}{\partial x} = 0 \]

Note that in these approximations, \( \epsilon \) gives us a measure of nonlinearity, and \( \delta \) in the appropriate limit gives us the limit of shallow water waves. We have to decide the relative size of \( \epsilon \) and \( \delta \); we assume \( \epsilon \sim \delta^2 \). We’ll take \( \epsilon << 1 \), but not zero, for weak nonlinearity, and \( \delta << 1 \), which gives us that the depth is small compared to the extent of the waves. This is the crucial approximation step, so we will emphasize it.

The Korteweg de Vries equations are recovered as the governing equations for hydrodynamic waves in the limit of weak nonlinearity and shallow waves, that is, the height of the waves is vanishing relative to their spatial extent.
To make physical sense of this approximation, another note is warranted: shallow water is about relative scale, not absolute. Thus, it turns out that soliton solutions of the KdV equations describe tsunamis, which most of us would think of as huge waves, quite well. This works out because tsunamis, while in the deep ocean before they break, are less than a meter tall, and even though they’re in ocean water of a depth up to 4 km, they are often hundreds of kilometers wide.

Carrying forward our approximations, we can measure \( \eta \), the displacement of the water, in units of \( \delta \), \( \varphi = \eta/\delta \), and we assume \( \varphi = O(1) \), which satisfies weak nonlinearity.

Now we finally seek a solution of our non-dimensionalized equations in the appropriate approximation limit. We have Laplace’s equation \( \nabla^2 \phi = 0 \) for the velocity potential by incompressibility; since our assumptions give \( z = \delta \), that is, small \( z \), we can do a power series solution of the equation, \( \phi = \sum z^n \phi_n \). We get a recurrence

\[
\phi_{n+2} = -\frac{1}{(n+2)(n+1)} \frac{\partial^2 \phi_n}{\partial x^2}
\]

for the \( \phi_n \) by taking the Laplacian of the power series for \( \phi \). Since the velocity vanishes on \( z = 0 \), we get \( \phi_1 = 0 \), and thus by recurrence, all \( \phi_{2k+1} = 0 \). Thus

\[
\phi(x, z, t) = \phi_0(x, t) - \frac{1}{2} z^2 \frac{\partial^2 \phi_0}{\partial x^2} + \frac{1}{4!} \frac{\partial^4 \phi_0}{\partial x^4} - ...
\]

and we can differentiate to get \( u \) and \( w \). Defining \( \frac{\partial \phi}{\partial x} = f(x, t) \), and choosing to do the perturbative calculation up to order \( \delta^3 \), we get

\[
\begin{align*}
  u(x, z, t) & = f(x, t) - \frac{1}{2} z^2 f_{xx} \\
  w(x, z, t) & = -zf_x + \frac{1}{6} z^3 f_{xxx}
\end{align*}
\]

Since this calculation is getting tedious and the physical situation and approximations are already clear, we’ll omit details going forward. The approach is to use the above equations in terms of \( f \) for the velocities and the surface kinematic conditions to get a relation between \( f \) and \( \varphi \), and then use and the physical boundary condition derived earlier in terms of \( F \epsilon, \delta \) and \( f \), and take terms to order \( \epsilon \delta \sim \delta^3 \) to get

\[
\begin{align*}
  \varphi_t + f_x + \epsilon \varphi f_x + \epsilon f \varphi_x - \frac{1}{6} \delta^2 f_{xxx} & = 0 \\
  f_t - \frac{1}{2} \delta^2 f_{xxt} + \epsilon f * f_x + \varphi_x & = 0
\end{align*}
\]

The four equations above (with \( u, w \) taken at the surface \( u_{\text{surface}} = f \), \( w(z = \delta(1 + \epsilon \varphi)) \), are the shallow water wave equations. These equations can be solved perturbatively to the same order we’ve been working in, and we end up with the KdV equation

\[
\varphi_t + \varphi_x + \frac{3}{2} \epsilon \varphi \varphi_x + \frac{1}{6} \delta^2 \varphi_{xxx} = 0
\]

We can write the equation in a moving frame with velocity \( 1 \equiv \sqrt{gh} \), moving with the wave, to recover the KdV equation we will use going forward.

3 General Situations Well-Modelled by the KdV equation

The above discussion gives the gritty details for how the KdV equation shakes out of the shallow water wave equations. The interest of the KdV equation is that it comes up naturally in all kinds of models, for reasons we’ll discuss in section 5. Each of the soliton references listed gives numerous examples of applications where the KdV equation comes up. Among them are internal gravity waves in a stratified fluid (highly relevant for geophysics), waves in an astrophysical plasma, electrical transmission line behavior, and even blood pressure waves—soliton solutions to the KdV equation explain why our pulse, coming from a localized
pressure wave in our arteries, is detectable all over our body, and persists despite changes in local conditions and artery geometry in the circulation system. Soliton models persist even outside of macroscopic physics—in fact, solitons were “discovered” in an atomic physics laboratory by Fermi, Pasta and Ulam when they were trying to model the thermalization of a solid: they had to explain why a burst of energy would traverse the possible energy levels and come back to its original state almost exactly, instead of thermalizing as equilibrium statistical mechanics predicts.

4 Soliton Solutions in General

In the remainder of this paper, we will be discussing a remarkable class of solutions to the KdV equation known as “solitons.” Now is the time to prevent myth-making: the KdV equation was derived for shallow water waves in 1895, and “solitons” were not an accepted physical phenomenon until Fermi and Ulam in the 1960s, though observations of one were first recorded in 1834. The KdV equation is not synonymous with the evolution equation of soliton waves. The equation admits other solutions, and there are solitary wave solutions of other equations. Notably, the sine-Gordon equation admits this special kind of solution, and it developed in the wholly different context of an evolution equation for constant negative curvature surfaces even before the KdV equation. Soliton solutions of this equation are now studied in quantum field theory. The Nonlinear Schrodinger Equation, another one that comes up in QFT, also admits soliton solutions, and is important for the very practical problem of modeling many nonlinear phenomena in fiber optics.

So what is a soliton? The name suggested itself because of a soliton’s particle-like properties: it is a localized disturbance, and it propagates while retaining its identity—its shape. On these points most sources that discuss solitons seem to agree. Beyond that, definitions vary, mostly according as whether the audience is mathematicians (where solitons are interesting as examples of integrable systems with infinitely many conserved quantities, amenable to solution by the Inverse Scattering Transform), physicists (where solitons are important as particle models, or theoretical entities postulated by field theorists for years), or fluid mechanicians, who were introduced to solitons by direct observation of these strange waves and attempted to describe what they saw.

It is not so important to prescribe exactly what a soliton is, since the remarkable things about solitons are evident in their properties, which we will discuss. But in short, they are, to a first pass, local, travelling wave solutions to nonlinear equations with excellent stability properties. An additional property, often used to identify solitons, is that, though the interaction of solitons is complex and nonlinear, their behavior after the collision is easy to predict—each solitary wave simply continues on exactly as it was, as if they passed through each other, with just phase shifts to indicate that an interaction took place at all.

From a mathematical perspective, the discovery of solitons was a watershed because it vastly expanded our repertoire of integrable systems, which previously were limited to the Harmonic oscillator, motion in a central force field, and other toy physical models, which limitedness suggested that analytic solutions to real problems are well-nigh impossible. Integrable systems that can be solved exactly, with tight connections to the Lagrangian and Hamiltonian formulations of mechanics and conserved quantities; in light of recent focus on chaos theory, perhaps the most important feature of integrable systems is that their dynamics are the simplest possible in a sense, and they do not exhibit sensitivity to initial conditions. Arnol’d’s mathematical mechanics book gives a good introduction to the fascinating geometry and physical uses of integrable systems, of which solitons are the most prominent infinite degrees-of-freedom example. Newell’s mathematical physics book gives a good phenomenological introduction in the preface and chapter one, and takes the reader to the frontiers of the mathematical theory by the end.

5 Soliton Solutions to the KdV Equation: Nonlinearity-Dispersion Balance and Stability

We will work with the KdV equation in its final non-dimensional form, now considering its properties in the abstract:

\[
\frac{\partial \phi}{\partial \tau} + 6 \phi \frac{\partial \phi}{\partial \xi} + \frac{\partial^3 \phi}{\partial \xi^3} = 0
\]
where $\phi$ is the velocity potential and $\xi, \tau$ are the non-dimensionalized positions and times, respectively.

As discussed, for example, in Acheson, finite amplitude waves tend to steepen: this is the result of nonlinearity, with parts of the wave packet at different amplitudes moving at different speeds. Most real waves are also subject to dispersion, which tends to “smooth out” the envelope of the wave. Soliton solutions of the KdV equation result from an exact balance between these nonlinearity and dispersion effects, as we will discuss.

More important, though, is the stability of the equilibrium reached by soliton solutions. This nonlinearity-dispersion balance persists, and it is insensitive to perturbations. Thus, solitons don’t just give us a fuzzy exact solution to nonlinear wave equations with transient applicability–they can model situations well even when exact soliton solutions do not obtain. Contrast this to the situation when we solve nonlinear equations by linearizing them and seeking perturbation solutions: this method often has limited stability properties, and is very sensitive to the particulars of the perturbative approximation, leaving us to do constant and diligent parameter bookkeeping.

For a discussion of the stability of solitons, see exercise (2d) of Newell. The text describes the Benjamin-Feir, or modulational, instability in nonlinear waves: for surface gravity waves, for example, the wave packet generated by a paddling motion breaks into a well-separated series of soliton waves governed by the Nonlinear Schrodinger Equation. If the paddling motion continues indefinitely, or is periodic the wave packet is instead periodic in time and quasiperiodic in space, and governed by the KdV equation: the result is that the packet breaks into separate soliton pulses that eventually regroup into a pulse resembling the initial condition. Both situations demonstrate the unique stability of solitons: in the case of a general wave packet, where dispersion and nonlinearity are not balanced, resonance will suck energy from the carrier wave into side bands–in one dimension, this process stops when a soliton solution is reached, which persists.

We can quantify this discussion of the nonlinearity and dispersion. In the limit where nonlinearity dominates, it’s the $\phi \frac{\partial \phi}{\partial \xi}$ term that dominates the KdV equation. Thus we have

$$\frac{\partial \phi}{\partial \tau} + \phi \frac{\partial \phi}{\partial \xi} = 0$$

By inspection, we can compare this to the linear wave equation

$$\frac{\partial \phi}{\partial \tau} + v \frac{\partial \phi}{\partial \xi} = 0$$

which has traveling wave solutions $\phi = f(\xi - v \tau)$, and note that $v$ determines the speed of this linear wave. Analogously, the amplitude $\phi$ is the speed of the component of the wave packet with amplitude $\phi$, so that large amplitude components travel fastest. As noted in the discussion in Acheson, and as can be verified numerically, this condition tends to lead to the formation of shock waves.

In the linear limit, the KdV equation becomes

$$\frac{\partial \phi}{\partial \tau} + \frac{\partial^3 \phi}{\partial \xi^3} = 0$$

which has plane wave solutions with dispersion relation $\omega = -k^3$ (which looks nonstandard only because it’s written in the moving frame). This dispersion relation shows that the phase velocity $\omega_k = \frac{k}{k}$ depends on $k$, which means that this is a dispersive medium equation, where the pulses broaden.

The stability of the nonlinearity-dispersion balance in soliton solutions makes them powerful. For example, it means that waves travelling over a rough ocean bottom can still be modelled as solitary waves–which is why we see them so often in the field. But the stability, of course, has its limits: most spectacularly, we see waves breaking on the shore because the perturbation drives constantly towards the domination of nonlinearity as the beach gets shallow: the dispersive term scales as $h^2$ in the dimensional version of the KdV equation, while the nonlinearity scales as $1/h$: as the depth decreases, nonlinearity dominates and the wave breaks.

6 Solving the KdV Equation for Localized Waves: One-Solitons

We will now develop the fundamental soliton solutions of the KdV equation, single solitary waves. (The stable solutions are in general multiple well-separated solitons, which are no longer localized: we won’t treat the general case, though we’ll look at how two one-solitons interact).
We seek a spatially localized solution that propagates at constant speed while maintaining its shape. This requires for $\phi$ the functional form $\phi(\xi, \tau) = \phi(\xi - v\tau)$. Traveling with the wave by taking $z = \xi - v\tau$, we get the governing equation:

$$-v\phi_z + 6\phi\phi_z + \phi_{zzz} = 0$$

Separating out a derivative with respect to $z$ and integrating, we get

$$\phi_{zz} + 3\phi^2 - v\phi + C = 0$$

for some constant $C$. This becomes, multiplying by $\phi_z$ and again integrating,

$$\frac{1}{2}\phi_z^2 + \phi^3 - \frac{1}{2}v\phi^2 = 0$$

where we’ve gotten rid of integration constants because of the condition that $\phi$ is localized, i.e. it decays as $z \to \infty$.

We can explicitly bring in the unity of soliton theory with classical mechanics by analyzing this as the equation of the Hamiltonian for a system of unit mass and position $\phi$ moving in an effective potential $V_{\text{eff}}(\phi) = \phi^3 - \frac{1}{2}v\phi^2$.

If we think about the potential diagram for such a system, we can confirm that there are no (localized) soliton solutions with negative amplitude, which happily agrees with John Scott Russell’s empirical observations in 1834: he saw negative amplitude waves, which are solutions of the KdV equation, but they did not persist like solitons do.

To see this, consider a particle moving in this potential, leaving the position $\phi = 0$ in the negative $\phi$ direction at time $z = -\infty$. Differentiating the potential and taking the negative, we see that the acceleration is in the $-\phi$ direction for $\phi < 0$, giving increasing kinetic energy at all times. Since the $V_{\text{eff}}$ must then decrease for all times $z > -\infty$ to preserve conservation of energy, $\phi$ must diverge, which violates the locality of the solution $\phi$. So there are no solutions with negative amplitude.

What solutions are there? Now that we know $\phi > 0$ always, the math works out nicely. We can do a trigonometric substitution, starting with the differential form of the governing equation

$$dz = \frac{d\phi}{\sqrt{v\phi^2 - 2\phi^3}}$$

and taking $\phi = \left(\frac{u + sech^2 u}{2}\right)$ (which we can do since we know $\phi$ is positive), integrating to get

$$\phi = \frac{v}{2}sech^2\left(\sqrt{\frac{v}{4}}z\right)$$

We have our first explicit localized soliton solutions to the KdV equations!

Now that we know they do exist, let’s briefly consider their form. Notice especially that we must have $v > 0$. Remembering that $v$ is the propagating velocity, we might worry that this breaks the symmetry between the $\pm x$ directions. But we have to remember that we’re not in the lab frame, but in a frame moving at speed $c_0 = \sqrt{gh}$. This simply confirms the experimental observation that all solitary waves move at speeds greater than the speed of long-wavelength linear waves in a channel: $v_{\text{lab-frame}} > \sqrt{gh}$. Transforming this local solution to the lab frame also shows that the difference $v - c_0$ varies with $\eta_0$, the height of the wave above the surface, in accordance with observations.

Aside: General, non-local solutions to the KdV equation can be expressed in terms of elliptic cosine functions, which interpolate between $\cos$ and $sech$. These functions are “sharper” than cosine functions, which accounts for any beachgoer’s observation that ocean waves away from the beach, where nonlinearity is relevant, have a “pointed” shape.

### 7 The Interaction between Two Solitons and its Signature

The shape of the interface when two solitons collide is in general complicated, and cannot be deduced easily from the profiles of the two 1-solitons, as we can do by superposition in a linear theory. Remarkable, then,
that solitons maintain their shape and speed after a “collision”–the interaction is “elastic.” They seem to pass right through each other.

The signature indicating that a collision occurred is a slight phase difference in the wave positions compared to where they would usually be. In the standard 2-soliton solution, the larger-amplitude wave is retarded from where it would be if there had been no interaction, and the smaller one jumps ahead. (Without this signature, we would not have nonlinearity).

In the standard solution, given in Dauxois, a few qualitative features of the collision can be adjudged. The most revealing is that the height of the 2-soliton at the point where the crests of the one-soliton overlap is less than the height of the larger one-soliton: a clear indication that the interaction is nonlinear.

A fuller analysis of the interaction problem can be found in any discussion of multi-soliton solutions and their explicit derivation. I don’t think the mathematics would add intuition here, since I’d be picking a multi-soliton solution out of a hat to analyze anyway–the development of these solutions requires the full force of the general theory.

Instead, we’ll note here the complex behavior of 2-solitons in observations. For one-dimensional soliton solutions, the behavior noted above obtains. It was assumed for a long time that the same interaction features hold for 2D solitons, observed for example in the interaction of ocean waves, especially since ocean waves are generally taken not to vary much by cross section in their dynamics. The following image from Ablowitz (2012) show how the 2D soliton behavior is qualitatively different from predictions when waves meet at an angle. We see that the height of the interaction region is greater than that of the two waves combined. Ablowitz notes that these types of interactions are more frequent than previously suspected, and posits that they could form part of an explanation for the formation of tsunami waves, by the buildup of the amplitude over many such interactions–these solitons are also notable for the persistence of this X-shape as the new soliton moves, instead of the waves passing through each other.

8 Conclusion: Physical Applicability and Current Research

One could delve into the extensive mathematics behind soliton solutions to the KdV and related equations fruitfully: for this Newell is particularly useful. Moving farther from the physical basis of solitons, we note that many soliton solutions were discovered only as a result of deep theorems in classical algebraic geometry. Even if the formal mathematical structure is not of interest, the general equilibrium solution for the travel of a nonlinear dispersive wave packet involves multi-soliton solutions to the governing equations, and so more advanced mathematical techniques must be mastered in order to develop realistic soliton-based models.

More important for us, though, are the physical avenues of inquiry. Besides the traditional area of water waves, solitons have been involved in hypotheses regarding the formation of ball lightning, the propagation of kinks in a solid, and modeling of the stastical mechanics of liquids, ferromagnets, semiconductors, and various phase transitions. If you have ever seen the twisting of a ladder of connected rods in a standing wave pattern, a common sculpture in kids’ science museums, you’ve seen a stationary soliton, which arises sometimes in non-KdV models like the sine-Gordon equations. The kinked shape of snow frozen on a branch can take the profile of a standing-wave soliton solution. There are some who are tackling the extreme nonlinearities and high-dimensionality of the problems of protein folding, DNA transcription, and how energy is transported in biological molecules using methods derived from the study of solitons. Not everything is a soliton–strictly, there are no physical solitons–but the stability properties and fundamental nonlinearity of solitons have made them vital components of a wide range of interesting models, and an active subject of research and point of contact across several fields.
References


