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Hierarchy of Chaotic Maps with an Invariant Measure and their Compositions

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Abstract

We give a hierarchy of many-parameter families of maps of the interval $[0, 1]$ with an invariant measure and using the measure, we calculate Kolmogorov–Sinai entropy of these maps analytically. In contrary to the usual one-dimensional maps these maps do not possess period doubling or period-n-tupling cascade bifurcation to chaos, but they have single fixed point attractor at certain region of parameters space, where they bifurcate directly to chaos without having period-n-tupling scenario exactly at certain values of the parameters.

1 Introduction

In the past twenty years dynamical systems, particularly one dimensional iterative maps have attracted much attention and have become an important area of research activity. One of the landmarks in it was introduction of the concept of Sinai–Ruelle–Bowen (SRB) measure or natural invariant measure [1, 2]. This is, roughly speaking, a measure that is supported on an attractor and also describes statistics of long time behavior of the orbits for almost every initial condition in corresponding basin of attractor. This measure can be obtained by computing the fixed points of the so called Frobenius–Perron (FP) operator which can be viewed as a differential-integral operator.The exact determination of invariant measure of dynamical systems is rather a nontrivial task taking into account that the invariant measure of few dynamical systems such as one-parameter family of one-dimensional piecewise linear maps [3, 4, 5] including Baker and tent maps or unimodal maps such as logistic map for certain values of its parameter can be derived analytically.

In most cases only numerical algorithms, as an example Ulam's method [6, 7, 8], are used for computation of fixed points of FP-operator.

Here in this paper we give a new hierarchy of many-parameter families of maps of the interval $[0,1]$ with an invariant measure. These maps are defined as ratios of polynomials where we have derived analytically their invariant measure for arbitrary values of the parameters. Using this measure, we have calculated analytically Kolomogorov–Sinai (KS) entropy of these maps. It is shown that they possess very peculiar property, that is, contrary to the usual maps, these do not possess period doubling or period-n-tupling cascade bifurcation to chaos, but instead they have single fixed point attractor at certain region of parameters space where they bifurcate directly to chaos without having period-n-tupling scenario.

The paper is organized as follows: In Section 2 we introduce the hierarchy of many-parameter families of chaotic maps. In Section 3 we show that the invariant measure is actually the eigenstate of the FP-operator corresponding to largest eigenvalue 1. Then in section IV using this measure we calculate KS-entropy of these maps for an arbitrary values of the parameters. Paper ends with a brief conclusion.

2 Hierarchy of chaotic maps with an invariant measure and their compositions

Let us first consider the one-parameter families of chaotic maps of the interval $[0,1]$ defined as the ratios of polynomials of degree N:

$$\begin{aligned}\Phi_N(x,\alpha) &= \frac{\alpha^2\left(1+(-1)^N {}_2F_1\left(-N,N,\frac{1}{2},x\right)\right)}{(\alpha^2+1)+(\alpha^2-1)(-1)^N {}_2F_1\left(-N,N,\frac{1}{2},x\right)} \\ &= \frac{\alpha^2(T_N(\sqrt{x}))^2}{1+(\alpha^2-1)(T_N(\sqrt{x})^2)},\end{aligned} \tag{2.1}$$

where N is an integer greater than 1. Also

$${}_2F_1\left(-N,N,\frac{1}{2},x\right) = (-1)^N\cos\left(2N\arccos\sqrt{x}\right) = (-1)^N T_{2N}(\sqrt{x})$$

are hypergeometric polynomials of degree N and $T_N(x)$ are Chebyshev polynomials of type I [9] respectively. Obviously these maps unit interval $[0,1]$ into itself. $\Phi_N(x,\alpha)$ is $(N-1)$-nodal map, that is it has $(N-1)$ critical points in unit interval $[0,1]$, since its derivative is proportional to derivative of hypergeometric polynomial ${}_2F_1\left(-N,N,\frac{1}{2},x\right)$ which is itself a hypergeometric polynomial of degree $(N-1)$, hence it has $(N-1)$ real roots in unit interval $[0,1]$. Defining Shwarzian derivative [10] $S\Phi_N(x,\alpha)$ as:

$$S\left(\Phi_N(x,\alpha)\right) = \frac{\Phi_N'''(x,\alpha)}{\Phi_N'(x,\alpha)} - \frac{3}{2}\left(\frac{\Phi_N''(x,\alpha)}{\Phi_N'(x,\alpha)}\right)^2 = \left(\frac{\Phi_N''(x,\alpha)}{\Phi_N'(x,\alpha)}\right)' - \frac{1}{2}\left(\frac{\Phi_N''(x,\alpha)}{\Phi_N'(x,\alpha)}\right)^2,$$

with a prime denoting differentiation with respect to variable x, one can show that (see Appendix A):

$$S\left(\Phi_N(x,\alpha)\right) = S\left({}_2F_1\left(-N,N,\frac{1}{2},x\right)\right) \le 0.$$

Therefore, the maps $\Phi_N^{\alpha}(x)$ have at most $N+1$ attracting periodic orbits [10]. Using the above hierarchy of family of one-parameter chaotic maps, we can generate new hierarchy of families of many-parameter chaotic maps with an invariant measure simply from the composition of these maps. Hence considering the functions $\Phi_{N_k}(x,\alpha_k)$, $k=1,2,\ldots,n$ we denote their composition by: $\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)$ which can be written in terms of them in the following form:

$$\begin{aligned}\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x) &= \overbrace{(\Phi_{N_1}\circ\Phi_{N_2}\circ\ldots\circ\Phi_{N_n}(x))}^{n} \\ &= \Phi_{N_1}\left(\Phi_{N_2}(\ldots(\Phi_{N_n}(x,\alpha_n),\alpha_{(n-1)})\ldots,\alpha_2),\alpha_1\right). \end{aligned} \tag{2.2}$$

Since these maps consist of composition of (N_k-1)-nodals $(k=1,2,\ldots,n)$ maps with negative Shwarzian derivative, they are $(N_1N_2\cdots N_n-1)$-nodals maps and their Shwarzian derivative is negative, too [10]. Therefore these maps have at most $N_1N_2\cdots N_n+1$ attracting periodic orbits [10]. As it is shown below in this section, these maps have only a single period one stable fixed points. Denoting m-composition of these functions by $\Phi^{(m)}$, it is straightforward to show that the derivative of $\Phi^{(m)}$ at its possible $m\times n$ periodic points of an m-cycle can be defined as: $x_{\mu,k+1}=\Phi_{N_k}(x_{\mu,k},\alpha_k)$, $x_{1,\mu+1}=\Phi_{N_n}(x_{n,\mu},\alpha_N)$, and $x_{1,1}=\Phi_{N_n}(x_{m,n},\alpha_n)$, $\mu=1,2\ldots,m$, $k=1,2,\ldots,n$ is

$$\left|\frac{d}{dx}\Phi^{(m)}\right| = \prod_{\mu=1}^{m}\prod_{k=1}^{n}\left|\frac{N_k}{\alpha_k}\left(\alpha_k^2+\left(1-\alpha_k^2\right)x_{\mu,k}\right)\right|, \tag{2.3}$$

since for $x_{\mu,k}\in[0,1]$ we have:

$$\min\left(\alpha_k^2+\left(1-\alpha_k^2 x_{\mu,k}\right)\right)=\min\left(1,\alpha_k^2\right),$$

therefore:

$$\min\left|\frac{d}{dx}\Phi^{(m)}\right| = \prod_{k=1}^{n}\left(\frac{N_k}{\alpha_k}\min\left(1,\alpha_k^2\right)\right)^m.$$

Hence, the above expression is definitely greater than 1 for $\prod\limits_{k=1}^{n}\frac{1}{N_k} < \prod\limits_{k=1}^{n}\alpha_k < \prod\limits_{k=1}^{n}N_k$, that is, these maps do not have any kind of m-cycle or periodic orbits in the region of the parameters space defined by $\prod\limits_{k=1}^{n}\frac{1}{N_k} < \prod\limits_{k=1}^{n}\alpha_k < \prod\limits_{k=1}^{n}N_k$, actually they are chaotic in this region of the parameters space. From (2.3) it follows that $\left|\frac{d}{dx}\Phi^{(m)}\right|$ at $m\times n$ periodic points of m-cycle belonging to interval $[0,1]$, vary between $\prod\limits_{k=1}^{n}(N_k\alpha_k)^m$ and $\prod\limits_{k=1}^{n}\left(\frac{N_k}{\alpha_k}\right)^m$ for $\prod\limits_{k=1}^{n}\alpha_k < \prod\limits_{k=1}^{n}\frac{1}{N_k}$ and between $\prod\limits_{k=1}^{n}\left(\frac{N_k}{\alpha_k}\right)^m$ and $\prod\limits_{k=1}^{n}(N_k\alpha_k)^m$ for $\prod\limits_{k=1}^{n}\alpha_k > \prod\limits_{k=1}^{n}N_k$, respectively.

Definitely from the definition of these maps, we see that $x=1$ and $x=0$ (in special case of odd integer values of $N-1,N_2,\ldots,N_n$) belong to one of m-cycles.

For $\prod\limits_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}\frac{1}{N_k}$ $\left(\prod\limits_{k=1}^{n}\alpha_k > \prod\limits_{k=1}^{n}N_k\right)$, the formula (2.3) implies that for those cases in which $x=1$ $(x=0)$ belongs to one of m-cycles, we have $\left|\frac{d}{dx}\Phi^{(m)}\right|<1$, hence the curve of $\Phi^{(m)}$ starts at $x = 1$ ($x = 0$) beneath the bisector and then crosses it at the previous (next) periodic point with slope greater than one, since the formula (2.3) implies that the slope of fixed points increases with the decreasing (increasing) of $|x_{\mu,k}|$, therefore at all periodic points of n-cycles except for $x = 1$ ($x = 0$) the slope is greater than one that is they are unstable, this is possible only if $x = 1$ ($x = 0$) is the only period one fixed point of these maps.

Hence, all m-cycles except for possible period one fixed points $x = 1$ and $x = 0$ are unstable.

Actually, the fixed point $x = 0$ is the stable fixed point of these maps in the regions of the parameters spaces defined by $\alpha_k > 0$, $k = 1, 2, \ldots, n$ and $\prod_{k=1}^{n} \alpha_k < \prod_{k=1}^{n} \frac{1}{N_k}$ only for odd integer values of $N_1, N_2, \ldots, N_n$, however, if one of the integers N_k, $k = 1, 2, \ldots, n$ happens to be even, then the $x = 0$ will not be a stable fixed point anymore. But the fixed point $x = 1$ is stable fixed point of these maps in the regions of the parameters spaces defined by $\prod_{k=1}^{n} \alpha_k > \prod_{k=1}^{n} N_k$ and $\alpha_k < \infty$, $k = 1, 2, \ldots, n$ for all integer values of $N_1, N_2, \ldots, N_n$.

As an example we give below some of these maps:

$$\Phi_2^{\alpha}(x) = \frac{\alpha^2(2x-1)^2}{4x(1-x)+\alpha^2(2x-1)^2}, \tag{2.4}$$

$$\Phi_3^{\alpha}(x) = \frac{\alpha^2 x(4x-3)^2}{\alpha^2 x(4x-3)^2+(1-x)(4x-1)^2}, \tag{2.5}$$

$$\Phi_4^{\alpha}(x) = \frac{\alpha^2\left(1-8x(1-x)\right)^2}{\alpha^2(1-8x(1-x))^2+16x(1-x)(1-2x)^2}, \tag{2.6}$$

$$\Phi_5^{\alpha}(x) = \frac{\alpha^2 x\left(16x^2-20x+5\right)^2}{\alpha^2 x(16x^2-20x+5)^2+(1-x)(16x^2-(2x-1))}, \tag{2.7}$$

$$\Phi_{2,2}^{\alpha_1,\alpha_2}(x) = \frac{{\alpha_1}^2\left(4\,x\,(x-1)+(2\,x-1)^2\,{\alpha_2}^2\right)^2}{{\alpha_1}^2\left(4\,x\,(x-1)+(2\,x-1)^2\,{\alpha_2}^2\right)^2-16x\alpha_2^2(2x-1)^2(x-1)}, \tag{2.8}$$

$$\begin{aligned}\Phi_{2,3}^{\alpha_1,\alpha_2}(x) &= {\alpha_1}^2\left((x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_2}^2\right)^2\\ &\times\left({\alpha_1}^2\left((x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_2}^2\right)^2\right.\\ &\left.\quad-4x\alpha_2^2(x-1)(4x-1)^2(4x-3)^2\right)^{-1},\end{aligned} \tag{2.9}$$

$$\begin{aligned}\Phi_{3,2}^{\alpha_1,\alpha_2}(x) &= {\alpha_2}^2\left((x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_1}^2\right)^2\\ &\times\left({\alpha_2}^2\left((x-1)\,(4\,x-1)^2+x\,(4\,x-3)^2\,{\alpha_1}^2\right)^2\right.\\ &\left.\quad+4x\alpha_1^2(x-1)(4x-1)^2(4x-3)^2\right)^{-1},\end{aligned} \tag{2.10}$$

$$\Phi_{3,3}^{\alpha_1,\alpha_2}(x) = {\alpha_1}^2{\alpha_2}^2 x\,(4\,x-3)^2\left(3\,(x-1)\,(4\,x-1)^2 + x\,(4\,x-3)^2\,{\alpha_2}^2\right)$$
$$\times\Big(-(x-1)^3\,(4\,x-1)^6 + 3\,x\left(3\,{\alpha_1}^2-2\right)(4\,x-3)^2$$
$$\times\,(x-1)^2\,(4\,x-1)^4\,{\alpha_2}^2 + h\Big)^{-1}, \tag{2.11}$$

where

$$h = 3x^2\alpha_2^4\left(-3+2\alpha_1^2\right)(x-1)(4x-1)^2(4x-3)^4 + \alpha_1^2\alpha_2^6x^3(4x-3)^6.$$

Below we also introduce their conjugate or isomorphic maps which will be very useful in derivation of their invariant measure and calculation of their KS-entropy in the next section. Conjugacy means that the invertible map $h(x) = \frac{1-x}{x}$ maps $I = [0,1]$ into $[0,\infty)$ and transforms maps $\Phi_{N_k}(x,\alpha_k)$ into $\tilde{\Phi}_{N_k}(x,\alpha_k)$ defined as:

$$\tilde{\Phi}_{N_k}(x,\alpha_k) = h\circ\Phi_{N_k}(x,\alpha_k)\circ h^{(-1)} = \frac{1}{\alpha_k^2}\tan^2\left(N_k\arctan\sqrt{x}\right),$$

Hence, this transforms the maps $\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)$ into $\tilde{\Phi}_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)$ defined as:

$$\tilde{\Phi}_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)$$
$$= \frac{1}{\alpha_1^2}\tan^2\left(N_1\arctan\sqrt{\circ}\frac{1}{\alpha_2^2}\tan^2\left(N_2\arctan\sqrt{\circ}\cdots\circ\ \frac{1}{\alpha_n^2}\tan^2\left(N_n\arctan\sqrt{x}\right)\right)\right)$$
$$= \frac{1}{\alpha_1^2}\tan^2\left(N_1\arctan\sqrt{\frac{1}{\alpha_2^2}\tan^2\left(N_2\arctan\sqrt{\cdots\frac{1}{\alpha_n^2}\tan^2\left(N_n\arctan\sqrt{x}\right)\cdots}\right)}\right). \tag{2.12}$$

3 Invariant measure

Dynamical systems, even apparently simple dynamical systems such as maps of an interval, can display a rich variety of different asymptotic behaviors. On measure theoretical level these types of behavior are described by SRB [1, 11] or invariant measure describing statistically stationary states of the system. The probability measure μ on $[0,1]$ is called an SRB or invariant measure of the maps $\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)$ given in (2.2), if it is $\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)$-invariant and absolutely continuous with respect to Lebesgue measure. For deterministic system such as these composed maps, the $\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)$-invariance means that its invariant measure $\mu_{\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}}(x)$ fulfills the following formal FP-integral equation:

$$\mu_{\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}}(y) = \int_0^1\delta\left(y-\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)\right)\mu_{\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}}(x)dx.$$

This is equivalent to:

$$\mu_{\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}}(y) = \sum_{x\in\ \Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(y)}\mu_{\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}}(x)\frac{dx}{dy}, \tag{3.1}$$

defining the action of standard FP-operator for the map $\Phi(x)$ over a function as:

$$P_{\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}}f(y) = \sum_{x\in\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(y)}f(x)\frac{dx}{dy}. \tag{3.2}$$

We see that, the invariant measure $\mu_{\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}}(x)$ is the eigenstate of the FP-operator $P_{\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}}$ corresponding to the largest eigenvalue 1.

As we will prove below, the measure $\mu_{\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}}(x,\beta)$ defined as:

$$\frac{1}{\pi}\frac{\sqrt{\beta}}{\sqrt{x(1-x)}(\beta+(1-\beta)x)}, \tag{3.3}$$

is the invariant measure of the maps $\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)$ provided that the parameter β is positive and fulfills the following relation:

$$\prod_{k=1}^{n}\alpha_k \times \frac{A_{N_n}\left(\frac{1}{\beta}\right)}{B_{N_n}\left(\frac{1}{\beta}\right)} \times \frac{A_{N_{n-1}}\left(\frac{1}{\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_{n-1}}\left(\frac{1}{\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right)}\right)}$$
$$\times \frac{A_{N_{n-2}}\left(\frac{1}{\eta^{\alpha_{n-1},\alpha_n}_{N_{n-1},N_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_{n-2}}\left(\frac{1}{\eta^{\alpha_{n-1},\alpha_n}_{N_{n-1},N_n}\left(\frac{1}{\beta}\right)}\right)} \times\cdot\times \frac{A_{N_1}\left(\frac{1}{\eta^{\alpha_2,\alpha_3,\ldots,a_n}_{N_2,N_3,\ldots,N_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_1}\left(\frac{1}{\eta^{\alpha_2,\alpha_3,\ldots,\alpha_n}_{N_2,N_3,\ldots,N_n}\left(\frac{1}{\beta}\right)}\right)} = 1, \tag{3.4}$$

where the polynomials $A_{N_k}(x)$ and $B_{N_k}(x)$ $(k = 1, 2, \ldots, n)$ are defined as:

$$A_{N_k}(x) = \sum_{l=0}^{\left[\frac{N_k}{2}\right]} C^{N_k}_{2l} x^l, \tag{3.5}$$

$$B_{N_k}(x) = \sum_{l=0}^{\left[\frac{N_k-1}{2}\right]} C^{N_k}_{2l+1} x^l, \tag{3.6}$$

where the symbol [ ] means greatest integer part. Also the functions $\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right), \eta^{\alpha_{n-1},\alpha_n}_{N_{n-1},N_n}\left(\frac{1}{\beta}\right)$, $\ldots$ and $\eta^{\alpha_2,\alpha_3,\ldots,\alpha_n}_{N_2,N_3,\ldots,N_n}\left(\frac{1}{\beta}\right)$ are defined in the following forms:

$$\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right) = \beta\left(\frac{\alpha_n A_{N_n}\left(\frac{1}{\beta}\right)}{B_{N_n}\left(\frac{1}{\beta}\right)}\right)^2, \quad \eta^{\alpha_{n-1},\alpha_n}_{N_{n-1},N_n}\left(\frac{1}{\beta}\right) = \beta\left(\frac{\alpha_{n-1}A_{N_{n-1}}\left(\frac{1}{\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_{n-1}}\left(\frac{1}{\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right)}\right)}\right)^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\eta^{\alpha_2,\alpha_3,\ldots,\alpha_n}_{N_2,N_3,\ldots,N_n}\left(\frac{1}{\beta}\right) = \beta\left(\frac{\alpha_2 A_{N_2}\left(\frac{1}{\eta^{\alpha_3,\alpha_4,\ldots,\alpha_n}_{N_3,N_4,\ldots,N_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_2}\left(\frac{1}{\eta^{\alpha_3,\alpha_4,\ldots,\alpha_n}_{N_3,N_4,\ldots,N_n}\left(\frac{1}{\beta}\right)}\right)}\right)^2.$$

As we see the above measure is defined only for $\beta > 0$, hence from the relations (3.4), it follows that these maps are chaotic in the region of the parameters space which lead to positive solution of β. Taking the limits of $\beta \longrightarrow 0_+$ and $\beta \longrightarrow \infty$ in the relation (3.4) respectively one can show that the chaotic regions is: $\prod_{k=1}^{n}\frac{1}{N_k} < \prod_{k=1}^{n}\alpha_k < \prod_{k=1}^{n}N_k$ for odd integer values of $N_1, N_2, \ldots, N_n$ and if one of the integers happens to become even, then the chaotic region in the parameter space is defined by $\alpha_k > 0$, for $k = 1, 2, \ldots, n$ and $\prod_{k=1}^{n} \alpha_k < \prod_{k=1}^{n} N_k$ if one of the integers happens to become even, respectively. Out of these regions they have only period one stable fixed points.

In order to prove that measure (3.3) satisfies equation (3.1), with β given by relation (3.4), it is rather convenient to consider the conjugate map $\tilde{\Phi}^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)$, with measure $\tilde{\mu}_{\tilde{\Phi}^{\alpha_1,\alpha_2\ldots,\alpha_n}_{N_1,N_2\ldots,N_n}}$ denoted by $\tilde{\mu}_{\tilde{\Phi}}$ related to the measure $\mu_{\Phi^{\alpha_1,\alpha_2\ldots,\alpha_n}_{N_1,N_2\ldots,N_n}}$ denoted by μ_Φ through the following relation:

$$\mu\tilde{\Phi}(x) = \frac{1}{(1+x)^2}\mu_\Phi\left(\frac{1}{1+x}\right).$$

Denoting $\tilde{\Phi}^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)$ by y and inverting it, we get:

$$\begin{aligned}
x_{k_1} &= \tan^2\left(\frac{1}{N_1}\arctan\sqrt{y\alpha_1^2} + \frac{k_1\pi}{N_1}\right),\\
x_{k_1,k_2} &= \tan^2\left(\frac{1}{N_2}\arctan\sqrt{x_{k_1}\alpha_2^2} + \frac{k_2\pi}{N_2}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{k_1,k_2,\ldots,k_n} &= \tan^2\left(\frac{1}{N_n}\arctan\sqrt{x_{k_1,k_2,\ldots,k_{n-1}}\alpha_n^2} + \frac{k_n\pi}{N_n}\right),
\end{aligned}$$

for $k_j = 1, \ldots, N_j$ and $j = 1, \ldots, n$.

Then by taking the derivative of $x_{k_1,k_2,\ldots,k_n}$ with respect to y, we obtain:

$$\left|\frac{dx_{k1,k_2,\ldots,k_n}}{dy}\right| = \left(\prod_{k=1}^{n}\frac{\alpha_k}{N_k}\right)\sqrt{\frac{x_{k_1,k_2,\ldots,k_n}}{y}}$$
$$\times\frac{(1+x_{k_1,k_2,\ldots,k_n})(1+x_{k_2,k_3,\ldots,k_n})\cdots(1+x_{k_{n-1},k_n})(1+x_{k_n})}{(1+\alpha_n^2 x_{k_2,k_3,\ldots,k_n})(1+\alpha_{n-1}^2 x_{k_3,k_4,\ldots,k_n})\cdots(1+\alpha_3^2 x_{k_{n-1},k_n})(1+\alpha_2^2 x_{k_n})(1+\alpha_1^2 y)} \tag{3.7}$$

to be substituted in equation (3.1). In derivation of above formula we have used chain rule property of the derivative of composite functions.

Substituting the above result in equation (3.1), we have:

$$\tilde{\mu}_{\tilde{\Phi}}(y)\sqrt{y}\left(1+\alpha_1^2 y\right) = \left(\prod_{k=1}^{n}\frac{\alpha_k}{N_k}\right)\sum_{k_1}\sum_{k_2}\cdots\sum_{k_n}\sqrt{x_{k_1,k_2,\ldots,k_n}}$$
$$\times\frac{(1+x_{k_1,k_2,\ldots,k_n})(1+x_{k_2,k_3,\ldots,k_n})\cdots(1+x_{k_{n-1},k_n})(1+x_{k_n})}{(1+\alpha_n^2 x_{k_2,k_3,\ldots,k_n})(1+\alpha_{n-1}^2 x_{k_3,k_4,\ldots,k_n})\cdots(1+\alpha_3^2 x_{k_{n-1},k_n})(1+\alpha_2^2 x_{k_n})}\tilde{\mu}_{\tilde{\Phi}}(x_{k_1,k_2,\ldots,k_n}).$$

Now considering the following equation for the invariant measure $\tilde{\mu}_{\tilde{\Phi}}(y)$:

$$\tilde{\mu}_{\tilde{\Phi}}(y) = \frac{1}{\sqrt{y}(1+\beta y)}, \tag{3.8}$$

then the FP-equation reduced to:

$$\frac{1+\alpha_1^2 y}{1+\beta y} = \left(\prod_{k=1}^{n} \frac{\alpha_k}{N_k}\right)$$
$$\times \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_n=1}^{N_n} \left(\frac{(1+x_{k_1,k_2,\ldots,k_n})(1+x_{k_2,k_3,\ldots,k_n})\cdots(1+x_{k_{n-1},k_n})(1+x_{k_n})}{(1+\alpha_n^2 x_{k_2,k_3,\ldots,k_n})(1+\alpha_{n-1}^2 x_{k_3,k_4,\ldots,k_n})\cdots(1+\alpha_3^2 x_{k_{n-1},k_n})(1+\alpha_2^2 x_{k_n})} \right).$$

Now using the follow identity:

$$\frac{\alpha}{N} \sum_{k=0}^{N} \frac{1+\alpha^2 x_k}{1+\beta x_k} = \frac{1+\alpha^2 y}{\left(\frac{B\left(\frac{1}{\beta}\right)}{\alpha A\left(\frac{1}{\beta}\right)} + \beta \left(\frac{\alpha A\left(\frac{1}{\beta}\right)}{B\left(\frac{1}{\beta}\right)} \right) y \right)}, \tag{3.9}$$

for a one-parameter chaotic map $y = \Phi_N^{\alpha}(x)$ (Its proof is given in Appendix B), we obtain:

$$\frac{\alpha_n}{N_n} \sum_{k_n=1}^{N_n} \frac{1+x_{k_1,k_2,\ldots,k_n}}{1+\beta x_{k_1,k_2,\ldots,k_n}} = \frac{\alpha_n A_{N_n}\left(\frac{1}{\beta}\right)}{B_{N_n}\left(\frac{1}{\beta}\right)} \frac{1+\alpha_n^2 x_{k_1,k_2,\ldots,k_{n-1}}}{1+\eta_{N_n}^{\alpha_n}\left(\frac{1}{\beta}\right) x_{k_1,k_2,\ldots,k_{n-1}}},$$

$$\frac{\alpha_{n-1}\alpha_n}{N_{n-1}N_n} \sum_{k_{n-1}=1}^{N_{n-1}} \sum_{k_n=1}^{N_n} \frac{(1+x_{k_1,k_2,\ldots,k_{n-1}})(1+x_{k_1,k_2,\ldots,k_n})}{(1+\alpha_{n-1} x_{k_1,k_2,\ldots,k_{n-1}})(1+\beta x_{k_1,k_2,\ldots,k_n})}$$
$$= \frac{\alpha_n \alpha_{n-1} A_{N_n}\left(\frac{1}{\beta}\right) A_{N_{n-1}}\left(\frac{1}{\eta_{N_n}^{\alpha_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_n}\left(\frac{1}{\beta}\right) B_{N_{n-1}}\left(\frac{1}{\eta_{N_n}^{\alpha_n}\left(\frac{1}{\beta}\right)}\right)} \times \frac{1+\alpha_{n-1}^2 x_{k_1,k_2,\ldots,k_{n-2}}}{1+\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}\left(\frac{1}{\beta}\right) x_{k_1,k_2,\ldots,k_{n-2}}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(\prod_{k=1}^{n} \frac{\alpha_k}{N_k}\right) \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{n_2} \cdots$$
$$\times \sum_{k_n=1}^{N_n} \left(\frac{(1+x_{k_1,k_2,\ldots,k_n})(1+x_{k_2,k_3,\ldots,k_n})\cdots(1+x_{k_{n-1},k_n})(1+x_{k_n})}{(1+\alpha_n^2 x_{k_2,k_3,\ldots,k_n})(1+\alpha_{n-1}^2 x_{k_3,k_4,\ldots,k_n})\cdots(1+\alpha_3^2 x_{k_{n-1},k_n})(1+\alpha_2^2 x_{kn})} \right)$$
$$= \prod_{k=1}^{n} \alpha_k \times \frac{A_{N_n}\left(\frac{1}{\beta}\right)}{B_{N_n}\left(\frac{1}{\beta}\right)} \times \frac{A_{N_{n-1}}\left(\frac{1}{\eta_{N_n}^{\alpha_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_{n-1}}\left(\frac{1}{\eta_{N_n}^{\alpha_n}\left(\frac{1}{\beta}\right)}\right)}$$
$$\times \frac{A_{N_{n-2}}\left(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_{n-2}}\left(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}\left(\frac{1}{\beta}\right)}\right)} \times \cdot \times \frac{A_{N_1}\left(\frac{1}{\eta_{N_2,N_3,\ldots,N_n}^{\alpha_2,\alpha_3,\ldots,a_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_1}\left(\frac{1}{\eta_{N_2,N_3,\ldots,N_n}^{\alpha_2,\alpha_3,\ldots,\alpha_n}\left(\frac{1}{\beta}\right)}\right)} \frac{1+\alpha_1^2 y}{1+\eta_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}\left(\frac{1}{\beta}\right) y}.$$

Now by inserting the right side of last relation in (3.5), we get:

$$\frac{1+\alpha_1^2 y}{1+\beta y} = \prod_{k=1}^{n} \alpha_k \frac{A_{N_n}\left(\frac{1}{\beta}\right)}{B_{N_n}\left(\frac{1}{\beta}\right)} \times \frac{A_{N_{n-1}}\left(\frac{1}{\eta_{N_n}^{\alpha_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_{n-1}}\left(\frac{1}{\eta_{N_n}^{\alpha_n}\left(\frac{1}{\beta}\right)}\right)}$$
$$\times \frac{A_{N_{n-2}}\left(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_{n-2}}\left(\frac{1}{\eta_{N_{n-1},N_n}^{\alpha_{n-1},\alpha_n}\left(\frac{1}{\beta}\right)}\right)} \times \cdot \times \frac{A_{N_1}\left(\frac{1}{\eta_{N_2,N_3,\ldots,N_n}^{\alpha_2,\alpha_3,\ldots,\alpha_n}\left(\frac{1}{\beta}\right)}\right)}{B_{N_1}\left(\frac{1}{\eta_{N_2,N_3,\ldots,N_n}^{\alpha_2,\alpha_3,\ldots,\alpha_n}\left(\frac{1}{\beta}\right)}\right)} \frac{1+\alpha_1^2 y}{1+\eta_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}\left(\frac{1}{\beta}\right) y}.$$

We see that the above relation holds true provided that the parameter β fulfills the relation (3.4).

4 Kolmogrov–Sinai entropy

KS-entropy or metric entropy [1, 11] measures how chaotic a dynamical system is and it is proportional to the rate at which information about the state of dynamical system is lost in the course of time or iteration. Therefore, it can also be defined as the average rate of information loss for a discrete measurable dynamical system $(\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x), \mu)$. By introducing a partition $\alpha = A_c(n_1, \ldots, n_\gamma)$ of the interval $[0, 1]$ into individual laps A_i, one can define the usual entropy associated with the partition by:

$$H(\mu, \gamma) = -\sum_{i=1}^{n(\gamma)} m(A_c) \ln m(A_c),$$

where $m(A_c) = \int_{n \in A_i} \mu(x) dx$ is the invariant measure of A_i. By defining n-th refining $\gamma(n)$ of γ as:

$$\gamma^n = \bigcup_{k=0}^{n-1} \left(\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)\right)^{-(k)}(\gamma),$$

then entropy per unit step of refining is defined by:

$$h\left(\mu, \Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x), \gamma\right) = \lim_{n\to\infty}\left(\frac{1}{n} H(\mu,\gamma)\right).$$

Now, if the size of individual laps of $\gamma(N)$ tends to zero as n increases, then the above entropy is known as KS-entropy, that is:

$$h\left(\mu, \Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)\right) = h\left(\mu, \Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x), \gamma\right).$$

KS-entropy, which is a quantitative measure of the rate of information loss with the refining, may also be written as:

$$h\left(\mu, \Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)\right) = \int \mu(x) dx \ln\left|\frac{d}{dx}\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x)\right|, \tag{4.1}$$

which is also a statistical mechanical expression for the Lyapunov characteristic exponent, that is the mean divergence rate of two nearby orbits. The measurable dynamical system $(\Phi_{N_1,N_2,\ldots,N_n}^{\alpha_1,\alpha_2,\ldots,\alpha_n}(x), \mu)$ is chaotic for $h > 0$ and predictive for $h = 0$.

In order to calculate the KS-entropy of the maps $\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)$, it is rather convenient to consider their conjugate maps given by (2.12), since it can be shown that KS-entropy is a kind of topological invariant, that is, it is preserved under conjugacy map. Hence, we have:

$$h\left(\mu,\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)\right)=h\left(\tilde{\mu},\tilde{\Phi}^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)\right).$$

Using the integral (4.1), the KS-entropy of $\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)$ can be written as:

$$\begin{aligned}&h\left(\mu,\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)\right)\\ &=\frac{1}{\phi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln\left|\frac{d}{dy_{N_2,N_3,\cdot,N_n}}\left(\frac{1}{\alpha_1^2}\tan^2\left(N_1\arctan\sqrt{y_{N_2,N_3,\cdot,N_n}}\right)\right.\right.\\ &\left.\left.\times\frac{d}{dy_{N_3,N_4,\cdot,N_n}}\left(\frac{1}{\alpha_2^2}\tan^2\left(N_2\arctan\sqrt{y_{N_3,N_4,\cdot,N_n}}\right)\cdots\frac{d}{dx}\frac{1}{\alpha_n^2}\tan^2(N_n\arctan\sqrt{x})\right)\right)\right|\end{aligned}$$

or

$$\begin{aligned}&h\left(\mu,\Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)\right)\\ &=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln\left|\frac{d}{dy_{N_2,N_3,\cdot,N_n}}\left(\frac{1}{\alpha_1^2}\tan^2\left(N_1\arctan\sqrt{y_{N_2,N_3,\cdot,N_n}}\right)\right)\right|\\ &+\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln\left|\frac{d}{dy_{N_3,N_4,\cdot,N_n}}\left(\frac{1}{\alpha_2^2}\tan^2\left(N_2\arctan\sqrt{y_{N_3,N_4,\cdot,N_n}}\right)\right)\right|\\ &+\cdots+\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln\left|\frac{d}{dy_{N_n}}\left(\frac{1}{\alpha_{n-1}^2}\tan^2\left(N_{n-1}\arctan\sqrt{y_{N_n}}\right)\right)\right|\\ &+\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln\left|\frac{d}{dx}\left(\frac{1}{\alpha_n^2}\tan^2\left(N_n\arctan\left(\sqrt{x}\,\right)\right)\right)\right|,\end{aligned}\tag{4.2}$$

where

$$y_{N_n}=\frac{1}{\alpha_n^2}\tan^2\left(N_n\arctan(\sqrt{x})\right),\tag{4.3}$$

$$y_{N_{n-1},N_n}=\frac{1}{\alpha_{n-1}}\tan^2\left(N_{n-1}\arctan(\sqrt{y_{N_n}})\right),\tag{4.4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_{N_2,N_3,\cdot,N_n}=\frac{1}{\alpha_1^2}\tan^2\left(N_1\arctan(\sqrt{y_{N_3,N_4,\cdot,N_n}})\right).\tag{4.5}$$

Now, we calculate the integrals appearing above in the expression for the entropy separately. The last integral is calculated in appendix C which reads:

$$\begin{aligned}&\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln\left|\frac{d}{dx}\left(\frac{1}{\alpha_n^2}\tan^2\left(N_n\arctan\left(\sqrt{x}\,\right)\right)\right)\right|\\ &=\ln\left(\frac{N_n\left(1+\beta+2\sqrt{\beta}\,\right)^{N_n-1}}{A_{N_n}\left(\frac{1}{\beta}\right)B_{N_n}\left(\frac{1}{\beta}\right)}\right).\end{aligned}\tag{4.6}$$

In order to calculate the integral before the last one in (4.2), that is the following integral,

$$\frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)} \ln\left|\frac{d}{dy_{N_n}}\left(\frac{1}{\alpha_{n-1}^2}\tan^2\left(N_{n-1}\arctan\sqrt{y_{N_n}}\right)\right)\right|, \tag{4.7}$$

first we make the following change of variable by inverting the relation (4.3)

$$x_{k_{n-1}} = \tan^2\left(\frac{1}{N_{n-1}}\arctan\left(\sqrt{y_{N_n}}\alpha_{n-1}^2\right) + \frac{k_{n-1}\pi}{N_{n-1}}\right), \qquad k_{n-1} = 1, \ldots, N_{n-1}.$$

Then the integral (4.7) is reduced to:

$$\sum_{k_{n-1}=1}^{N_{n-1}} \frac{1}{\pi}\int_{x^i_{k_{n-1}}}^{x^f_{k_{n-1}}} \frac{\sqrt{\beta}dx_{k_{n-1}}}{\sqrt{x_{k_{n-1}}}(1+\beta x_{k_{n-1}})} \ln\left|\frac{d}{dy_{N_n}}\left(\frac{1}{\alpha_{n-1}^2}\tan^2\left(N_{n-1}\arctan\sqrt{y_{N_n}}\right)\right)\right|,$$

where $x^i_{k_{n-1}}$ and $x^f_{k_{n-1}}$ $(k_{n-1} = 1, 2, \ldots, N_{n-1})$ denote the initial and end points of k-th branch of the inversion of function $y_{N_n} = \left(\frac{1}{\alpha_n^2}\tan^2\left(N_n \arctan\sqrt{x}\right)\right)$ respectively. Now, by inserting the derivative of $x_{k_{n-1}}$ with respect to y_{N_n} in the above relation and changing the order of sum and integration, we get:

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \sum_{k_{n-1}=1}^{N_{n-1}} \sqrt{\beta}dy_{N_n} &\frac{\alpha_{n-1}\sqrt{x_{k_{n-1}}}(1+x_{k_{n-1}})}{N_{n-1}\sqrt{y_{n_n}}\left(1+\alpha_{n-1}^2 y_{N_n}\right)\sqrt{x_{k_{n-1}}}(1+\beta x_{k_{n-1}})} \\ &\times \ln\left|\frac{d}{dy_{N_n}}\left(\frac{1}{\alpha_{n-1}^2}\tan^2(N_{n-1}\arctan\sqrt{y_{N_n}})\right)\right|.\end{aligned}$$

By using the formula (B.6) of Appendix B, we get:

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}dy_n}{\sqrt{y_n}} &\left(\frac{B_{N_n}\left(\frac{1}{\beta}\right)}{\alpha_n A_{N_n}\left(\frac{1}{\beta}\right)} + \beta\frac{\alpha_n A_{N_n}\left(\frac{1}{\beta}\right)}{B_{N_n}\left(\frac{1}{\beta}\right)} y_{N_n}\right) \\ &\times \ln\left|\frac{d}{dy_{N_n}}\left(\frac{1}{\alpha_{n-1}^2}\tan^2\left(N_{n-1}\arctan\sqrt{y_{N_n}}\right)\right)\right|.\end{aligned}$$

Finally, through calculating the above integral with the prescription of Appendix C, we obtain:

$$\ln\left(\frac{N_{n-1}\left(1+\eta_{N_n}^{\alpha_n} + 2\sqrt{\eta_{N_n}^{\alpha_n}}\right)^{N_{n-1}-1}}{A_{N_{n-1}}\left(\eta_{N_n}^{\alpha_n}\right) B_{N_{n-1}}\left(\eta_{N_n}^{\alpha_n}\right)}\right).$$

Similarly, we can calculate the other integrals appearing in the expression for the entropy of the composed maps given in (4.2):

$$h\left(\mu, \Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)\right)$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)} \ln\left|\frac{d}{dy_{N_k,N_{k+1},\cdot,N_n}}\left(\frac{1}{\alpha_{k-1}^2}\tan^2\left(N_{k-1}\arctan\sqrt{y_{N_k,N_{k+1},\cdot,N_n}}\right)\right)\right|$$
$$= \ln\left(\frac{N_{k-1}\left(1+\eta^{\alpha_k,\alpha_{k+1},\ldots,\alpha_n}_{N_k,N_{k-1},\ldots,N_n}\left(\frac{1}{\beta}\right)+2\sqrt{\eta^{\alpha_k,\alpha_{k+1},\ldots,\alpha_n}_{N_k,N_{k-1},\ldots,N_n}\left(\frac{1}{\beta}\right)}\right)^{(N_{k-1})-1}}{A_{N_{k-1}}\left(\eta^{\alpha_k,\alpha_{k+1},\ldots,\alpha_n}_{N_k,N_{k-1},\ldots,N_n}\left(\frac{1}{\beta}\right)\right)B_{N_{k-1}}\left(\eta^{\alpha_k,\alpha_{k+1},\ldots,\alpha_n}_{N_k,N_{k-1},\ldots,N_n}\right)\left(\frac{1}{\beta}\right)}\right)$$

for $k = 1, 2, \ldots, n$.

Finally, summing the above integral, we get the following expression for the entropy of these maps:

$$h\left(\mu, \Phi^{\alpha_1,\alpha_2,\ldots,\alpha_n}_{N_1,N_2,\ldots,N_n}(x)\right) = \ln\left\{\left[(N_1N_2\cdots N_n)\left(1+\sqrt{\beta}\right)^{2(N_n-1)}\right.\right.$$
$$\times\left(1+\sqrt{\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right)}\right)^{2(N_{n-1}-1)}\cdots\left(1+\sqrt{\eta^{\alpha_2,\alpha_3,\ldots,\alpha_n}_{N_2,N_3,\ldots,N_n}\left(\frac{1}{\beta}\right)}\right)^{2(N_1-1)}\Bigg]\Bigg/\Bigg[A_{N_n}(\beta)$$
$$\times B_{N_n}(\beta)A_{N_{n-1}}\left(\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right)\right)B_{N_{n-1}}\left(\eta^{\alpha_n}_{N_n}\left(\frac{1}{\beta}\right)\right)\cdots$$
$$\left.\left.\times A_{N_1}\left(\eta^{\alpha_2,\alpha_3,\ldots,\alpha_n}_{N_2,N_3,\ldots,N_n}\left(\frac{1}{\beta}\right)\right)B_{N_1}\left(\eta^{\alpha_2,\alpha_3,\ldots,\alpha_n}_{N_2,N_3,\ldots,N_n}\left(\frac{1}{\beta}\right)\right)\right]\right\}. \tag{4.8}$$

Using the formulas (4.8), one can show that KS-entropy of one parameter families has the following asymptotic behavior:

$$h\left(\mu, \Phi_N\left(x, \alpha = N + 0^-\right)\right) \sim (N-\alpha)^{\frac{1}{2}},$$
$$h\left(\mu, \Phi_N\left(x, \alpha = \frac{1}{N} + 0^+\right)\right) \sim \left(\alpha - \frac{1}{N}\right)^{\frac{1}{2}},$$

near the bifurcation points, that is $\beta \longrightarrow 0$ as $\beta \longrightarrow \infty$. The above asymptotic behaviors indicate that one-parameter maps $\Phi^{\alpha}_N(x)$ belong to the same universality class which are different from the universality class of pitch fork bifurcating maps, but their asymptotic behavior is similar to class of intermittent maps [12]. Even though, intermittency can not occur in these maps for any values of parameter α, the maps $\Phi^{\alpha}_N(x)$ and their n-composition $\Phi^{(n)}$ do not have minimum values other than zero and maximum values other than 1 in the interval $[0, 1]$.

Also by imposing the relations between the parameters α_k, $k = 1, 2, \ldots, n$ which are consistent with the relation (3.4), we can reduce these maps to other many-parameter families of maps with the number of the parameters less than n. Particularly by imposing enough relations, we can reduce them to one-parameter families of chaotic maps with an arbitrary asymptotic behavior as the parameter takes the limiting values. Hence we can construct chaotic maps with arbitrary universality class. As an illustration, we consider the chaotic map $\Phi_{2,2}^{\alpha_1,\alpha_2}(x)$. Using the formula (4.8), we have:

$$h\left(\mu, \Phi_{2,2}^{\alpha_1,\alpha_2}(x)\right) = \ln \frac{(1+\sqrt{\beta})^2\left(2\sqrt{\beta}+\alpha_2(1+\beta)\right)^2}{(1+\beta)\left(4\beta+\alpha_2^2(1+\beta)^2\right)}$$

with the following relation among the parameters α_1, α_2 and β:

$$\alpha_1\left(4\beta+\alpha_2^2(1+\beta)^2\right) = 4\alpha_2\beta(1+\beta)$$

which is obtained from the relation (3.4). Now choosing $\beta = \alpha_2^{\nu}$, $0 < \nu < 2$, the above relation reduces to:

$$\alpha_1 = \frac{4\alpha_2^{1+\nu}(1+\alpha_2^{\nu})}{\alpha_2^2(1+\alpha_2^{\nu})^2+4\alpha_2^{\nu}}$$

and entropy given by (4.8) reads:

$$h\left(\mu, \Phi_{2,2}^{\alpha_2}(x)\right) = \ln \frac{\left(1+\alpha_2^{\frac{\nu}{2}}\right)^2\left(2\alpha_2^{\frac{\nu}{2}}+\alpha_2(1+\alpha_2^{\nu})\right)^2}{(1+\alpha_2^{\nu})\left(4\alpha_2^{\nu}+\alpha_2^2(1+\alpha_2^{\nu})^2\right)}$$

which has the following asymptotic behavior near $\alpha_2 \longrightarrow 0$ and $\alpha_2 \longrightarrow \infty$:

$$h\left(\mu, \Phi_{2,2}^{\alpha_2}(x)\right) \sim \alpha_2^{\frac{\nu}{2}} \qquad \text{as} \quad \alpha_2 \longrightarrow 0,$$
$$h\left(\mu, \Phi_{2,2}^{\alpha_2}(x)\right) \sim \left(\frac{1}{\alpha_2}\right)^{\frac{\nu}{2}} \qquad \text{as} \quad \alpha_2 \longrightarrow \infty.$$

The above asymptotic behaviours indicate that for an arbitrary value of $0 < \nu < 2$ the maps $\Phi_{2,2}^{\alpha_2}(x)$ belong to the universality class which is different from the universality class of one-parameter chaotic maps of $\Phi_N(x)$ (2.1) or the universality class of pitch fork bifurcating maps.

5 Conclusion

We have given hierarchy of exactly solvable many-parameter families of one-dimensional chaotic maps with an invariant measure, that is measurable dynamical system with an interesting property of being either chaotic or having stable fixed point, and they bifurcate from a stable single periodic state to chaotic one and vice-versa without having usual period doubling or period-n-tupling scenario.

Again this interesting property is due to the existence of invariant measure for a region of the parameters space of these maps. Hence, to support this conjecture, it would be interesting to find the other measurable one-parameter maps, specially coupled or higher dimensional maps, which are under investigation.

Appendix

A Shwartzian derivative

The Shwarzian derivative $S\Phi_N(x)$ [10] is defined as:

$$S\left(\Phi_N(x)\right) = \frac{\Phi_N'''(x)}{\Phi_N'(x)} - \frac{3}{2}\left(\frac{\Phi_N''(x)}{\Phi_N'(x)}\right)^2 = \left(\frac{\Phi_N''(x)}{\Phi_N'(x)}\right)' - \frac{1}{2}\left(\frac{\Phi_N''(x)}{\Phi_N'(x)}\right)^2,$$

with a prime denoting a single differential. One can show that:

$$S\left(\Phi_N(x)\right) = S\left({}_2F_1\left(-N, N, \frac{1}{2}, x\right)\right) \leq 0,$$

since $\frac{d}{dx}\left({}_2F_1\left(-N, N, \frac{1}{2}, x\right)\right)$ can be written as:

$$\frac{d}{dx}\left({}_2F_1\left(-N, N, \frac{1}{2}, x\right)\right) = A\prod_{i=1}^{N-1}(x - x_i)$$

with $0 \leq x_1 < x_2 < x_3 < \cdots < x_{N-1} \leq 1$, then we have:

$$S\left({}_2F_1\left(-N, N, \frac{1}{2}, x\right)\right) = \frac{-1}{2}\sum_{J=1}^{N-1}\frac{1}{(x - x_j)^2} - \left(\sum_{J=1}^{N-1}\frac{1}{(x - x_j)}\right)^2 < 0.$$

Also, one can show the shwartzian derivative of composition of function with negative shwartzian derivatives is negative too.

B Derivation of the formula (3.9)

In order to drive formula (3.9), we write the summation in its left side as:

$$\frac{\alpha}{N}\sum_{k=0}^{N}\frac{1 + \alpha^2 x_k}{1 + \beta x_k} = \frac{\alpha}{\beta} + \left(\frac{\beta - 1}{\beta^2}\right)\frac{\partial}{\partial\beta^{-1}}\left(\ln\left(\prod_{k=1}^{N}(\beta^{-1} + x_k)\right)\right). \tag{B.1}$$

To evaluate the second term in the right side of (B.1), we denote $\tilde{\Phi}_N^{\alpha}(x)$ by y therefore, the map $\tilde{\Phi}_N^{(1)}(x, \alpha) = \frac{1}{\alpha^2}\tan^2(N\arctan\sqrt{x})$ can be written as:

$$0 = \alpha^2 y\cos^2(N\arctan\sqrt{x}) - \sin^2(N\arctan\sqrt{x}). \tag{B.2}$$

Now, we can write the equation (B.1) in the following form:

$$\frac{\alpha}{N}\sum_{k=0}^{N}\frac{1 + \alpha^2 x_k}{1 + \beta x_k} = \frac{(-1)^N}{(1 + x)^N}\left(\alpha^2 y\left(\sum_{k=0}^{[\frac{N}{2}]} C_{2k}^N(-1)^N x^k\right)^2 \right.$$
$$\left. - x\left(\sum_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N(-1)^N x^k\right)^2\right) = \frac{\text{constant}}{(1 + x)^N}\prod_{k=1}^{N}(x - x_k),$$

where x_k are the roots of equation (B.2) and they are given by

$$x_k = \tan^2\left(\frac{1}{N}\arctan\sqrt{y\alpha^2} + \frac{k\pi}{N}\right), \qquad k = 1,\ldots,N. \tag{B.3}$$

Therefore, we have:

$$\begin{aligned} &\frac{\partial}{\partial\beta^{-1}}\ln\left(\prod_{k=1}^{N}(\beta^{-1}+x_k)\right) \\ &= \frac{\partial}{\partial\beta^{-1}}\ln\left[\left(1-\beta^{-1}\right)^N\left(\alpha^2 y\cos^2\left(N\arctan\sqrt{-\beta^{-1}}\right) - \sin^2\left(N\arctan\sqrt{-\beta^{-1}}\right)\right)\right] \\ &= -\frac{N\beta}{\beta-1} + \frac{\beta N\left(1+\alpha^2 y\right)A\left(\frac{1}{\beta}\right)}{\left(A\left(\frac{1}{\beta}\right)\right)^2\beta^2 y + \left(B\left(\frac{1}{\beta}\right)\right)^2}, \end{aligned} \tag{B.4}$$

with polynomials $A(x)$ and $B(x)$ given in (3.5) and (3.6) where in derivation of above formulas we have used the following identities:

$$\cos\left(N\arctan\sqrt{x}\right) = \frac{A(-x)}{(1+x)^{\frac{N}{2}}}, \qquad \sin\left(N\arctan\sqrt{x}\right) = \sqrt{x}\frac{B(-x)}{(1+x)^{\frac{N}{2}}}. \tag{B.5}$$

By inserting the results (B.4) in (B.1), we get:

$$\frac{\alpha}{N}\sum_{k=0}^{N}\frac{1+\alpha^2 x_k}{1+\beta x_k} = \frac{1+\alpha^2 y}{\left(\frac{B\left(\frac{1}{\beta}\right)}{\alpha A\left(\frac{1}{\beta}\right)} + \beta\left(\frac{\alpha A\left(\frac{1}{\beta}\right)}{B\left(\frac{1}{\beta}\right)}\right)y\right)}. \tag{B.6}$$

C Derivation of the last integral of (4.2)

Using the relations (3.5) and (3.6) the last integral of (4.2) can be written as:

$$h(\mu,\Phi_N^{\alpha}(x)) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln\left(\frac{N}{\alpha^2}\left|\frac{(1+x)^{N-1}B(-x)}{(A(-x))^3}\right|\right). \tag{C.1}$$

We see that polynomials appearing in the numerator (denominator) of integrand appearing on the right side of equation (C.1) have $\frac{[N-1]}{2}$ $\left(\frac{[N]}{2}\right)$ simple roots denoted by x_k^B, $k = 1,\ldots,\left[\frac{N-1}{2}\right]$ $\left(x_k^A,\ k=1,\ldots,\left[\frac{N}{2}\right]\right)$ in the interval $[0,\infty)$. Hence, we can write the above formula in the following form:

$$h\left(\mu,\Phi_N^{(\alpha)}(x)\right) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\beta}dx}{\sqrt{x}(1+\beta x)}\ln\left(\frac{N}{\alpha^2}\times\frac{(1+x)^{N-1}\prod_{k=1}^{\left[\frac{N-1}{2}\right]}\left|x-x_k^B\right|}{\prod_{k=1}^{\left[\frac{N}{2}\right]}\left|x-x_k^A\right|^3}\right).$$

Now, making the following change of variable $x = \frac{1}{\beta}\tan^2\frac{\Theta}{2}$, and taking into account that degree of numerators and denominator are equal for both even and odd values of N, we get:

$$h(\mu, \Phi_N^{(\alpha)}(x)) = \frac{1}{\pi}\int_0^{\infty} d\theta \Bigg\{ \ln\left(\frac{N}{\alpha^2}\right) + (N-1)\ln|\beta + 1 + (\beta-1)\cos\theta|$$

$$+ \sum_{k=1}^{[\frac{N-1}{2}]} \ln\left|1 - x_k^B\beta + \left(1 + x_k^B\beta\right)\cos\theta\right| - 3\sum_{k=1}^{[\frac{N}{2}]} \ln\left|1 - x_k^A\beta + \left(1 + x_k^A\beta\right)\cos\theta\right| \Bigg\}.$$

Using the following integrals:

$$\frac{1}{\pi}\int_0^{\pi} \ln|a + b\cos\theta| = \begin{cases} \ln\left|\dfrac{a + \sqrt{a^2 - b^2}}{2}\right|, & |a| > |b|, \\ \ln\left|\dfrac{b}{2}\right|, & |a| \le |b|, \end{cases}$$

we get:

$$h\left(\mu, \Phi_N^{(\alpha)}(x)\right) = \ln\left(\frac{N\left(1 + \beta + 2\sqrt{\beta}\right)^{N-1}}{\left(\sum\limits_{k=0}^{[\frac{N}{2}]} C_{2k}^N \beta^k\right)\left(\sum\limits_{k=0}^{[\frac{N-1}{2}]} C_{2k+1}^N \beta^k\right)}\right). \tag{C.2}$$

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