Abstract—We consider the problem of node localization in sensor networks, and we focus on networks in which the ranging measurements are subject to errors and anchor positions are subject to uncertainty. We consider a statistical model for the uncertainty in the anchor positions and formulate the robust localization problem that finds a maximum likelihood estimation of the node positions. To overcome the non-convexity of the resulting optimization problem, we obtain a convex relaxation that is based on the second order cone programming (SOCP). We also propose a possible distributed implementation using the SOCP convex relaxation. We present numerical studies that compare the presented approach to other existing convex relaxations for the robust localization problem in terms of positioning error and computational complexity.

I. INTRODUCTION

The availability of accurate information about the location of nodes is essential in many sensor network applications such as target tracking and detection, cooperative sensing, and energy-efficient routing. A common approach for sensor localization is to utilize the (noisy) ranging information between the sensor node and the \textit{a priori} known location of the anchor nodes in order to estimate the position of the rest of the sensor nodes. This ranging information is measured between nodes which are in communication range of each other by using received signal strength information [1], angle of arrival [2], or time of arrival [3].

Different properties of the localization problem, such as conditions for unique localizability, are investigated in some recent studies such as [4]–[8], where it is also shown that the sensor localization problem is a difficult and computationally involved problem in its general form. Several approaches have been proposed to reduce the localization complexity, e.g. [9]. One approach is to relax the original problem to obtain a convex optimization problem which can be efficiently solved using existing algorithms such as interior point methods [10]. Two main convex relaxation techniques have been considered for sensor localization problem. The first one is based on second order cone programming (SOCP) [11], [12], while the second one is based on semidefinite programming (SDP) [13]–[15].

There is a tradeoff between the SDP and SOCP techniques in terms of computational complexity and localization accuracy. It is well known that SDP provides a tighter relaxation and hence results in a better localization accuracy compared to SOCP [12]. On the other hand, SOCP has a lower computational complexity and thus it can be solved faster compared to SDP.

Sensor localization algorithms require accurate anchor positions in order to estimate the location of the rest of the sensor nodes. However, in many scenarios anchor positions may not be accurately known. This uncertainty in turn significantly affects the quality of the estimated sensor positions.

In this paper, we consider robust localization for wireless sensor networks in the presence of anchor position uncertainty. Using a statistical model for the uncertainty in the anchor positions, we study a robust sensor localization that maximizes the likelihood estimation for all nodes. To overcome the non-convexity of the problem, we obtain a convex relaxation of the problem based on SOCP. We also demonstrate how the distributed localization approach for the case of perfectly known anchor positions [11] can be generalized to robust scenarios using SOCP convex relaxation. Robust sensor localization under anchor position uncertainty is also studied in [14], but there a convex relaxation based on SDP is developed.

Our motivation behind choosing SOCP is that it is significantly faster to solve compared to SDP. In other words, the robust SDP approach typically provides a tighter convex relaxation, while the presented robust SOCP approach has a significantly lower complexity compared to the robust SDP approach. Our numerical results also confirm that the robust SOCP is scalable to the network size, and it can be solved significantly faster than the robust SDP at the cost of a small performance loss.

The rest of this paper is organized as follows. The SOCP formulations with and without anchor position uncertainty are derived in Section II. Section III extends the SOCP formulation to a distributed implementation. Numerical results demonstrating the performance of the proposed method are presented in Section IV, followed by the conclusions in Section V.

II. THE SENSOR LOCALIZATION PROBLEM

We consider a sensor network that consists of a set of anchors, \(N_a\), and a set of general sensor nodes \(N_s\). We also
denote $\mathcal{N} = \mathcal{N}_a \cup \mathcal{N}_n$ as the set of all nodes in the network, and $x_i^0$ as the vector that contains the actual coordinates of a node $i \in \mathcal{N}$.

The sensor network can be represented by an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is the set of all sensors as defined above, and $\mathcal{E}$ is the set of links. A link $(i, j) \in \mathcal{E}$ if nodes $i, j$ are in the communication range, $R_c$, of each other, i.e., $\|x_i^0 - x_j^0\| \leq R_c$. Similar to [14], we assume that $\mathcal{E}$ does not include anchor-anchor links.

Because of the errors in the ranging measurements, the distance between nodes $i, j$ can be written as

$$d_{ij} = \|x_i^0 - x_j^0\| + e_{ij},$$

where $e_{ij}$ is the measurement error that can be modeled as a zero-mean Gaussian random variable with variance $\sigma_{ij}^2$ [14]. We assume that the values of $e_{ij}$ are independent for different links. We will denote $\mathcal{D}$ as the set of all available noisy measurements

$$\mathcal{D} = \{d_{ij} \mid (i, j) \in \mathcal{E}\}.$$

### A. Localization with Perfectly Known Anchor Positions

Let us first assume a sensor network in which the anchor positions are perfectly known. Given the anchor positions $x_i^0$, $i \in \mathcal{N}_a$, and the set of noisy measurements $\mathcal{D}$, the goal of the sensor localization problem is to obtain the estimates of general sensor node positions, $x_i$, $i \in \mathcal{N}_n$, that minimize the sum of squared measurement errors:

$$\min_{x_i \in \mathcal{N}_n} \sum_{(i,j)\in \mathcal{E}} (\|x_i - x_j\| - d_{ij})^2. \tag{1}$$

The above-mentioned optimization problem is non-convex. Convex SOCP and SDP relaxation techniques have been applied to (1) and its variants in [11]–[13], [15].

### B. Localization with Anchor Position Uncertainty

In practice, perfect knowledge of anchor positions may not be available. In many scenarios, the anchor positions are obtained using global positioning system (GPS) which can be subject to errors. Let $a_i$, $i \in \mathcal{N}_a$, denote the given uncertain anchor positions and

$$a_i = x_i^0 + w_i, \quad i \in \mathcal{N}_a, \tag{2}$$

where $w_i$ is the uncertainty in the position of the $i$th anchor which is modeled as a zero-mean Gaussian random vector with covariance matrix $\Phi_i$. We assume that $w_i$ are independent of $e_{ij}$.

Using the anchor position uncertainty in (2), our goal is to obtain a robust counterpart of the localization problem in (1) that explicitly takes into account the uncertainty in the anchor positions. In order to obtain estimates $x_i$, $i \in \mathcal{N}_n$, of the uncertain anchor positions as well as general node positions, we will consider a maximum likelihood estimation (MLE) approach. While the MLE approach has been considered in [14] to obtain a convex SDP relaxation of the robust localization problem, we will obtain a convex SOCP relaxation which is characterized by significantly lower computational complexity.

To obtain an MLE of $x_i^0$, $i \in \mathcal{N}$, we examine the probability

$$P(D, \{a_i\} \mid \{x_i\}) = \prod_{(i,j)\in \mathcal{E}} P(D \mid \{x_i\}) \times \prod_{i \in \mathcal{N}_a} P(\{a_i\} \mid \{x_i\})$$

$$= \prod_{(i,j)\in \mathcal{E}} \frac{1}{\sqrt{2\pi\sigma_{ij}}} \exp\left(-\frac{(\|x_i - x_j\| - d_{ij})^2}{\sigma_{ij}^2}\right) \times \prod_{i \in \mathcal{N}_a} \frac{1}{\sqrt{2\pi\sigma_{ij}}} \exp\left(-\frac{(a_i - x_i)^T\Phi_i^{-1}(a_i - x_i)}{2}\right). \tag{3}$$

By taking the logarithm of (3), the robust localization problem can be written as

$$\min_{x_i, i \in \mathcal{N}_n} \sum_{(i,j)\in \mathcal{E}} g_{ij} (\|x_i - x_j\| - d_{ij})^2 + \sum_{i \in \mathcal{N}_a} (a_i - x_i)^T\Phi_i^{-1}(a_i - x_i), \tag{4a}$$

where $g_{ij} \doteq \frac{1}{\sigma_{ij}}$. Similar to the optimization problem in (1), the robust localization problem (4) is non-convex. To obtain an SOCP relaxation of (4), we first write the optimization problem as

$$\min_{x_i, a_i, s_i} \sum_{(i,j)\in \mathcal{E}} t_{ij}^2 + \sum_{i \in \mathcal{N}_a} s_i^2, \tag{5a}$$

subject to $g_{ij} (\|x_i - x_j\| - d_{ij}) \leq t_{ij}$, $(i,j) \in \mathcal{E}$, $(5b)$

$$\|\Phi_i^{-1/2}(a_i - x_i)\| \leq s_i, \quad i \in \mathcal{N}_a, \tag{5c}$$

By defining a vector $u$ as the concatenation of variables in the objective, i.e.,

$$u \doteq [t_{ij} \mid (i,j) \in \mathcal{E}, s_i \in \mathcal{N}_a],$$

we can write (5) as the following equivalent epigraph form:

$$\min_{x_i, u_i, y_{ij}, v} v, \tag{6a}$$

subject to $\|u\| \leq v$, $(6b)$

$$g_{ij} |y_{ij} - d_{ij}| \leq t_{ij}, \quad (i,j) \in \mathcal{E}, \tag{6c}$$

$$\|\Phi_i^{-1/2}(a_i - x_i)\| \leq s_i, \quad i \in \mathcal{N}_a, \tag{6d}$$

$$\|x_i - x_j\| \leq y_{ij}, \quad (i,j) \in \mathcal{E}. \tag{6e}$$

Finally, a convex SOCP relaxation of the robust localization problem in (6) can be obtained by relaxing the equality constraint (6c) to inequality. The resulting SOCP optimization can be efficiently solved using an interior point algorithm [10].

### III. A Possible Distributed Implementation

Another attractive feature of the SOCP relaxation for the robust localization problem is its potential for distributed implementation, which allows the optimization problem to be divided into a number of smaller subproblems that can be solved locally at each node. These subproblems will rely on the position of the neighboring nodes. Hence, after the step of solving the subproblems locally, the nodes will be allowed to exchange their position estimates with their neighboring nodes.
A synchronous distributed algorithm was proposed for network localization with perfectly known anchor positions in [11]. In this case, the local subproblems are solved only at the general sensor nodes and not at the anchors. In this section we generalize this distributed algorithm to the robust localization problem with uncertain anchor positions.

Consider the robust localization problem in (4). It can be written as

$$
\min_{\mathbf{x}_i, i \in \mathcal{N}} \sum_{(i,j) \in \mathcal{E}} g_{ij} (y_{ij} - d_{ij})^2 + \sum_{i \in \mathcal{N}_a} (\mathbf{a}_i - \mathbf{x}_i)^T \Phi_i^{-1} (\mathbf{a}_i - \mathbf{x}_i),
$$

subject to $y_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|,$ $(i, j) \in \mathcal{E},$ (7b)

where constraint (7b) can be replaced by an inequality constraint for the SOCP relaxation. Using the barrier function approach, cf. [16], the relaxed constrained problem can be written as the following unconstrained problem:

$$
\min_{\mathbf{x}_i, i \in \mathcal{N}} \sum_{(i,j) \in \mathcal{E}} g_{ij} (y_{ij} - d_{ij})^2 + \sum_{i \in \mathcal{N}_a} (\mathbf{a}_i - \mathbf{x}_i)^T \Phi_i^{-1} (\mathbf{a}_i - \mathbf{x}_i) + \sum_{(i,j) \in \mathcal{E}} B(\|\mathbf{x}_i - \mathbf{x}_j\| - y_{ij}),
$$

where $B(.)$ is a properly chosen barrier function, such as the logarithmic barrier function $B(z) = -\frac{1}{c} \log(-z)$ for a large constant $c \gg 1$ [11]. Now, the problem in (8) is separable and each term in the summation can be minimized independently at each node $i$ using only information about the positions, $\mathbf{x}_j$, and ranging information $d_{ij}$, of the set of neighboring nodes, $\mathcal{K}_i$. Using an approach similar to the one used in the previous section, we can formulate the local subproblems to be solved at each anchor or general sensor node. Specifically, for each general node $i \in \mathcal{N}_a$, the local subproblem is

$$
\min_{\mathbf{x}_i, \mathbf{u}_{ij}, v_i} v_i, \quad \text{(9a)}
$$

subject to $\|\mathbf{u}_{ij}\| \leq v_i,$ $\mathbf{g}_{ij} [y_{ij} - d_{ij}] \leq \mathbf{t}_{ij}$, $j \in \mathcal{K}_i,$ $\|\mathbf{x}_i - \mathbf{x}_j\| \leq y_{ij}$, $j \in \mathcal{K}_i,$ (9c) (9d)

where $\mathbf{u}_{ij} = [t_{ij} \ j \in \mathcal{K}_i]$. Moreover, for each anchor node $i \in \mathcal{N}_a$, the local subproblem is

$$
\min_{\mathbf{x}_i, \mathbf{u}_{is}, v_i} v_i, \quad \text{(10a)}
$$

subject to $\|\mathbf{u}_{is}\| \leq v_i,$ $\|\mathbf{a}_i - \mathbf{x}_i\| \leq s_i$, $\mathbf{g}_{ij} [y_{ij} - d_{ij}] \leq \mathbf{t}_{ij}$, $j \in \mathcal{K}_i,$ $\|\mathbf{x}_i - \mathbf{x}_j\| \leq y_{ij}$, $j \in \mathcal{K}_i,$ (10c) (10d) (10e)

where $\mathbf{u}_{is} = [t_{ij} \ j \in \mathcal{K}_i, \ s_i]$ for $i \in \mathcal{N}_a$.

We observe that the local subproblems are also solved at the anchors for the robust localization problem. Furthermore, the local subproblem of each anchor is different from that of a general sensor node due to the robust formulation of the problem.

In an attempt to treat the case of uncertain anchor positions, a three-phase distributed heuristic is proposed in [11]. In this method, the sensors first perform a local SOCP phase to find their locations based on the uncertain anchor positions. Then, the anchors use this information to refine their location information. At the third phase, the general sensors run a new round of local SOCP based on the refined location of anchors. In this method, however, the subproblems solved in the anchors are similar to that of general sensors and do not explicitly take into account the anchor uncertainties. Hence, our proposed distributed localization algorithm outperforms the three-phase method, as will be demonstrated in Section IV.

### IV. Performance Evaluation

In this section we examine the performance of the proposed distributed algorithms by means of simulations. We use the CVX Matlab toolbox [17] for solving the devised robust SOCP (RSOCP) and the benchmark robust SDP (RSDP) optimizations. For the simulations, nodes are uniformly-randomly deployed in an area of $40 \times 40$ m. Unless specified otherwise in a simulation setting, the communication range $R_c$ is adjusted based on $|\mathcal{V}|$ for obtaining the average connectivity of 4. Let $p = \frac{|\mathcal{N}_a|}{|\mathcal{V}|}$ denote the fraction of anchors in the network. We consider a fixed link error model with equal $\sigma_{ij}^2$ for all links, and also a variable link error model, where the noise variance for link $(i, j) \in \mathcal{E}$ is set to $\sigma_{ij}^2 = \sigma_j^2 d_{ij}^2$ and $\sigma_j^2$ is the noise variance for the unit distance. In the latter link error model, a link with a longer distance has a larger noise variance [14]. To compare the performance of different methods, we consider the positioning mean square error (MSE), in dBm²,

$$
\eta = 10 \log_{10} \left( \frac{1}{|\mathcal{N}|} \sum_{i \in \mathcal{N}} \|\mathbf{x}_i - \mathbf{x}^*_i\|^2 \right),
$$

as the performance criterion. A smaller positioning MSE indicates a better localization performance.

Figure 1 shows the estimated locations for a sample scenario from [14] with $|\mathcal{N}_a| = 8$ anchors and $|\mathcal{N}_a| = 10$ general sensors.
anchors in an area of 40 m × 40 m. In this figure, the blue circles and diamonds represent the actual position of general nodes and anchors; the red asterisks and ‘+’ signs show the estimated locations of general nodes and anchors using RSDP [14]; and the black dots and crosses show the estimated locations of general nodes and anchors using the proposed RSOCP (6), respectively. The communication range is set to $R_c = 25$ m. Also, the unit noise variance and anchor uncertainty covariance matrix are set to $\sigma^2_d = -20$ dBm$^2$ and $\Phi_i = -10 I_2$ dBm$^2$, respectively.

As can be seen, both RSOCP and RSDP methods are able to locate both anchors and general sensors with a small error. For the nodes located close to the origin $(0, 0)$, the RSOCP estimations are slightly more accurate than RSDP. For the farther nodes, the RSOCP error increases and becomes larger than the RSDP error. This trend demonstrates that the nodes located closer to the center of the anchors’ convex hull are generally localized more accurately in RSOCP. Note that this property is because of the special structure in RSOCP optimization and might not be generally true for RSDP. For example, the minimum RSDP localization error in Figure 1 belongs to the node located at $(-11.6, -10.8)$, which is far from the origin. The total MSE for RSDP for this scenario is $\eta_{\text{RSDP}} = 0.50$ dBm$^2$, which is slightly better than that for RSOCP, $\eta_{\text{RSOCP}} = 1.35$ dBm$^2$.

Figure 2 shows the average value of positioning MSE, as a function of $\sigma^2_{ij}$, for RSOCP and RSDP in 1000 uniformly-random network topologies with $|N| = 50$ nodes and with $p = 30\%$ anchors. In order to observe the effect of anchor position uncertainties on the performance, we also plot the $\text{MSE}$ of the standard SOCP, which does not formulate the uncertainties [11], [12]. The Cramer-Rao lower bound (CRLB) [14] is also shown. Note that the CRLB indicates the lowest achievable values for $\eta$. Here, the value of $\sigma^2_{ij}$ varies from $-35$ to $-5$ dBm$^2$ and the uncertainty covariance is set to $\Phi_i = -10 I_2$ dBm$^2$.

As can be seen, the RSOCP, which considers uncertainties, outperforms the standard SOCP formulation. In addition, RSDP has a smaller MSE compared to the standard SOCP. We also observe that RSDP performs slightly better than RSOCP for the smaller values of noise due to the tighter relaxation on the problem. However, as the noise increases, the MSE of both robust methods tend to converge to the same value. This is due to the fact that the performance CRLB is achieved when the noise variance is large.

The total time spent for the optimization in the RSOCP, RSDP and standard SOCP methods for network sizes of 20 to 100 with $p = 30\%$ anchors are shown in Figure 3. Here, a PC with four 3.6 GHz processors and 2.0 GB of RAM is used for performing the simulations. As can be seen, both SOCP and RSOCP methods are significantly faster than RSDP. Particularly, for a number of $|N| = 100$ nodes, RSOCP is twice as fast as RSDP. This figure shows that the proposed robust SOCP model is a suitable tool for fast and accurate localization in sensor networks containing a large number of nodes.

Now we consider the larger network sizes where due to computational complexity, it is preferred to use the distributed methods for solving the localization problem. In Figure 4, we compare the performance of the distributed SOCP with the three-phase iterative distributed method [11], for $|N| = 100$ to 500 nodes and $p = 40\%$ anchors. We consider 10 different topologies for each value of $|N|$ and show the scatter plots for $\eta$ and computation time in Figures 4(a) and 4(b), respectively. For obtaining results in this figure, we set $\sigma^2_d = -20$ dBm$^2$ and $\Phi_i = -10 I_2$ dBm$^2$. For a fair comparison, both algorithms are iteratively executed for six rounds, i.e. the three-phase algorithm is run twice and our sensor - anchor refinement procedure in (9), (10) is executed three times. For measuring the computation time, we sum the maximum time taken by the nodes to solve their local problems in each phase. Note that the communication time between nodes is not taken into account.

As can be seen, in most of the cases, the proposed distributed RSOCP algorithm outperforms the three-phase algorithm in terms of $\eta$. This is due to the fact that our anchor
refinement step takes into account the uncertainty and is able to provide a better position estimates for anchors. Moreover, the time taken for solving distributed RSOCP is less than that for the three-phase algorithm. This may be because of the simpler structure of our anchor update problem compared to the proposed heuristic in [11]. In addition, by comparing Figures 4(b) and 3, we observe that the time taken for solving distributed RSOCP is significantly lower than its centralized version. For example, for a network of $|N| = 100$ nodes, distributed RSOCP always takes less than 10 seconds, whereas the centralized RSOCP takes around 150 seconds, using the same computer. This is because in the distributed RSOCP, small subproblems are simultaneously solved in different nodes and each phase is completed when the subproblem with the maximum computation time is solved.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have considered the localization problem for networks in which the anchor positions are not perfectly known. We have formulated a robust localization problem that explicitly takes into account the uncertainty in anchor positions. The robust formulation tries to obtain an MLE estimation of the node positions using a statistical model of the anchor position uncertainty. We have obtained a convex relaxation of the resulting non-convex problem using SOCP. The proposed approach offers lower computational complexity compared to SDP relaxation of the robust localization problem which makes it an attractive choice for networks with large number of nodes.

REFERENCES