Adaptive NN Backstepping Output-Feedback Control for Stochastic Nonlinear Strict-Feedback Systems With Time-Varying Delays

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Abstract—For the first time, this paper addresses the problem of adaptive output-feedback control for a class of uncertain stochastic nonlinear strict-feedback systems with time-varying delays using neural networks (NNs). The circle criterion is applied to designing a nonlinear observer, and no linear growth condition is imposed on nonlinear functions depending on system states. Under the assumption that time-varying delays exist in the system output, only an NN is employed to compensate for all unknown nonlinear terms depending on the delayed output, and thus, the proposed control algorithm is more simple even than the existing NN backstepping control schemes for uncertain systems described by ordinary differential equations. Three examples are given to demonstrate the effectiveness of the control scheme proposed in this paper.

Index Terms—Adaptive output-feedback control, neural network (NN), nonlinear observer, stochastic nonlinear strict-feedback systems, time-varying delays.

I. INTRODUCTION

I

T IS well known that stochastic disturbances often exist in many practical systems. Their existence is a source of instability of control systems; thus, the investigation on stability analysis and control design of stochastic systems has received increasing attention in the past decade [1]–[27], [48], [50], [51]. In particular, for a class of stochastic nonlinear strict-feedback (or output-feedback) systems, many interesting control schemes [3]–[27] have been proposed by using the well-known backstepping technique. Pan and Basar [3] first derived a backstepping design approach for stochastic nonlinear strict-feedback systems motivated by a risk-sensitive cost criterion. Since then, a series of extensions has been made under different assumptions or for different systems [4]–[9]. By using the quartic Lyapunov functions instead of the classical quadratic functions, Deng and Krstić [10]–[14] solved the (adaptive) stabilization problem of stochastic strict-feedback (or output-feedback) systems, and then, this design idea was extended to several different cases, such as tracking control [15], decentralized control [16], [20], and control of high-order systems [17]–[19]. Recently, an output-feedback control scheme has been proposed for stochastic nonminimum-phase nonlinear systems [22]. However, in the foregoing control schemes, the output-feedback backstepping methods are only for stochastic systems in the standard output-feedback form, where the nonlinear functions depend on the system outputs or inverse dynamics instead of all system states. To the best of the authors’ knowledge, only a result was reported on the output-feedback control for stochastic nonlinear strict-feedback systems [21], but the linear growth condition was imposed on the nonlinear functions due to using the high-gain linear observer.

Recently, the control problem for stochastic time-delay strict-feedback (or output-feedback) systems has also received more and more attention. In [24], the authors attempted to solve the stabilization problem for stochastic time-delay strict-feedback systems, but the results cannot hold [26]. Fortunately, the observer-based control problem can be solved for time-delay stochastic systems in the standard output-feedback form [23], [50].

On the other hand, neural networks (NNs) have been proved to be very useful tools for solving the control problem of uncertain systems. In practical applications, NNs are often used as approximators to model unknown system functions, and many important results on adaptive backstepping NN (or fuzzy [49]) control have been obtained in the past years, e.g., [28]–[43], just to name a few. However, little work has been done to address the NN control problem of stochastic systems with unknown system functions [25], [27], [51]. In [27], the unknown system functions and the control input were assumed to be only in the last system equation, that is, the uncertainties must satisfy the matching condition. In [25] and [51], the authors did not investigate the time-delay case.

Motivated by the aforementioned discussion, we will investigate the more challenging problem in this paper, i.e., the output-feedback adaptive NN control for a class of uncertain stochastic nonlinear strict-feedback systems with time-varying delays. Two main difficulties are how to design a suitable observer and how to deal with the unknown and unmatched nonlinear time-delay functions. As shown in this
paper, these difficulties can be solved under some suitable assumptions on system nonlinear functions. The main contributions lie in the following.

1) Compared with the existing works on stochastic control systems, where the output-feedback control schemes are designed only for systems with the standard output-feedback form [12], [13], [16], [20], [22], [23], [25] in which nonlinear functions depend only on the system output or inverse dynamic instead of the system states, this paper will investigate the output-feedback control for a more general class of strict-feedback systems in which nonlinear functions depend not only on the system output but also on the system states. In general, this class of systems can be controlled only by the state-feedback control schemes [10], [14].

2) From the viewpoint of design techniques, the circle criterion [44] is introduced to solve the problem of nonlinear observer design; thus, the linear growth condition imposed on nonlinear functions is not required, which is different from [21], where a high-gain linear observer is designed for stochastic strict-feedback systems under the assumption of linear growth condition.

3) We lump all unknown output-dependent functions into a suitable unknown function that is compensated only by an NN. This is different from the existing work on NN control [28]–[43], where each unknown function is usually approximated by a sole NN; thus, the design procedure is very complex, particularly for high-order systems because of using numerous NNs. The main benefit of only adopting an NN is to simplify the design procedure and to reduce the computation loads. Moreover, the requirement that the output-dependent system functions are known or bounded [12], [13], [16], [20], [22], [23], [25] is removed due to the use of NN.

The rest of this paper is organized as follows. In Section II, we present mathematical preliminaries, stochastic stability, and NN approximation. Section III gives problem formulation. We present the adaptive NN control design procedure and give the stability analysis in Section IV. In Section V, three simulation examples are provided to illustrate the effectiveness of the proposed controller. In Section VI, we conclude the work of this paper.

II. NOTATIONS, STOCHASTIC STABILITY, AND NN APPROXIMATION

A. Notations

This section closely follows [23]. Throughout this paper, the following notations are adopted.

1) \( \mathbb{R}_+ \) denotes the set of all nonnegative real numbers, \( \mathbb{R}_n \) denotes the real \( n \)-dimensional space, and \( \mathbb{R}_n^{m 	imes r} \) denotes the real \( n \times r \) matrix space.

2) \( \text{Tr}(X) \) denotes the trace for square matrix \( X \), and \( \lambda_{\min}(X) \) and \( \lambda_{\max}(X) \) denote the minimal and maximal eigenvalues of the symmetric real matrix, respectively.

3) \( |X| \) denotes the Euclidean norm of a vector \( X \), and the corresponding induced norm for a matrix \( X \) is also denoted by \( |X| \); \( \|X\|_F \) denotes the Frobenius norm of \( X \) defined by \( \|X\|_F = \sqrt{\text{Tr}(X^TX)} \).

4) \( C([-d, 0]; \mathbb{R}^n) \) denotes the space of continuous \( \mathbb{R}^n \)-valued functions on \([-d, 0]\) endowed with the norm \( \| \cdot \| \) defined by \( \|f\| = \sup_{t \in [-d, 0]} |f(t)| \) for \( f \in C([-d, 0]; \mathbb{R}^n) \), and \( C^0([-d, 0]; \mathbb{R}^n) \) denotes the family of all \( F_0 \)-measurable bounded \( C([-d, 0]; \mathbb{R}^n) \)-valued random variables \( \xi = (\xi_\theta : -d \leq \theta \leq 0) \).

5) \( C^1 \) denotes the set of all functions with continuous first partial derivatives; \( C^{2,1}(\mathbb{R}^n \times [-d, \infty); R_+) \) denotes the family of all nonnegative functions \( V(x, t) \) on \( \mathbb{R}^n \times [-d, \infty) \), which are \( C^2 \) in \( x \) and \( C^1 \) in \( t \); and \( C^{2,1} \) denotes the family of all functions, which are \( C^2 \) in the first argument and \( C^1 \) in the second argument.

6) \( \mathcal{K} \) denotes the set of all functions: \( R_+ \to R_+ \), which are continuous, strictly increasing, and vanish at zero; \( \mathcal{K}_\infty \) denotes the set of all functions that are of class \( \mathcal{K} \) and unbounded; and \( \mathcal{KL} \) denotes the set of all functions \( \beta(s, t) : R_+ \times R_+ \to R_+ \), which are of class \( \mathcal{K} \) for each fixed \( t \) and decrease to zero as \( t \to \infty \) for each fixed \( s \).

B. Stochastic Stability

Consider an \( n \)-dimensional stochastic time-delay system

\[
\begin{align*}
\dot{x}(t) &= f(x(t), x(t - d(t)), t) dt + g(x(t), x(t - d(t)), t) d\omega, \quad \forall t \geq 0 (1)
\end{align*}
\]

with initial data \( \{x(\theta) : -d \leq \theta \leq 0\} = \xi \in C^0_{\mathcal{F}_0} \times \{[-d, 0]; \mathbb{R}^n\} \), where \( d(t) : R_+ \to [0, d) \) is a Borel measurable function; \( f : \mathbb{R}^n \times \mathbb{R}^n \times R_+ \to \mathbb{R}^n \) is a Borel measurable function; \( g : \mathbb{R}^n \times \mathbb{R}^n \times R_+ \to \mathbb{R} \times R_+ \) are locally Lipschitz; and \( \omega \) is an \( r \)-dimensional standard Brownian motion defined on the complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\), with \( \Omega \) being a sample space, \( \mathcal{F} \) being a \( \sigma \)-field, \( (\mathcal{F}_t)_{t \geq 0} \) being a filtration, and \( P \) being a probability measure.

Define a differential operator \( \mathcal{L} \) as follows:

\[
\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x(t), x(t - d(t)), t) + \frac{1}{2} \text{Tr}\left\{ g^T \frac{\partial^2 V}{\partial x^2} g \right\}
\]

where \( V(x, t) \in C^{2,1} \).

Definition 1: The equilibrium \( x = 0 \) of system (1) with \( f(0, 0, t) = 0 \) and \( g(0, 0, t) = 0 \) is said to be globally stable in probability if, for any \( \varepsilon > 0 \), there exists a function \( \gamma(\cdot) \in \mathcal{K} \) such that

\[
P\{|x(t)| \leq \gamma(\|\xi\|)\} \geq 1 - \varepsilon, \quad \forall t \geq 0,
\]

\[
\quad \forall \xi \in C^0_{\mathcal{F}_0}([-d, 0]; \mathbb{R}^n) \setminus \{0\} (3)
\]

where \( \|\xi\| = \sup_{\theta \in [-d, 0]} |x(\theta)| \).

Lemma 1 [23]: For system (1), assume that both terms \( f(x, y, t) \) and \( g(x, y, t) \) are locally Lipschitz in \((x, y)\) and \( f(0, 0, t) = 0, g(0, 0, t) = 0 \). If there exist a function \( V_1(x, t) \in C^{2,1}((\mathbb{R}^n \times [-d, \infty); R_+) \) and two \( \mathcal{K}_\infty \) functions \( \gamma_1 \) and \( \gamma_2 \) such that

\[
\gamma_1(|x(t)|) \leq V_1(x(t), t) \leq \gamma_2 \left( \sup_{-d \leq s \leq 0} |x(t + s)| \right)
\]

\[
\mathcal{L}V_1 \leq -V_2(x(t))
\]

(4)

(5)
where $V_2(x)$ is continuous and nonnegative, then 1) there exists a unique solution on $[-d, \infty)$, and 2) the solution $x = 0$ of system (1) is globally stable in probability, and moreover
\[
P \left\{ \lim_{t \to \infty} V_2(x(t)) = 0 \right\} = 1.
\] (6)

The following lemma will be used in this paper.

**Lemma 2 (Young’s Inequality [10]):** For $\forall(x, y) \in \mathbb{R}^2$, the following inequality holds:
\[
XY \leq \frac{e^p}{p} |x|^p + \frac{1}{qe^q} |y|^q
\] (7)
where $\epsilon > 0$, $p > 1$, $q > 1$, and $(p - 1)(q - 1) = 1$.

### 3. NN Approximation

In this paper, an unknown smooth nonlinear function $\Psi(y) : R \to R$ will be approximated on a compact set $D$ by the following radial basis function NN [32, 33]:
\[
\Psi(y) = W^T S(y) + \sigma(y)
\] (8)
where $S(y) = [s_1(y), \ldots, s_l(y)]^T : D \to R^l$ is a known smooth vector function with the NN node number $l > 1$. The basis function $s_i(y)$, $1 \leq i \leq l$, is chosen as the commonly used Gaussian function with the form
\[
s_i(y) = \exp\left( -\frac{(y - \mu_i)^2}{\varsigma^2} \right)
\]
where $\mu_i \in D$ and $\varsigma > 0$ are the center and the width of the basis function $s_i(y)$, respectively. The optimal weight vector $W = [w_1, \ldots, w_l]^T$ is defined as
\[
W := \arg \min_{W \in \mathbb{R}^l} \left\{ \sup_{y \in D} |\Psi(y) - W^T S(y)| \right\}
\]
and $\sigma(y)$ denotes the NN inherent approximation error. In many previously published papers, the approximation error is assumed to be bounded by a fixed constant. However, this may not be true in many cases since it is not guaranteed that the compact set $D$ can easily be identified before the stability of the closed-loop system is established. Hence, we instead make the following assumption on the approximation error $\sigma(y)$.

**Assumption 1 [28]:** There exist a known positive function $\psi(y)$ and an unknown positive constant $\theta$ such that
\[
|\sigma(y)| \leq \psi(y)\theta.
\] (9)

Similar to the analysis in [28], if the inequality (9) only holds on the compact set $D$, then the results obtained in this paper are semiglobal. However, in the special case that the inequality (9) holds for all $y \in R$, the stability results become global. To simplify the analysis, in this paper, we assume that the bounding condition (9) holds globally.

**III. Problem Formulation**

Consider the following stochastic nonlinear strict-feedback system with a time-varying delayed output:
\[
\begin{align*}
&dx_i = \left\{ \sum_{j=1}^{i+1} a_{i,j} x_j + \phi_i(x_i) 
+ f_i(y, y(t - d(t))) \right\} dt \\
&\quad + g_i(y, y(t - d(t))) d\omega, \quad i = 1, \ldots, n - 1 \\
&dx_n = \left\{ \sum_{j=1}^{n} a_{n,j} x_j + \phi_n(x) 
+ f_n(y, y(t - d(t))) + \rho u \right\} dt \\
&\quad + g_n(y, y(t - d(t))) d\omega, \quad y = x_1
\end{align*}
\] (10)
where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, $y \in \mathbb{R}$, and $u \in \mathbb{R}$ are the system state vector, output, and control input, respectively; $\bar{x}_i = [x_1, \ldots, x_i]^T$; $a_{i,j}$ and $\rho$ are known constants; $\phi_i(\bar{x}_i)$ are known smooth nonlinear functions with $\phi_i(0) = 0$, but $f_i : \mathbb{R}^2 \to \mathbb{R}$ and $g_i^T : \mathbb{R}^2 \to \mathbb{R}^r$ are unknown locally Lipschitz smooth functions with $f_i(0, 0) = 0$ and $g_i(0, 0) = 0$; $d(t) : R_+ \to [0, d]$ is an uncertain time-varying delay satisfying $\dot{d}(t) \leq \varsigma < 1$, with $\varsigma$ being an unknown constant; $\omega$ is defined as in system (1); and the initial condition $\{x(\theta) : -d \leq \theta \leq 0\} = x(\theta) \in \mathbb{C}_{\infty} \times \{[-d, 0] : \mathbb{R}^n\}$ is unknown. Only the system output $y$ can be available for measurement.

**Remark 1:** The system model (10) is an extension of the system addressed in [23], where $\phi_i(\bar{x}_i) = 0$, and
\[
\begin{align*}
&|f_i(y, y(t - d(t)))| \leq l_i^f \varphi_i d, \quad |g_i(y, y(t - d(t)))| \leq l_i^g \varphi_i d,
\end{align*}
\]
with $l_i^f, l_i^g > 0$ being unknown constants, and $\varphi_i, \psi_i \geq 0$ being known smooth functions satisfying $\varphi_i(0) = \varphi_i(0) = \psi_i(0) = 0$. However, in this paper, on one hand, the appearance of $\varphi_i(\bar{x}_i)$ makes the observer design more difficult than that in [23]. On the other hand, the time-delay terms $f_i(y, y(t - d(t)))$ and $g_i(y, y(t - d(t)))$ are assumed to be completely unknown, which further relaxes the requirement on nonlinear time-delay functions.

Since $f_i(0, 0) = 0$ and $g_i(0, 0) = 0$, according to the well-known mean value theorem, the following equalities hold:
\[
\begin{align*}
&f_i(y, y(t - d(t))) = y(t - d(t)) \bar{f}_i(y, y(t - d(t))) \\
&g_i(y, y(t - d(t))) = y(t - d(t)) \bar{g}_i(y, y(t - d(t)))
\end{align*}
\] (11, 12)
where the unknown functions
\[
\begin{align*}
&\bar{f}_i(y, y(t - d(t))) = \frac{\partial f_i(y, s)}{\partial s} |_{s = y(t - d(t))} \\
&\bar{g}_i(y, y(t - d(t))) = \frac{\partial g_i(y, s)}{\partial s} |_{s = y(t - d(t))}
\end{align*}
\]
with $0 < \vartheta_{f_i}, \vartheta_{y_i} < 1$. These unknown nonlinear functions will be lumped in a suitable unknown function that is compensated using only an NN in this paper.

The objective of this paper is to design an output-feedback control scheme by using only an NN such that the closed-loop system is asymptotically stable in probability while all states converge to zero in probability. To this end, we rewrite the system in (10) into the following matrix form:

$$
\begin{aligned}
\begin{cases}
    dx = (Ax + \Phi(x) + F(y, y(t - d(t))) + Bu) \, dt \\
y = Cx
\end{cases}
\end{aligned}
$$

where

$$
A = \begin{bmatrix}
    a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\
a_{2,1} & a_{2,2} & a_{2,3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n}
\end{bmatrix}
$$

$$
\Phi(x) = \begin{bmatrix}
    \phi_1(x_1) \\
    \phi_2(x_2) \\
    \vdots \\
    \phi_n(x_n)
\end{bmatrix}
$$

$$
F(\cdot, \cdot) = \begin{bmatrix}
    f_1(\cdot, \cdot) \\
    f_2(\cdot, \cdot) \\
    \vdots \\
    f_n(\cdot, \cdot)
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
    0 \\
    \vdots \\
    0
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
    1 \\
    0 \\
    \vdots \\
    0
\end{bmatrix}^T
$$

Noticing (11) and (12), the unknown time-delay vector-valued functions $F(y, y(t - d(t)))$ and $G(y, y(t - d(t)))$ can further be written as

$$
\begin{aligned}
    F(y, y(t - d(t))) &= y(t - d(t)) \bar{F}(y, y(t - d(t))) \\
    G(y, y(t - d(t))) &= y(t - d(t)) \bar{G}(y, y(t - d(t)))
\end{aligned}
$$

where

$$
\bar{F}(y, y(t - d(t))) = \begin{bmatrix}
    \bar{f}_1(y, y(t - d(t))) \\
    \vdots \\
    \bar{f}_n(y, y(t - d(t)))
\end{bmatrix}
$$

$$
\bar{G}(y, y(t - d(t))) = \begin{bmatrix}
    \bar{g}_1(y, y(t - d(t))) \\
    \vdots \\
    \bar{g}_n(y, y(t - d(t)))
\end{bmatrix}
$$

The main results of this paper are based on the following further assumptions.

**Assumption 2:** There exist a matrix $H$ and a known vector-valued function $J(x)$ such that $\Phi(x) = HJ(x)$, where $J(x)$ satisfies

$$
\frac{\partial J(x)}{\partial x} + \left(\frac{\partial J(x)}{\partial x}\right)^T \geq 0 \quad \forall x \in \mathbb{R}^n.
$$

**Assumption 3:** Matrices $A$ and $H$ satisfy the following linear matrix inequality (LMI):

$$
\begin{bmatrix}
    (A + LC)^T P + P(A + LC) + Q PH + (I + KC)^T \\
    H^T P + (I + KC)
\end{bmatrix} \leq 0
$$

where $P = P^T > 0$, $Q = Q^T > 0$, $K = [k_1, \ldots, k_n]^T$, and $L = [l_1, \ldots, l_n]^T$.

**Remark 2:** Assumption 2 also implies that $\phi_i(x_i)$ in system (10) satisfies the multivariable analog of the monotonicity property described in [44]. From this assumption together with Assumption 3, we can construct a nonlinear observer applying the circle criterion [44]. Moreover, Assumption 2 does not require that $\phi_i(x_i)$ must satisfy the linear growth condition, which is different from [21].

**Remark 3:** It is easily verified that the LMI [see (17)] in Assumption 3 is equivalent to the following inequality and equality:

$$
\begin{bmatrix}
    (A + LC)^T P + P(A + LC) - Q \\
    PH - (I + KC)^T
\end{bmatrix} \leq 0
$$

Obviously, conditions (18) and (19) are more easily verified than condition (17).

**IV. ADAPTIVE NN CONTROL DESIGN**

In this section, we will employ the circle criterion [44] to design a nonlinear observer for system (13) and utilize the adaptive backstepping technique to design the adaptive NN controller. To simplify the design procedure, some derivations are omitted, and only the main design procedures are given.

**A. Nonlinear Observer Design**

Using the circle criterion [44], we design the following nonlinear delay-independent observer for system (13):

$$
d\hat{x} = (Ax + L(C\hat{x} - y) + \Phi(x) + K(C\hat{x} - y) + Bu) \, dt
$$

where $K$ and $L$ satisfy the LMI (17) in Assumption 3.

Define the observer error $\hat{x} = x - \hat{x}$. From (13) and (20), it follows that

$$
d\hat{x} = ((A + LC)\hat{x} + \Phi(x) - \Phi(v) + F(y, y(t - d(t)))) \, dt + G(y, y(t - d(t))) \, d\omega
$$

where $v = \hat{x} + K(C\hat{x} - y)$.

**Theorem 1:** Consider the following Lyapunov candidate for the observer error system (21):

$$
V_0 = \frac{1}{2} (\hat{x}^T P\hat{x})^2
$$
where $b$ is a positive constant, and then, $\mathcal{L}V_0$ is bounded by
\[
\mathcal{L}V_0 \leq \left(-b\lambda_{\min}(P)\lambda_{\min}(Q) + \frac{3b}{2} \left(\frac{4}{3}P^{\frac{4}{3}} + 3bn\sqrt{n}c_2^2\right)|\dot{x}|^4 + y^4(t - d(t)) \right) \Psi_1(y, y(t - d(t)))
\]
where $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are the design constants, and
\[
\Psi_1(y, y(t - d(t))) = \frac{b}{2\epsilon_1^3} |\dot{F}(y, y(t - d(t)))|^4 + \frac{3bn\sqrt{n}}{\epsilon_2^2} |\ddot{G}(y, y(t - d(t)))|^4.
\]

**Proof:** Define $\mu = x - v$ and $\Lambda(x, \mu) = J(x) - J(x - \mu)$. From this, together with $\Phi(x) = HJ(x)$ by Assumption 2, it follows that
\[
\Phi(x) - \Phi(v) = H(J(x) - J(x - \mu))
\]
and then, by the mean value theorem, we have $\Lambda(x, \mu) = \int_0^1 (\partial J/\partial x)|_{x=x-\tau \mu}d\tau$, which, together with Assumption 2, implies that
\[
\mu^T \Lambda(x, \mu) = \frac{1}{2} \mu^T \left( \int_0^1 \left[ \frac{\partial J}{\partial x} + \left( \frac{\partial J}{\partial x} \right)^T \right]_{x=x-\tau \mu} d\tau \right) \mu \geq 0.
\]

Then, we have
\[
\ddot{x}^T(P(\Phi(x) - \Phi(v))) = \ddot{x}^T PH\Lambda(x, \mu)
\]
and
\[
= -\mu^T \Lambda(x, \mu) \leq 0.
\]

Along the trajectory of (21), one has
\[
\mathcal{L}V_0 = b(\ddot{x}^T P \ddot{x} + 2\dddot{x}^T P \dddot{x} + 2\dddot{x}^T P \dddot{x} P F) + 2b\text{Tr} \left\{ G^T (2P\dddot{x}^T P + \dddot{x}^T P \dddot{x}) G \right\}.
\]

Substituting (18) and (27) into (28) yields
\[
\mathcal{L}V_0 \leq -b(\ddot{x}^T P \ddot{x} + 2\dddot{x}^T P \dddot{x} P F) + 2b\text{Tr} \left\{ G^T (2P\dddot{x}^T P + \dddot{x}^T P \dddot{x}) G \right\}.
\]

Using Young’s inequality [see (7)], together with (14) and (15), we have
\[
2b(\ddot{x}^T P \dddot{x}) \leq 2b|P|^2 |\dot{x}|^4|P|^{\frac{4}{3}} \leq 3b\epsilon_1^3 |\dot{x}|^4 |\dot{x}|^{4/3}
\]
and
\[
2b(\ddot{x}^T P \dddot{x}) \leq 2b|P|^2 |\dot{x}|^4|P|^{\frac{4}{3}} \leq 3b\epsilon_1^3 |\dot{x}|^4 |\dot{x}|^{4/3}
\]
where $\epsilon_1$ and $\epsilon_2$ are the positive design constants. The detailed derivation of the inequality (31) is similar to [12, eq. (A.7)]. Substituting (30) and (31) back into (29) yields (23).

**B. Controller Design**

We give the following overall system consisting of the first equation of the system (10) and the observer (20):
\[
dy = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}\bar{x}_2 + a_{1,2}\bar{x}_2 + \varphi_1(y) \\ + f_1(y, y(t - d(t))) \\ + g_1(y, y(t - d(t))) dt + g_1(y, y(t - d(t))) d\omega \\ + \varphi_1(\bar{x}_1 - \bar{k}_1x_1) dt, \quad i = 2, \ldots, n - 1 \\ + \varphi_1(\bar{x}_1 - \bar{k}_1x_1) dt \end{bmatrix}
\]
where $\bar{x}_i = [\bar{x}_1, \ldots, \bar{x}_i]^T$, and $\bar{k}_i = [k_1, \ldots, k_i]^T$. Obviously, the controller of the aforementioned system can be designed by the backstepping technique. Following the adaptive backstepping design idea, we define the following coordinate transformation:
\[
z_i = y = x_1
\]
\[
z_i = \bar{x}_i - a_{i-1}(y, \bar{x}_{i-1}, \hat{W}, \hat{\theta}), \quad i = 2, \ldots, n
\]
where $\hat{W}$ and $\hat{\theta}$ are the estimates of the unknown NN weight vector $W$ and the unknown upper bound of NN approximation error (see (51) later), respectively, and $a_i$’s are the stabilizing functions to be designed later. Under the transformation (33) and (34), system (32) is changed into (35), shown at the bottom of the next page, where for $i = 2, \ldots, n$, we define $\Delta_i$ as follows:
\[
\Delta_i = \sum_{j=1}^i a_{i,j} \bar{x}_j - l_{i-1} \bar{x}_1 + \varphi_i(\bar{x}_1 - \bar{k}_1x_1)
\]
\[
- \frac{\partial a_{i-1}}{\partial y} \left( a_{1,1}x_1 + a_{1,2}\bar{x}_2 + \varphi_1(y) \right)
\]
\[
- \frac{\partial a_{i-1}}{\partial x} \left( \sum_{j=1}^{m-1} a_{m,j} \bar{x}_j - l_{m-1} \bar{x}_1 + \varphi_m(\bar{x}_m - \bar{k}_m \bar{x}_1) \right)
\]
\[
- \frac{\partial a_{i-1}}{\partial \hat{W}} \hat{W} - \frac{\partial a_{i-1}}{\partial \hat{\theta}} \hat{\theta}.
\]
From (35), we design the stabilizing functions and the control law as (36), shown at the bottom of the page, where $\Xi_t$ is defined as

$$\Xi_t = \frac{3}{4} (\eta_t, a_{1,2})^{4/3} \left( \frac{\partial \alpha_{t-1}}{\partial y} \right)^{4/3} + \frac{3}{4} \xi_t \left( \frac{\partial \alpha_{t-1}}{\partial y} \right)^3 + \frac{1}{4 \delta_t^{1/3}}, \quad i = 2, \ldots, n. \quad (37)$$

ci, $\lambda_i, \delta_i, \eta_i$, and $\xi_i$ are positive design parameters, and $S(y)$ is the vector-valued basis function, and the adaptive laws are designed as

$$\dot{\gamma} = \gamma \psi(y) y^4 \quad \dot{W} = \Gamma S(y) y^4 \quad (38)$$

where $\gamma > 0$ and $\Gamma > 0$ are the adaptive gains.

Remark 4: It is well known that the adaptive law [see (38)] may demonstrate divergence (also called drift) of the tunable parameters in the case of the presence of measurement errors. To overcome this drawback, many good methods are utilized, such as the $\sigma$ modification technique [45]. Based on this method, the adaptive law [see (38)] will be modified as

$$\dot{\gamma} = \gamma \left[ \psi(y) y^4 - \sigma \right] \quad \dot{W} = \Gamma \left[ S(y) y^4 - \sigma W \right]$$

where $\gamma > 0$ and $\Gamma > 0$ are defined as in (38), and $\sigma > 0$ is a small constant. The foregoing modification is referred to as $\sigma$ modification [45] or as leakage. This modification method has widely been used in the existing work, for example, adaptive control for systems described by the ordinary differential equation [28]–[33], the discrete difference equation [34]–[37], the time-delay functional differential equation [38]–[42], or the Itô stochastic differential equation [15]. To simplify the analysis and avoid repeating the previous work, in this paper, we only consider the ideal case of no measurement errors. For the case of the presence of measurement errors, the readers may be referred to [15].

Substituting (36) into (35) results in (39), shown at the bottom of the next page.

### C. Stability Analysis

Consider the following Lyapunov function for system (39):

$$V_1 = \frac{1}{4} y^4 + \frac{1}{4} \sum_{i=2}^{n} \Sigma_i^4 + \frac{1}{2} \hat{W}^{T} \Gamma^{-1} \hat{W} + \frac{1}{2} \gamma^{-1} \hat{\theta}^2 \quad (40)$$

where $b$ is a positive design constant, and $\hat{W} = W - \bar{W}$ and $\hat{\theta} = \theta - \bar{\theta}$ denote the estimate errors of $W$ and $\theta$, respectively. Along the solutions of (38) and (39), we have

$$\mathcal{L} V_1 = - c_1 y^4 - y^4 \left( \hat{W}^{T} S(y) + \psi(y) \hat{\theta} \right)$$

$$+ \frac{3}{4} y^2 q_1(y, y(t-d(t))) g_0^T(y, y(t-d(t)))$$

$$+ \frac{3}{2} y^2 q_1(y, y(t-d(t))) g_0^T(y, y(t-d(t)))$$

$$- \frac{1}{2} \sum_{i=2}^{n} \Sigma_i^4 \left( \frac{\partial^2 \alpha_{i-1}}{\partial y^2} \right) g_1^T g_1^T$$

$$+ \frac{3}{2} \sum_{i=2}^{n} \Sigma_i^2 \left( \frac{\partial \alpha_{i-1}}{\partial y} \right)^2 g_1^T g_1^T$$

$$\mathcal{L} V_1 = - \hat{W}^{T} S(y) y^4 - \hat{\theta} \psi(y) y^4. \quad (41)$$

\begin{align*}
\begin{cases}
\frac{dy}{dt} = [a_{1,2} \alpha_1 + a_{1,2} \bar{z}_2 + a_{1,1} x_1 + a_{1,2} \bar{x}_2 + \varphi_1(y) + f_1(y, y(t-d(t)))] dt + g_1(y, y(t-d(t))) d\omega \\
\frac{dz_i}{dt} = [a_{i+1,2} \alpha_1 + a_{i+1,1} \bar{z}_i + a_{1,2} \bar{x}_2 + \varphi_1(y) + f_1(y, y(t-d(t)) - \frac{1}{2} \left( \frac{\partial^2 \alpha_{i-1}}{\partial y^2} \right) g_1^T g_1^T] dt \\
\frac{dz_n}{dt} = [a_{2,2} \alpha_1 + a_{1,1} x_1 + a_{1,2} \bar{x}_2 + \varphi_1(y) + f_1(y, y(t-d(t)))] dt + g_1(y, y(t-d(t))) d\omega
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
\alpha_1 = \frac{1}{a_{1,2}} \left[ -c_1 y - a_{1,1} x_1 - \varphi_1(y) - y \left( \hat{W}^{T} S(y) + \psi(y) \hat{\theta} \right) \right] \\
\alpha_i = \frac{1}{a_{i+1,1}} \left[ -c_i \bar{z}_i - \Xi_i \bar{z}_i - \frac{1}{4} \lambda_i \left( \frac{\partial \alpha_{i-1}}{\partial y} \right)^2 \bar{z}_i^2 - \frac{1}{4} \lambda_i (a_{i+1,1} \bar{\theta}_i) \frac{1}{4} \bar{z}_i - \Delta_i \right], \quad i = 2, \ldots, n - 1 \\
u = \frac{1}{\rho} \left[ -c_n \bar{z}_n - \Xi_n \bar{z}_n - \frac{1}{4} \lambda_n \left( \frac{\partial \alpha_{n-1}}{\partial y} \right)^2 \bar{z}_n - \Delta_n \right]
\end{cases}
\end{align*}
Using Young’s inequality [see (7)] and noting (11) and (12), each underlined term in (41) is dealt with as follows:

I:
\[
y^3(a_{1,2} \bar{x}_2 + a_{1,2} \tilde{x}_2 + f_1) 
\leq \frac{3}{4}(\delta_1 a_{1,2})^{4/3} y^{4} + \frac{3}{4}(\epsilon_3 a_{1,2})^{4/3} y^{4} + \frac{3}{4}(\epsilon_1 a_{1,2})^{4/3} y^{4} + \frac{1}{4} \delta_1 \bar{x}_2^2 + \frac{1}{4} \epsilon_1 |\tilde{x}|^4 
+ \frac{1}{4} \epsilon_1 y^4 (t - d(t)) \left| \tilde{f}_1 (y, y (t - d(t))) \right|^4. \quad (42)
\]

II:
\[
\frac{3}{2} y^2 g_1 (y, y (t - d(t))) g_1^T (y, y (t - d(t))) 
\leq \frac{3}{4} y^4 + \frac{3}{4} y^4 (t - d(t)) |\tilde{g}_1 (y, y (t - d(t)))|^4. \quad (43)
\]

III:
\[
\sum_{i=2}^{n-1} a_{i,i+1} z_{i+1}^2 \leq \frac{3}{4} \sum_{i=2}^{n-1} (\delta_i a_{i,i+1})^{4/3} z_i^4 + \frac{1}{4} \sum_{i=3}^{n} \frac{1}{\eta_i} z_i^4. \quad (44)
\]

IV:
\[
- \sum_{i=2}^{n} z_i^3 \frac{\partial \alpha_{i-1}}{\partial y} (a_{1,2} \bar{x}_2 + f_1 (y, y (t - d(t)))) 
\leq \frac{3}{2} \sum_{i=2}^{n} (\eta_i a_{1,2})^{4/3} \left( \frac{\partial \alpha_{i-1}}{\partial y} \right)^{4/3} z_i^4 + \frac{1}{4} \sum_{i=2}^{n} \frac{1}{\eta_i} |\tilde{x}|^4 
+ \frac{1}{4} y^4 (t - d(t)) \sum_{i=2}^{n} \frac{1}{\eta_i} |\tilde{g}_1 (y, y (t - d(t)))|^4. \quad (45)
\]

V:
\[
- \frac{1}{2} \sum_{i=2}^{n} z_i^3 \left( \frac{\partial^2 \alpha_{i-1}}{\partial y^2} \right) g_1 g_1^T \leq \frac{1}{4} \sum_{i=2}^{n} \lambda_i \left( \frac{\partial^2 \alpha_{i-1}}{\partial y^2} \right)^2 z_i^6 
+ \frac{1}{4} y^4 (t - d(t)) \sum_{i=2}^{n} \frac{1}{\lambda_i} |\tilde{g}_1 (y, y (t - d(t)))|^4. \quad (46)
\]

VI:
\[
\frac{3}{2} \sum_{i=2}^{n} z_i^2 \left( \frac{\partial \alpha_{i-1}}{\partial y} \right)^2 g_1 g_1^T \leq \frac{3}{4} \sum_{i=2}^{n} \xi_i \left( \frac{\partial \alpha_{i-1}}{\partial y} \right)^4 z_i^4 
+ \frac{3}{4} y^4 (t - d(t)) \sum_{i=2}^{n} \frac{1}{\xi_i} |\tilde{g}_1 (y, y (t - d(t)))|^4. \quad (47)
\]

Substituting (42)–(47) back into (41) leads to
\[
\mathcal{L} V_1 \leq \left( \frac{1}{4} \sum_{i=2}^{n} \frac{1}{\eta_i} + \frac{1}{4} \epsilon_3 \right) |\bar{x}|^4 - \sum_{i=1}^{n} c_i z_i^4 
- y^4 (W^T S(y) + \psi(y) \theta) 
+ y^4 \beta + y^4 (t - d(t)) \Psi_2 (y, y (t - d(t))) \quad (48)
\]

where
\[
\beta = \frac{3}{4} (\delta_1 a_{1,2})^{4/3} + \frac{3}{4} (\epsilon_3 a_{1,2})^{4/3} + \frac{3}{4} (\epsilon_1 a_{1,2})^{4/3} + \frac{3}{4} \epsilon_5 
\Psi_2 (y, y (t - d(t))) 
= \frac{1}{4} \sum_{i=2}^{n} \frac{1}{\eta_i} |\tilde{f}_1 (y, y (t - d(t)))|^4 
+ \frac{1}{4} \sum_{i=2}^{n} \frac{1}{\eta_i} |\tilde{g}_1 (y, y (t - d(t)))|^4 
+ \frac{3}{4} \sum_{i=2}^{n} \frac{1}{\eta_i} |\tilde{g}_1 (y, y (t - d(t)))|^4.
\]

Now, we consider the following positive definite function for the whole closed-loop system:
\[
V = V_0 + V_1 + \frac{1}{1 - \zeta} \int_{t-d(t)}^{t} y^4 (\tau) \Psi (y, y (\tau)) \, d\tau \quad (49)
\]

where the positive function \( \Psi (y, y (\tau)) = \Psi_1 (y, y (\tau)) + \Psi_2 (y, y (\tau)). \) From (23) and (48), we have
\[
\mathcal{L} V \leq \left( -b \lambda_{\min} (P) \lambda_{\min} (Q) + \frac{3b}{2} \epsilon_3^{4/3} |P|^{8/3} \right) 
+ 3bn \sqrt{\pi} \epsilon_2 |P|^4 + \frac{1}{4} \sum_{i=2}^{n} \frac{1}{\eta_i} + \frac{1}{4} \epsilon_3^4 |\bar{x}|^4 
- \sum_{i=1}^{n} c_i z_i^4 
- y^4 (W^T S(y) + \psi(y) \theta) 
+ y^4 \left( \beta + \frac{\Psi (y, y)}{1 - \zeta} \right). \quad (50)
\]
The unknown function \( \beta + (\Psi(y, y)/1 - \zeta) \) can be approximated by an NN as follows:
\[
\beta + \frac{\Psi(y, y)}{1 - \zeta} = W^T S(y) + \sigma(y) \quad (51)
\]
with the approximation error \( |\sigma(y)| \leq \psi(y) \theta \). Substituting (51) into (50) results in
\[
\mathcal{L}V \leq \left( - b\lambda_{\min}(P)\lambda_{\min}(Q) + \frac{3b}{2} c_1^{4/3} |P|^{8/3} \right.
\]
\[
+ 3b \sqrt{n} \epsilon_2 |P|^4 + \frac{1}{4} \sum_{i=2}^{n} \frac{1}{\eta_i} + \frac{1}{4c_3} \right) |\dot{x}|^2 - \sum_{i=1}^{n} c_i z_i^4.
\]
(52)

Given \( 0 < \nu < 1 \), the parameters \( \epsilon_1, \epsilon_2, \epsilon_3, \eta_i \), and \( b \) are selected so that the following inequality holds:
\[
b\lambda_{\min}(P)\lambda_{\min}(Q) + \frac{3b}{2} c_1^{4/3} |P|^{8/3} + 3b \sqrt{n} \epsilon_2 |P|^4
\]
\[
+ \frac{1}{4} \sum_{i=2}^{n} \frac{1}{\eta_i} + \frac{1}{4c_3} \leq -\nu \quad (53)
\]
and we have
\[
\mathcal{L}V \leq -\nu |\dot{x}|^2 - \sum_{i=1}^{n} c_i z_i^4. \quad (54)
\]

The main results are stated as follows.

**Theorem 2:** Under Assumptions 1–3, consider the closed-loop system consisting of system (10), observer (20), control law (36), and adaptive laws (38). For bounded initial conditions, the closed-loop system has a unique solution on \([-d, \infty)\), and the closed-loop equilibrium of interest is globally stable in probability. Moreover, the solution process can be regulated to the origin almost surely, i.e., \( P(\lim_{t \to \infty} |x(t)| = 0) = 1 \).

**Proof:** Similar to [23], from (54) together with Lemma 1, the conclusion is obvious. \( \square \)

V. SIMULATION STUDY

In this section, we give three simulation examples to illustrate the proposed control method.

**Example 1:** Consider the following second-order system:
\[
\begin{aligned}
\dot{x}_1 &= \left[ 0.5x_1 + 1.5x_2 - x_1^3 + f_1(y, y(t - d(t))) \right] dt + g_1(y, y(t - d(t))) dw \\
\dot{x}_2 &= \left[ u + 2x_1 - 2x_2 + x_1^2 - x_2^2 + f_2(y, y(t - d(t))) \right] dt + g_2(y, y(t - d(t))) dw \\
y &= x_1
\end{aligned}
\]
(55)

where
\[
\begin{aligned}
f_1(y, y(t - d(t))) &= yy(t - d(t)) \\
g_1(y, y(t - d(t))) &= y(t - d(t)) / (1 + y^2) \\
f_2(y, y(t - d(t))) &= \sin(yy(t - d(t))) \\
g_2(y, y(t - d(t))) &= y(t - d(t)) \ln(1 + y^2)
\end{aligned}
\]
are unknown functions satisfying Assumption 2. The time-varying delay is specified as \( d(t) = 1 + 0.8 \sin(t) \). Compared with system (13), it can easily be shown that
\[
A = \begin{bmatrix} 0.5 & 1.5 \\ 2 & -2 \end{bmatrix}
\]
\[
\Phi(x) = \begin{bmatrix} -x_1^3 \\ x_1^2 - x_2^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} = HJ(x)
\]
where \( J(x) \) satisfies Assumption 2. It is easy to verify that, when
\[
P = \begin{bmatrix} 1.5 & 1 \\ 1 & 1 \end{bmatrix} \
L = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \
K = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}
\]
the LMI (17) holds. Based on the control method proposed in Section III, the nonlinear observer is designed as
\[
\begin{aligned}
\dot{x}_1 &= 0.5\dot{x}_1 + 1.5\dot{x}_2 - (\dot{x}_1 - y) - 51(\dot{x}_1 - y)^3 \\
\dot{x}_2 &= 2\dot{x}_1 - 2\dot{x}_2 - 2(\dot{x}_1 - y) + [\dot{x}_1 - 0.5(\dot{x}_1 - y)]^5 \\
&\quad + [\dot{x}_2 + (\dot{x}_1 - y)]^5 + u
\end{aligned}
\]
(56)
and the controller is given by
\[
u = -\left[ c_2 + \frac{3}{2} (1.5\eta_2)^{4/3} \frac{\partial \alpha_1}{\partial y} \right]^{4/3} + 3 \frac{\partial \alpha_1}{\partial y} + \frac{1}{4\delta_1} z_2^4
\]
\[
- \frac{1}{4} \lambda_2^2 \frac{\partial \alpha_1}{\partial y^2} z_2^3 - 2\dot{x}_1 + 2\dot{x}_2 + 2(\dot{x}_1 - y)
\]
\[
+ \frac{1}{4} \frac{\partial \alpha_1}{\partial y} (1.5\dot{x}_2 + 0.5y - y^3) + \frac{\partial \alpha_1}{\partial W} W + \frac{\partial \alpha_1}{\partial \theta} \theta
\]
(57)

with the stabilizing function
\[
\alpha_1 = \frac{2}{3} (-c_1 y - 0.5y + y^3 - y \left[ W^T S(y) + \hat{\theta} \psi(y) \right])
\]
where \( S(y) = [s_1(y), \ldots, s_L(y)]^T \).

In the simulation, the design parameters are chosen as \( c_1 = c_2 = 0.5 \) and \( \delta_1 = \lambda_2 = \eta_2 = \xi_2 = 1 \). The initial conditions are set to be \( y(t) = 0.5, t \in [-d, 0], x_2(0) = 0, \dot{x}_1(0) = 0, \dot{x}_2(0) = 0, \dot{W}(0) = 0, \) and \( \dot{\theta}(0) = 0 \). The basis functions are specified as \( s_i(y) = \exp(-y - \mu_i^2/2) \), \( i = 1, \ldots, L \), with the node number \( l = 9 \), the width \( \zeta = 0.5 \), and the centers \( \mu_i \) evenly spaced in \([-2, 2]\). It is assumed that \( \psi(y) = 1 \). The parameter adaptive laws are given by (38) with the adaptive gains \( \gamma = 1 \) and \( \Gamma = 0.5I \).

The simulation results are shown in Fig. 1, from which we can see that the controller renders the resulting closed-loop system asymptotically stable and the limits of the estimated parameters exist and are finite.
In fact, this example is an extension of the example studied in [46] cannot be applied to solving the aforementioned control problem is easily solved. First, the observer is designed as follows:

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 - \frac{9}{2}(\hat{x}_1 - y) \\
\dot{\hat{x}}_2 &= -\hat{x}_2 + \hat{x}_3 - \frac{1}{2}x_2^3 - \frac{15}{8}(\hat{x}_1 - y) \\
\dot{\hat{x}}_3 &= u - \frac{1}{2}\hat{x}_3 - \frac{1}{2}\hat{x}_2^2\hat{x}_3 - \frac{15}{8}(\hat{x}_1 - y)
\end{align*}
\]

and the controller is given by

\[
u = -\left[c_3 + \frac{3}{2}(1.5\eta_3)^{4/3} \left( \partial\alpha_2 \over \partial y \right)^{4/3} + \frac{3}{4c_2^3} \left( \partial\alpha_2 \over \partial y \right)^4 + \frac{1}{4\delta_3^2} \right] z_3
\]

with the error variables \(z_2 = \hat{x}_2 - \alpha_1\) and \(z_3 = \hat{x}_3 - \alpha_2\), and the stabilizing functions are given as follows:

\[
\begin{align*}
\alpha_1 &= -c_1 y - y \left[\hat{W}^T S(y) + \hat{\theta}\psi(y)\right] \\
\alpha_2 &= -\left[c_2 + \frac{3}{2}(1.5\eta_2)^{4/3} \left( \partial\alpha_1 \over \partial y \right)^{4/3} + \frac{3}{4c_2^3} \left( \partial\alpha_1 \over \partial y \right)^4 + \frac{1}{4\delta_3^2} \right] z_2 \\
&- \frac{1}{4}\lambda_2 \left( \partial\alpha_1 \over \partial y \right)^2 z_2 - \frac{3}{4}\delta_2^{4/3} z_2 + \hat{x}_2 + \frac{1}{3} \hat{x}_2^3 \\
&+ \frac{15}{4}(\hat{x}_1 - y) + \hat{\alpha}_1 \hat{y} + \hat{\theta}_2 \hat{W} + \hat{\theta}_3 \hat{\theta}
\end{align*}
\]

where \(S(y)\) is defined as before.

In the simulation, we take the design parameters \(c_1 = c_2 = c_3 = 0.5\) and \(\delta_1 = \delta_2 = \delta_3 = \lambda_3 = \eta_3 = \eta_3 = \xi_2 = \xi_3 = 1\). The initial conditions are set to be \(y(t) = 0.5, t \in [-d, 0]\), \(x_2(0) = -0.5, x_3(0) = 0, \hat{x}_1(0) = 0, \hat{x}_2(0) = 0, \hat{x}_3(0) = 0, W(0) = 0, \) and \(\hat{\theta}(0) = 0\). The basis functions \(s_i(y)\) are designed as in Example 1. It is still assumed that \(\psi(y) = 1\). The parameter adaptive laws are given by (38) with the adaptive gains \(\gamma = 1\) and \(\Gamma = 0.5\). The simulation results are shown in Fig. 2, from which we can see that the control performance is still very well.

**Example 3:** In this example, we consider a one-link manipulator with the inclusion of motor dynamics and stochastic
disturbances to illustrate the application of the control method proposed in this paper. The dynamic equation of such system is given by [47]
\[
\begin{align*}
D\dot{q} + B\dot{q} + N\sin(q) &= \tau + \tau_d \\
M\ddot{\theta} + H_m\dot{\theta} &= u - K_m\dot{q}
\end{align*}
\]
where \(q, \dot{q}\), and \(\ddot{q}\) denote the link position, velocity, and acceleration, respectively. \(\tau\) is the torque produced by the electrical subsystem, and \(\tau_d = g(q, q(t - d(t)))\dot{w}\), where \(g(q, q(t - d(t))) = \sin(yy(t - d(t)))\) represents the uncertainty with delay \(d(t) = 2 + 0.5\cos(t)\), and \(w\) represents the torque stochastic disturbance defined as in system (10). \(u\) is the control input used to represent the electromechanical torque. \(D = 1\) kg m\(^2\) s\(^{-1}\) is the mechanical inertia, \(B = 1\) Nm s rad\(^{-1}\) is the coefficient of viscous friction at the joint, \(N = 10\) is a positive constant related to the mass of the load and the coefficient of gravity, \(M = 0.1\) H is the armature inductance, \(H = 1.03\) is the armature resistance, and \(K_m = 0.2\) Nm/A is the back electromotive force coefficient. Let \(x_1 = q\), \(x_2 = \dot{q}\), and \(x_3 = \dot{\dot{\theta}}\), and then, we can express the preceding equation in the form of (10) as follows:
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-\sin(x_1) - g(y, y(t - d(t)))}{y} \\
\dot{x}_3 &= 2x_2 - 10x_3 + 10u
\end{align*}
\]
where \(H = -I\), and \(J(x) = [0, 0, 0]^T\). It is easy to verify that, when
\[
P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
the LMI (17) holds. Based on the proposed control scheme, the foregoing control problem is easily solved. First, the observer is designed as follows:
\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 - (\hat{x}_1 - y) \\
\dot{\hat{x}}_2 &= -\hat{x}_2 + \hat{x}_3 + 0.5(\hat{x}_1 - y) \\
\dot{\hat{x}}_3 &= 10u - 2\hat{x}_2 - 10\hat{x}_3 + (\hat{x}_1 - y)
\end{align*}
\]
and the controller is given by
\[
u = -\frac{1}{10}\begin{bmatrix} c_3 + \frac{3}{2}(1.5\eta_3)^{1/3} \left(\frac{\partial^2\alpha_2}{\partial y^2}\right)^{4/3} \\
+ \frac{3}{4\xi_3^2} \left(\frac{\partial^2\alpha_2}{\partial y^2}\right)^4 + \frac{1}{4\delta_2^3} \end{bmatrix} z_3
\]
where \(S(y)\) is defined as before. In the simulation, the design parameters are still chosen as \(c_1 = c_2 = c_3 = 0.5\) and \(\delta_1 = \delta_2 = \lambda_2 = \lambda_3 = \eta_2 = \eta_3 = \xi_2 = \xi_3 = 1\), and the initial conditions are set to be \(y(t) = 0.5, t \in [-d, 0], x_2(0) = -0.5, x_3(0) = 0, \hat{x}_1(0) = 0, \hat{x}_2(0) = 0, \hat{x}_3(0) = 0, \hat{W}(0) = 0, \) and \(\partial(0) = 0\). The basis functions \(s_i(y)\) are designed as before. It is still assumed that \(\psi(y) = 1\). The parameter adaptive laws are given by (38) with the adaptive gains \(\gamma = 1\) and \(\Gamma = 0.5I\). The simulation results are shown
in Fig. 3, from which we can see that the control performance is still very well.

The simulation results of the foregoing examples accord with the main results given in Theorem 2, which sufficiently demonstrates the effectiveness of the control method proposed in this paper.

VI. CONCLUSION

In this paper, we have investigated the adaptive NN control problem for a more general class of strict-feedback stochastic nonlinear systems with time-varying delays. The nonlinear observer is introduced to remove the strict restrictions on nonlinear functions, such as linear growth condition [22]. Only an NN is employed to compensate for all system uncertainties, which further relaxes the assumptions on nonlinear delayed functions while simplifying the adaptive NN control design procedure.

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REFERENCES


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