Some non-trivial families of symplectic structures

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1. Introduction

It has recently been shown by Seidel [10] that many symplectic 4-manifolds admit symplectic automorphisms which are differentiably isotopic to the identity but which cannot be deformed to the identity through symplectomorphisms. Seidel detected this phenomenon using the Floer homology groups that are associated to pairs of Lagrangian surfaces. The purpose of this note is to show that a related but weaker manifestation of the same phenomenon can be detected using invariants derived from the Seiberg-Witten equations.

Let (X, ω_0) be a closed symplectic manifold. Let Diff be the group of diffeomorphisms of X isotopic to the identity, and let Symp be the subgroup consisting of symplectomorphisms. The map

$$\Psi : f \mapsto (f^{-1})^*(\omega_0)$$

gives an injection from the quotient space Diff/Symp to the space

$$\Lambda_0 = \{\omega \in \Omega^2(X) \mid \omega \text{ is symplectic and cohomologous to } \omega_0\}.$$

The basic result of Moser [8] states that a family of symplectic forms is locally trivial provided only that the cohomology class of the forms is constant. It follows that the image of Ψ is the connected component of Λ_0 containing ω_0. Whereas the results of [10] provide examples where Symp is disconnected, the phenomenon detected by the Seiberg-Witten equations is the non-trivial fundamental group of Diff/Symp, or equivalently of Λ_0.

The Seiberg-Witten equations, in fact, furnish a simple tool for detecting homology in such a space of symplectic forms. For each positive integer n, we shall exhibit a 4-dimensional example (X_n, ω_n) with Diff/Symp having non-trivial homology in dimension $2n - 1$. More particularly, there is a locally trivial family of

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symplectic forms on X_n, parameterized by a sphere S^{2n-1}, which is not the boundary of any locally trivial $2n$-dimensional family, despite the fact that the family is homotopically trivial as a family of non-degenerate 2-forms. The principle is a simple extension of the one established in [12], whereby the basic classes of a 4-manifold, defined using the Seiberg-Witten equations, constrain the cohomology class of a symplectic form.

The above examples arise as algebraic surfaces of general type, and for $n = 1$ they coincide with the sorts of examples considered in [10]. As a particular case, and following [10], one may consider a smooth quintic surface in $\mathbb{CP}^3$, carrying the symplectic form obtained by restricting the Kähler form of the Fubini-Study metric. As symplectic manifolds, all smooth quintics are isomorphic. The reason is that, by varying the coefficients of the quintic equation, one may join any two by a family of smooth quintics, and since the cohomology class of the symplectic form is constant along the path, the family must be locally trivial by Moser's result. The same reasoning shows that if one has a family γ of smooth quintic surfaces in $\mathbb{CP}^3$ parameterized by the circle $S^1 = [0, 1]/(0 \sim 1)$, then the monodromy of the family gives rise to a symplectic automorphism of the surface $\gamma(0)$ and that this symplectomorphism is well-defined to within symplectic isotopy. Inside the projective space $\mathbb{P}$ which parameterizes *all* quintics in $\mathbb{CP}^3$, there is a codimension-1 locus Δ which parameterizes singular quintics with a single ordinary double-point. Let δ be a small circle linking Δ in $\mathbb{P}$ (the boundary of a small disk transverse to Δ), and let γ be the family obtained by running around δ twice. The monodromy of this family is a symplectomorphism μ which is isotopic to the identity through diffeomorphisms, but not through symplectomorphisms. The automorphism is described quite explicitly in [10]; it is supported in the neighborhood the Lagrangian 2-sphere which is the vanishing cycle at the double-point. Choosing a particular isotopy from μ to the identity, one obtains an element of Diff/Symp which may be detected by the methods of this paper.

The examples here seem closely related to Gromov's application of pseudo-holomorphic curves in the study of symplectomorphisms of $S^2 \times S^2$ (see [4, 7, 1]).

Acknowledgment. It is rather straightforward to use the Seiberg-Witten equations to detect the phenomenon just described. Before hearing of [10], however, the author was unaware that the phenomenon was there to be detected.

2. Homology in spaces of symplectic forms

Let X be a closed, oriented 4-manifold carrying a Riemannian metric g and a 2-form η which is self-dual with respect to g. Let s be a Spin^c structure and $W^{\pm}$ the two spin bundles. The Seiberg-Witten equations, perturbed by η, are the following pair of equations for a spin-connection A and a section Φ of W^+:

$$\begin{aligned} \rho(F^+_{\hat{A}} + i\eta) - \{\Phi \otimes \Phi^*\} &= 0 \\ D^+_A \Phi &= 0. \end{aligned} \tag{1}$$

Here $\hat{A}$ denotes the connection on $\det W^+$ obtained from A, and ρ denotes Clifford multiplication. The braces denote the trace-free part of the endomorphism $\Phi \otimes \Phi^*$ of W^+. Our conventions here follow [6].

In order to work with Banach spaces rather than spaces of smooth functions, we introduce $R(X)$ to denote the space of all Riemannian metrics g on X of class C^l, and $\Omega^2(X)_k$ for the space of 2-forms of class L^2_k. We fix l and k; the values $l = 5$ and $k = 3$ are suitable, but as usual, other choices of topology are possible. Let $\mathcal{P}$ be the subset of $R(X) \times \Omega^2(X)_k$ consisting pairs (g, η) such that η is self-dual with respect to g:

$$\mathcal{P} = \{(g, \eta) \mid *_g \eta = \eta\}.$$

This space is a Banach manifold, for it is a locally trivial Banach vector bundle over $R(X)$. Inside $\mathcal{P}$ there is a Banach submanifold $\mathcal{P}_{\text{red}}$ consisting of pairs (g, η) for which the corresponding equations (1) have a *reducible* solution, i.e. a solution with $\Phi = 0$. The codimension of $\mathcal{P}_{\text{red}}$ is equal to $b^+(X)$. Let $\mathcal{P}^*$ be the complement of $\mathcal{P}_{\text{red}}$.

There is now a family of equations (1) parameterized by $\mathcal{P}$, and we let $\mathcal{M}$ denote the parameterized space of solutions modulo gauge transformations. The gauge group here is the group $\text{Map}(X, S^1)_{k+2}$ (the maps of class L^2_{k+2}) acting on $W^\pm$ as scalars, and the solutions in question are those for which A and Φ have class L^2_{k+1}. Let $\mathcal{M}^*$ be the part of $\mathcal{M}$ lying above $\mathcal{P}^*$. The basic transversality and compactness results for the Seiberg-Witten equations, described for example in [5], imply that $\mathcal{M}^*$ is a Banach manifold and that the map

$$\pi : \mathcal{M}^* \to \mathcal{P}^*$$

is smooth and proper. The index of π is given by

$$\begin{aligned} \text{ind} &= \frac{1}{4}(c_1(W^+)^2 - 2\chi - 3\sigma) \\ &= c_2(W^+) \end{aligned}$$

where χ and σ are the Euler number and signature of X (see [6]), and the real determinant line of the differential $d\pi$ is an orientable, in other words trivial, line bundle on $\mathcal{M}^*$. A trivialization for the determinant line can be specified by giving a homology orientation for X [3]. We suppose such an orientation is chosen.

We are interested in the case that the index is negative, and we therefore write $\mathrm{ind} = -d$. We impose however the condition that the index is not *too* negative: we require

$$0 < d < b^+(X) - 1. \tag{2}$$

Under this hypothesis, letting $\Delta \subset \mathcal{P}^*$ denote the image of π, we obtain a well-defined homomorphism

$$Q : H_{d-1}(\mathcal{P}^* \setminus \Delta; \mathbb{Z}) \to \mathbb{Z}$$

as follows. Since $\mathcal{P}$ is contractible and $\mathcal{P}_{\mathrm{red}}$ has codimension $b^+(X)$, any closed chain S of dimension $d-1$ in $\mathcal{P}^* \setminus \Delta$ bounds a singular d-chain T in $\mathcal{P}^*$. We can arrange that T is transverse to π, and an integer $(\mathcal{M}^* \cdot T)$ is then obtained by counting the points of $\mathcal{M}^*$ lying over T, using the chosen orientation. We define Q by specifying

$$Q(S) = (\mathcal{M}^* \cdot T).$$

To see that the result is independent of the choice of T, let T' be another choice. The difference $T - T'$ is the boundary of some $(d+1)$-chain U, which can still be taken to avoid $\mathcal{P}_{\mathrm{red}}$ by our hypothesis (2). If U is transverse to π, the pull-back of $\mathcal{M}^*$ over U is a 1-chain whose boundary is the difference of $(\mathcal{M}^* \cdot T)$ and $(\mathcal{M}^* \cdot T')$.

We now apply the above construction in the special case of a 4-manifold X admitting a symplectic structure ω_0 compatible with the orientation (that is, $\omega_0^2 > 0$). Let s_0 be the Spin^c structure determined by ω_0 (see [11] for example), and let s be any other Spin^c structure. There is a difference element $e = s - s_0$ in $H^2(X;\mathbb{Z})$. We continue to suppose that the index $-d$ corresponding to s satisfies (2); in particular, the condition $d \neq 0$ implies that s and s_0 are distinct, so e is non-zero. The formula for d in this setting is

$$d = -e \cdot e - c_1 \cdot e,$$

where c_1 is the first Chern class of an almost-complex structure compatible with ω_0. Define $\mathcal{P}^*$, π and Q as above, using s.

In the space $\Omega^2(X)$ of all 2-forms on X, consider now the subset

$$\begin{aligned}\Lambda &= \Lambda(e, s_0)\\ &= \{\omega \in \Omega^2(X) \mid \omega \text{ is symplectic, } [\omega] \smile e \le 0 \text{ and } s_\omega \cong s_0\}.\end{aligned}$$

The last condition means that ω determines the same Spin^c structure as ω_0, but we could replace this by the condition that the two have the same first Chern class, or that they are in the same component of the space of non-degenerate 2-forms. For each compact subset $K \subset \Lambda$, we shall now give a preferred homotopy class of maps

$$\tau_K : K \to \mathcal{P}^* \setminus \Delta.$$

These will have the property that, if $K \subset K'$, then $\tau_{K'}|_K$ is homotopic to τ_K; thus we are specifying an element

$$\tau \in \lim_{K} \, [K, \mathcal{P}^* \setminus \Delta].$$

To define τ_K, we recall the deformation of the Seiberg-Witten equations exploited in [13]. Let ω be a symplectic form belonging to Λ, and let g_ω be a compatible metric, i.e. a metric for which ω is self-dual and of constant length $\sqrt{2}$. The spin-bundle W_0^+ associated to the canonical Spin^c-structure s_0 has a preferred section Φ_0 of length 1, and there is a preferred spin connection A_0 for which Φ_0 satisfies the Dirac equation,

$$D^+_{A_0}\Phi_0 = 0.$$

There is then a unique self-dual form η_0 on (X, g_ω) for which the remaining part of the Seiberg-Witten equations is satisfied:

$$\rho(F^+_{\hat{A}_0} + i\eta_0) - \{\Phi_0 \otimes \Phi_0^*\} = 0.$$

Taubes considered the equations (1) with perturbing term $\eta = \eta_0 + r\omega$ for r large, and proved among other things the following lemma:

Lemma 2.1 (Taubes, [13, 12]). *Under the hypotheses $e \neq 0$ and $[\omega] \smile e \leq 0$, there exists a constant $r_0 = r_0(\omega, g_\omega, e)$ such that for all $r \geq r_0$, the equations* (1) *have no solution for the metric g_ω on X with perturbing term $\eta_0 + r\omega$.* □

This vanishing theorem says that $(g_\omega, \eta_0 + r\omega)$ belongs to $\mathcal{P}^* \setminus \Delta$ when $r \geq r_0$. Taubes' argument supplies explicit estimates of a suitable r_0, in terms of the geometry of X and its almost-complex structure. From this one can see that if ω varies over any compact set K and the g_ω vary continuously with ω in our chosen topology, then there will be a single r_0 which is large enough for the lemma to hold for all ω in K. This gives us a map τ_K from K to $\mathcal{P}^* \setminus \Delta$ as required:

$$\tau_K : \omega \mapsto (g_\omega, \eta_0 + r_0\omega).$$

(Our notation hides the fact that η_0 is a function of ω and g_ω.) It is straightforward to see that the homotopy class of this map does not depend on the choice of the metrics g_ω; and the compatibility condition under restriction to compact subsets is automatic.

Although there is no map on all of Λ, a family of maps on compact subsets, such as τ, does induce a well-defined map on homology,

$$\tau_* : H_i(\Lambda; \mathbb{Z}) \to H_i(\mathcal{P}^* \setminus \Delta; \mathbb{Z}).$$

By composing with τ_*, we can use the Q defined above to obtain a function, also called Q, on the homology of Λ:

$$Q : H_{d-1}(\Lambda; \mathbb{Z}) \to \mathbb{Z}.$$

We summarize this construction as a definition:

Definition 2.2. Let X be a closed oriented 4-manifold with a homology orientation. Let s_0 be the Spin^c structure determined by some symplectic form on X, let e be an element of $H^2(X; \mathbb{Z})$ for which the integer

$$d = -e \cdot e - c_1 \cdot e$$

lies in the range $0 < d < b^+(X) - 1$, and let Λ be the space of all symplectic forms which determine the same Spin^c structure and which satisfy also $[\omega] \smile e \leq 0$. Then there is a well-defined homomorphism

$$Q : H_{d-1}(\Lambda; \mathbb{Z}) \to \mathbb{Z}$$

constructed as above using the Seiberg-Witten equations for the Spin^c structure $s = s_0 + e$. □

Our remaining task is to exhibit the non-triviality of Q by describing explicit cycles of symplectic forms on which Q is non-zero.

3. An example: quotient singularities

Let Y_0 be the quotient $\mathbb{C}^2/C_m$, where C_m is the cyclic subgroup of order m inside the group of scalars. This variety has a resolution

$$\sigma_0 : \tilde{Y}_0 \to Y_0,$$

in which $\tilde{Y}_0$ is the total space of the line bundle $\mathcal{O}(-m)$ of degree $-m$ over $\mathbb{CP}^1$. Write E for the exceptional curve of the resolution. The space $\tilde{Y}_0$ has a natural $(m-1)$-parameter family of deformations, $\tilde{Y}_u$, $u \in \mathbb{C}^{m-1}$. The total space of the family forms the complex manifold

$$\tilde{\mathbf{Y}} = \mathcal{O}(-1)^m$$

(i.e. the total space of the bundle), and $\tilde{Y}_u$ is the fiber $\tilde{q}^{-1}(u)$ of the map

$$\tilde{q} : \tilde{\mathbf{Y}} \to \mathbb{C}^{m-1}$$

obtained by evaluating any generic collection of $m-1$ sections of the dual bundle of $\tilde{\mathbf{Y}}$. The appearance of $\tilde{Y}_0$ as the fiber $\tilde{q}^{-1}(0)$ is explained by an exact sequence

$$\mathbb{O}(-m) \to \mathbb{O}(-1)^m \to \mathbb{O}^{m-1}, \tag{3}$$

which is unique to within automorphisms of the bundles involved. This viewpoint makes clear that $\tilde{\mathbf{Y}}$ is a C^∞ product,

$$\tilde{\mathbf{Y}} \overset{C^\infty}{\cong} \tilde{Y}_0 \times \mathbb{C}^{m-1}.$$

A product structure can be defined by splitting the sequence using a standard hermitian metric on $\mathbb{O}(-1)^m$. There are no compact curves in $\tilde{\mathbf{Y}}$ other than E, and this curve is contracted to a point by the map

$$\sigma : \tilde{\mathbf{Y}} \to \mathbf{Y} \subset \mathbb{C}^{2m}$$

obtained by evaluating all sections of the dual bundle. (The image $\mathbf{Y}$ can be identified with the subset of $(\mathbb{C}^2)^m$ consisting of m-tuples of vectors in $\mathbb{C}^2$ whose span is either 0 or a line.) The sections defining $\tilde{q}$ also define a map $q : \mathbf{Y} \to \mathbb{C}^{m-1}$, and there is a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbf{Y}} & \xrightarrow{\sigma} & \mathbf{Y} \\ \tilde{q}\downarrow & & \downarrow q \\ \mathbb{C}^{m-1} & = & \mathbb{C}^{m-1}. \end{array}$$

Let y_0 be the image of E: this is the unique singular point of $\mathbf{Y}$.

Suppose now that we are given an analytic family of projective varieties X_u, arising as the fibers of a map $p : \mathbf{X} \to U$, where U is an open ball about 0 in $\mathbb{C}^{m-1}$. We suppose this family is embedded in $\mathbb{CP}^N \times U$, so that p is the projection on the second factor. Suppose that all fibers X_u are smooth, except for X_0 which has a single singular point x_0. Suppose finally that the germ of the map

$$p : (\mathbf{X}, x_0) \to (U, 0)$$

at x_0 is isomorphic to the germ of q at y_0. In particular then, X_0 has a quotient singularity at x_0. Using the homomorphism Q defined in the previous section (Definition 2.2), we shall prove:

Theorem 3.1. *Suppose p is as above, let $u \in U \setminus \{0\}$, and consider the fiber X_u as a symplectic manifold, with the form inherited from $\mathbb{CP}^N$. Let* Diff *and* Symp *be its group of diffeomorphisms and symplectomorphisms respectively. Then $\pi_{2m-3}(\mathrm{Diff}/\mathrm{Symp})$ is non-zero, provided that $b^+(X_u)$ is greater than $2m-1$. Indeed, there is a sphere representing a non-trivial homology class in $H_{2m-3}(\mathrm{Diff}/\mathrm{Symp}; \mathbb{Q})$.*

Proof. Fix an isomorphism of the germs, as just described, and use the isomorphism to resolve the singularity $(\mathbf{X}, x_0)$ using $\tilde{\mathbf{Y}}$ as a model. This gives a smooth family of varieties $\tilde{\mathbf{X}}$ and a commutative diagram,

$$\begin{array}{ccc} \tilde{\mathbf{X}} & \xrightarrow{\sigma} & \mathbf{X} \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ U & = & U. \end{array}$$

The family $\tilde{\mathbf{X}}$ is a C^∞ product, and we fix a trivialization

$$\tau : \tilde{\mathbf{X}} \xrightarrow{C^\infty} \tilde{X}_0 \times U,$$

where $\tilde{X}_0$ is the minimal resolution of X_0. The fibers $\tilde{X}_u$ and X_u are isomorphic for $u \neq 0$; so using the trivialization τ we can regard the Fubini-Study forms on X_u as giving a family of symplectic forms, ω_u, on the fixed C^∞ manifold $X = \tilde{X}_0$.

Let S^{2m-3} be a small sphere about 0 inside U, and consider the family of symplectic structures which it parameterizes:

$$(X, \omega_u), \quad u \in S^{2m-3}. \tag{4}$$

All these forms have the same cohomology class, and are therefore isotopic, by Moser's result. All have zero pairing with the homology class represented by the 2-sphere E in $\tilde{X}_0$, so writing e for the Poincaré dual of $[E]$, we have a $(2m-3)$-sphere S in the space

$$\Lambda = \{\omega \in \Omega^2(X) \mid \omega \text{ is symplectic, } [\omega] \smile e \leq 0 \text{ and } s_\omega \cong s_0\}.$$

Here s_0 is the canonical Spin^c structure of the complex manifold $\tilde{X}_0$. Note that the index $-d$ for the Seiberg-Witten equations with the Spin^c structure $s = s_0 + e$ is given by

$$\begin{aligned} d &= -e \cdot e - c_1 \cdot e \\ &= 2m - 2, \end{aligned}$$

so the homomorphism Q of Definition 2.2 can be evaluated on S. (The condition on $b^+(X)$ in Definition 2.2 is the same as the condition in the Theorem.) We shall show that S is essential, and carries a non-zero class in homology, by showing that

$$Q(S) = \pm 1. \tag{5}$$

From this it follows that S is also essential in the smaller set of symplectic forms

$$\Lambda_0 = \{\omega \in \Omega^2(X) \mid \omega \text{ is isotopic to } \omega_u\}.$$

In other words, the family ω_u ($u \in S^{2m-3}$) cannot be extended to a family of cohomologous symplectic forms on X parameterized by the ball. The obstruction to such an extension lives in $\pi_{2m-3}(\mathrm{Diff}/\mathrm{Symp})$, so (5) will prove the Theorem.

Let ν_u be any smooth family of Kähler forms on the fibers $\tilde{X}_u$ of $\tilde{p}$. Let $\sigma^*\omega_u$ be the pull-back of the forms ω_u via σ. For $u \neq 0$, the form $\sigma^*\omega_u$ coincides with ω_u on $\tilde{X}_u = X_u$, but for $u = 0$ the form $\sigma^*\omega_u$ is degenerate along the complex curve E. Let $\beta : U \to \mathbb{R}^+$ be a smooth cut-off function supported near 0 and equal to zero on the small sphere S^{2m-3}. Define

$$\tilde{\omega}_u = \sigma^*\omega_u + \beta(u)\nu_u.$$

This is a family of Kähler forms on the fibers of $\tilde{p}$, coinciding with the Fubini-Study forms on the fibers over S^{2m-3}. Using the trivialization τ, it can also be viewed as a family of symplectic forms on the fixed manifold X, extending the family (4) from the sphere S to the ball $B^{2m-2} \subset U$ which S bounds. This family is not locally trivial, because the cohomology class $[\tilde{\omega}_u]$ varies with u in the interior of the ball.

Writing $\tilde{g}_u$ for the Kähler metric corresponding to $\tilde{\omega}_u$, we have now a map

$$\begin{aligned} T : & B^{2m-2} \to \mathcal{P}^* \\ & u \mapsto (\tilde{g}_u, \eta_0 + r\tilde{\omega}_u). \end{aligned} \tag{6}$$

Proposition 3.2. *For r sufficiently large, the image of T meets $\Delta = \pi(\mathcal{M}^*)$ only at $T(0)$. There is exactly one solution in $\mathcal{M}^*$ over $T(0)$, and the map π is transverse to T at this point.*

Solutions of the perturbed Seiberg-Witten equations on the Kähler manifold $(X, \tilde{g}_u)$ correspond to algebraic curves in X homologous to E. There is one such curve for $u = 0$ and no such curve for $u \neq 0$, and this is the basis of the proposition above. We shall go through this argument carefully in the next section: the main point is to establish transversality. Once this is done, equation (5) follows from the definition of Q, and this establishes Theorem 3.1, which is our main goal. □

Note that the argument establishes a little more. If the family of symplectic forms $\{\omega_s\}_{s\in S}$ is exhibited as the boundary of a family of symplectic forms parameterized by the ball (or any other manifold), then there is at least one symplectic form in the larger family whose cohomology class has positive pairing with e.

4. Transversality: comparing deformations

Consider a Kähler surface (X, g, ω). The spin bundles for the canonical Spin^c structure s_0 are

$$W_0^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$$
$$W_0^- = \Lambda^{0,1}.$$

The distinguished spinor Φ_0 is the section 1 of $\Lambda^{0,0}$. The canonical spin connection A_0 is the standard one on these bundles, and

$$F^+_{\hat{A}_0} = \frac{1}{4i} s\omega$$

where s is the scalar curvature. On the two summands of W_0^+, Clifford multiplication by ω is $-2i$ and $+2i$ respectively, so the canonical perturbation η_0 for which the equations are satisfied is given by

$$\eta_0 = \tfrac{1}{4}(1+s)\omega.$$

If $s = s_0 + e$ and E is the complex line bundle with $c_1 = e$, then the spin bundle $W^+ = W_s^+$ for s is $\Lambda^{0,0}(E) \oplus \Lambda^{0,2}(E)$, and a typical spin connection can be written $A = A_0 + B$, where B is a connection in E.

For the Spin^c structure s, the Seiberg-Witten equations with perturbing term $\eta_0 + r\omega$ thus become the following equations for a pair $\Phi = (\alpha, \beta) \in \Lambda^{0,0}(E) \oplus \Lambda^{0,2}(E)$ and a connection B in E:

$$\begin{aligned} \bar\partial_B \alpha + \bar\partial_B^* \beta &= 0 \\ 2i\Lambda F_B^{1,1} - 2r - \tfrac{1}{2}(1 - |\alpha|^2 + |\beta|^2) &= 0 \\ 4F_B^{0,2} - \bar\alpha\beta &= 0. \end{aligned} \tag{7}$$

Here ΛF denotes the contraction of F with ω. The standard device now is to act on the first equation with $\bar\partial_B$ and then use the last equation to obtain

$$\bar\partial_B \bar\partial_B^* \beta + \tfrac{1}{4}|\alpha|^2 \beta = 0,$$

and hence deduce that either $\alpha = 0$ or $\beta = 0$. It then follows from the last equation that $F^{0,2} = 0$, so that E obtains from B the structure of a holomorphic bundle.

Finally, by integrating the second equation, one sees that α must be non-zero if $r > r_0$, where r_0 is defined by

$$\left(2r_0 + \tfrac{1}{2}\right) \times \mathrm{vol}(X) = 4\pi(c_1(E) \smile [\omega]).$$

When $r > r_0$, the zero-set of α is a curve C, from which the holomorphic bundle $(E, \bar\partial_B)$ and the section α can be recreated up to isomorphism. The main fact about these equations, which in this form goes back to Bradlow [2] but has its roots rather earlier, is that the procedure can be reversed, constructing a unique solution for each C:

Proposition 4.1. *When $r > r_0$, the above construction yields a one-to-one correspondence between solutions of the Seiberg-Witten equations with perturbing term $\eta_0 + r\omega$ and holomorphic curves (possibly non-reduced), $C \subset X$, whose fundamental class is Poincaré dual to e.* □

What is less well documented in the literature is the comparison of the deformation theory of the solution (B, α, β) with the deformation theory of the curve $C = \alpha^{-1}(0)$ (though see [9]). The linearization of the equations (7), combined with the natural gauge-fixing condition, yields the following equations for a deformation $(\dot B, \dot\alpha, \dot\beta)$ of a solution with $\beta = 0$:

$$\begin{aligned} \bar\partial_B \dot\alpha + \bar\partial_B^* \dot\beta + \dot B^{0,1}\alpha &= 0 \\ 4\bar\partial^* \dot B^{0,1} - \bar\alpha\dot\alpha &= 0 \\ 4\bar\partial \dot B^{0,1} - \bar\alpha\dot\beta &= 0. \end{aligned} \tag{8}$$

The second of these equations incorporates the gauge-fixing condition. Let us write

$$\mathcal{H}^1 \subset \Omega^1(i\mathbb{R}) \oplus \Omega^{0,0}(E) \oplus \Omega^{0,2}(E)$$

for the solution space of the linearized equations, and let

$$\mathcal{H}^2 \subset \Omega^{0,1}(E) \oplus \Omega^{0,0}(\mathbb{C}) \oplus \Omega^{0,2}(\mathbb{C})$$

be the cokernel of the operator described by the left-hand side of the equations (identified here with the kernel of the formal adjoint.)

Proposition 4.2. *There are isomorphisms*

$$\begin{aligned} \mathcal{H}^1 &\cong H^0(C, \mathcal{E}) \\ \mathcal{H}^2 &\cong H^1(C, \mathcal{E}) \end{aligned}$$

induced by maps $(\dot B, \dot\alpha, \dot\beta) \mapsto \dot\alpha|_C$ and $(\gamma, f, \nu) \mapsto \gamma|_C$ respectively. Here $\mathcal{E} \cong \mathcal{O}[C]$ is the sheaf of holomorphic sections of E with the holomorphic structure defined by $\bar\partial_B$.

Proof. Consider $(\dot{B}, \dot{\alpha}, \dot{\beta})$ in $\mathcal{H}^1$. If one applies $\bar{\partial}_B$ to the first equation of (8) and then uses the last equation, one obtains

$$\bar{\partial}_B \bar{\partial}_B^* \dot{\beta} + \tfrac{1}{4}|\alpha|^2 \dot{\beta} = 0,$$

much as before. Since α is non-zero, $\dot{\beta}$ must vanish. From the last equation, it follows that $\bar{\partial} \dot{B}^{0,1} = 0$. There is therefore a map

$$\begin{aligned} w : \mathcal{H}^1 &\to H^{0,1}(X, \mathcal{O}) \\ (\dot{B}, \dot{\alpha}, \dot{\beta}) &\mapsto [\dot{B}^{0,1}]. \end{aligned}$$

If $(\dot{B}, \dot{\alpha}, \dot{\beta}) \in \mathrm{Ker}(w)$, then $\dot{B} = \bar{\partial} h$ for some h. Any such h satisfies

$$\bar{\partial}_B(\dot{\alpha} + h\alpha) = 0,$$

by the first equation.

Lemma 4.3. *There is an isomorphism* $\mathrm{Ker}(w) = \mathrm{Coker}(m_\alpha)$*, where* m_α *denotes the multiplication map* $H^0(X, \mathcal{O}) \to H^0(X, \mathcal{E})$.

Proof. The element $\dot{\alpha} + h\alpha$ is in $H^0(X, \mathcal{E})$. The ambiguity in h is to add an element of $H^0(X, \mathcal{O})$, so there is a well-defined map

$$\begin{aligned} v : \mathrm{Ker}(w) &\to \mathrm{Coker}(m_\alpha), \\ (\dot{B}, \dot{\alpha}, \dot{\beta}) &\mapsto \dot{\alpha} + h\alpha \end{aligned}$$

with $\bar{\partial} h = B^{0,1}$. The map v is injective, for if we have $\dot{\alpha} + h\alpha = 0$, then the second of the linearized equations gives

$$4\bar{\partial}^* \bar{\partial} h + |\alpha|^2 h = 0$$

and hence $h = 0$ and $\dot{\alpha} = 0$. The map v is also surjective, for given $s \in H^0(X, \mathcal{E})$ the equation to be solved for h is

$$4\bar{\partial}^* \bar{\partial} h + |\alpha|^2 h = \bar{\alpha} f.$$

By the positivity of the operator on the right-hand side, a solution h exists, and $\dot{\alpha}$ and $\dot{B}^{0,1}$ can then be recovered as $-h\alpha$ and $\bar{\partial} h$. □

In a similar vein, we have the following Lemma, whose proof we omit.

Lemma 4.4. *The image of* w *is* $\mathrm{Ker}(n_\alpha)$*, where* n_α *denotes the multiplication map* $H^1(X, \mathcal{O}) \to H^1(X, \mathcal{E})$. □

These two lemmas show that the following sequence is exact at the middle three terms:

$$H^0(X,\mathcal{O}) \xrightarrow{m_\alpha} H^0(X,\mathcal{E}) \xrightarrow{\nu^{-1}} \mathcal{H}^1 \xrightarrow{w} H^1(X,\mathcal{O}) \xrightarrow{n_\alpha} H^1(X,\mathcal{E}).$$

Comparing this to the long exact sequence in cohomology obtained from the sequence $\mathcal{O} \to \mathcal{E} \to \mathcal{E}|_C$ (the first map being defined by α), one sees that there is at least an abstract isomorphism between $\mathcal{H}^1$ and $H^0(C,\mathcal{E})$. Chasing the definitions, we find that the map is given by

$$(\dot{B},\dot{\alpha},\dot{\beta}) \mapsto (\dot{\alpha} + h\alpha)|_C,$$

where $\bar{\partial} h = \dot{B}^{0,1}$. Since α vanishes on C, the right hand side is simply $\dot{\alpha}|_C$. So this proves the statements about $\mathcal{H}^1$ in the Proposition. The case of $\mathcal{H}^2$ is rather similar, and we will omit the argument. □

We continue to suppose that (B,α,β) is a solution of the perturbed Seiberg-Witten equations on the Kähler manifold (X,ω,g) with $\beta = 0$, and we continue to use $\mathcal{H}^1$, $\mathcal{H}^2$ to denote the kernel and cokernel of the linearized equations. We now suppose in addition that ω and g occur at $u = u_0$ in an analytic family of Kähler structures (ω_u, g_u) on X, parameterized by an open subset U of $\mathbb{C}^n$. There is a natural map

$$\delta : T_{u_0}U \to \mathcal{H}^2$$

in this situation. The occurrence of the solution (B,α,β) at $u = u_0$ in this family is a transverse phenomenon if δ is surjective. To define δ, let $(\dot{g},\dot{\omega})$ be the variation of (g,ω) along some direction in $T_{u_0}U$, and let $(\dot{B},\dot{\alpha},\dot{\beta})$ be any corresponding variation of the variables (not necessarily solving the equations). From these one calculates the first variation in the left-hand-sides of the equations (7), and the projection of this variation onto the cokernel $\mathcal{H}^2$ is δ.

We now obtain an explicit expression. Proposition 4.2 identifies the projection onto $\mathcal{H}^2$ with the map

$$\begin{aligned} \Omega^{0,1}(E) \oplus \Omega^{0,0}(\mathbb{C}) \oplus \Omega^{0,2}(\mathbb{C}) &\to H^1(C,\mathcal{E}) \\ (\gamma, f, \nu) &\mapsto \gamma|_C. \end{aligned}$$

Since only γ is involved, we need only examine the variation of the first equation of (7) as (g,ω) are varied. Let J be the the complex structure corresponding to (g,ω), so that the first equation can be written long-hand as

$$(1 + iJ)d_B\alpha + (1 + iJ) *_g d_B *_g \beta = 0.$$

Because β is zero, the variation of the left-hand side corresponding to a variation $(\dot{g}, \dot{\omega})$ of the Kähler structure contains only one term that does not already appear in (8), namely the term

$$i\dot{J} d_B\alpha,$$

where $\dot{J}$ is the variation of J. The linearization of the condition $J^2 = -1$, and the fact that J is real, tells one that $\dot{J}$ carries $\Omega^{1,0}$ to $\Omega^{0,1}$. Since $\bar{\partial}_B\alpha = 0$, the above term is therefore $i\dot{J}\partial_B\alpha$, an element of $\Omega^{0,1}(E)$. An explicit Dolbeault representative for $\delta \in H^1(C, \mathcal{E})$ is therefore

$$\delta = [i\dot{J}\partial_B\alpha]|_C \in H^{0,1}(C, E).$$

If Θ denotes the sheaf of holomorphic sections of $T^{1,0}X$ for the complex structure J, then the linearization of the integrability condition $\bar{\partial}^2 = 0$ says that $\dot{J}$ is $\bar{\partial}$-closed when interpreted as an element of $\Omega^{0,1}(X, \Theta)$. Thus $\dot{J}$ represents an element of $H^1(X, \Theta)$, and we have the Kodaira-Spencer map

$$\kappa : T_{u_0}U \to H^1(X, \Theta).$$

The derivative of α identifies $\mathcal{E}|_C$ with the normal bundle $\nu(C) = \mathcal{O}[C]|_C$ in X. The above expression for δ thus identifies δ with the composite

$$T_{u_0}Y \xrightarrow{i\kappa} H^1(X, \Theta) \xrightarrow{\rho} H^1(C, \nu(C)),$$

where ρ is obtained from restriction to C. We have therefore established:

Proposition 4.5. *Let (B, α, β) be a solution of the perturbed Seiberg-Witten equations on a Kähler manifold X, with β zero and α non-zero. Let C be the curve defined by the vanishing of α. Suppose that $X = X_{u_0}$ belongs to a family of Kähler manifolds X_u, parameterized by a smooth space U, and let $\kappa : T_{u_o}U \to H^1(X, \Theta)$ be the Kodaira-Spencer map for this family. Then the occurrence of the solution (B, α, β) at $u = u_0$ is transverse provided that the composite $\rho \circ \kappa$ is surjective, where ρ is the restriction map $H^1(X, \Theta) \to H^1(C, \nu(C))$.* □

The condition on κ in this proposition is precisely the condition that the occurrence of the curve C in X_{u_0} is a transverse phenomenon in the family of complex spaces X_u.

Now we return to the particular family constructed in the previous section, to prove Proposition 3.2. At $u = 0$, the class dual to e is represented by the curve E in $\tilde{X}_0$. This representative is unique, because $E \cdot E$ is negative and E is irreducible. For $u \neq 0$, the complex structure of $\tilde{X}_u = X_u$ carries a Kähler form ω_u (distinct from $\tilde{\omega}_u$

in general) whose pairing with the class dual to e is zero. It follows that for $u \neq 0$ this class cannot be represented by a curve. The restriction of the Kodaira-Spencer map to the normal bundle of the curve E is a map

$$\mathbb{C}^{m-1} \to H^1(\mathbb{CP}^1, \mathcal{O}(-m))$$

which can be identified with the extension class of the extension (3). This map is an isomorphism, or the middle term of (3) would not be $\mathcal{O}(-1)^m$. The occurrence of the curve E in the family is therefore transverse, and so too is the occurrence of the corresponding solution to the Seiberg-Witten equations, by the above proposition. □

5. Discussion

The model for an occurrence of a double-point in a family of complex surfaces is the double-point which appears at $v = 0$ in the family of affine quadrics

$$x^2 + y^2 + z^2 = v, \quad v \in D^2,$$

where D^2 is the unit disk in $\mathbb{C}$. Pulling back this family by the branched double-covering $u \mapsto u^2$ of the disk, one obtains a family $x^2 + y^2 + z^2 = u^2$ which is precisely the family $\mathbf{Y}$ in the case $m = 2$. The smooth family $\tilde{\mathbf{Y}}$ is the simultaneous resolution of the universal unfolding of the double-point. Our picture is therefore the same as the example from [10] in this case. The universal unfolding of the quotient singularity $\mathbb{C}^2/C_m$ for m greater than 2 is more complicated. But the Kuranishi space has a component (the Artin component) which admits a simultaneous resolution. This is the space $\tilde{\mathbf{Y}}$ for $m > 2$ which we have described. There is no particular reason why the curve which occurs in these examples need be rational or smooth. We have merely presented a case which is easy to understand quite explicitly.

Using the invariants defined in [6] for 4-manifolds with contact boundary, one can also obtain more local versions of these results. For example let B^4 be the unit ball in $\mathbb{C}^2$, let X' be the quotient B^4/C_m (the same action as before), and let ξ be the contact structure on the lens space at the boundary obtained from the embedding in $\mathbb{C}^2/C_m$. Let X be obtained from X' by replacing the singular point with a sphere C of self-intersection $-m$. Then we have:

Theorem 5.1. *In the space of all exact symplectic symplectic forms on X compatible with ξ on ∂X, there is a family ω_u parameterized by $u \in S^{2m-3}$ which represents a non-trivial class in homology and which, in particular, cannot be extended to a family parameterized by the ball. Indeed, if ω_v ($v \in B^{2m-2}$) is any family of (not necessarily exact) symplectic forms,* *compatible with ξ and extending the given family on S^{2m-3}, then there exists at least one $v \in B^{2m-2}$ for which the pairing of ω_v with C is positive.* □

In the examples of section 3, we stuck to the case of a Kähler manifold because the analysis of the Seiberg-Witten equations in this case is so much more elementary than for a general symplectic manifold. However, the invariants of [6] have an excision property that allows one to pass from one case to another, via the theorem just stated. For example, suppose (X, ω) is a closed Kähler manifold containing a Stein domain W with boundary. Suppose that (X', ω') is another symplectic manifold containing a copy of $(W, \omega|_W)$. Given a family of symplectic forms ω_u ($u \in S$) on X which differ from ω only in the interior of W, we can transfer the family to X'. The excision property then implies that the value of Q on the two families is the same.

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