
Michael Heindl∗ Angela Kunoth†

Abstract

For the Burgers’ equation as an example for a hyperbolic conservation law, we have considered in our previous paper [CGK] a weak formulation with a stabilization to handle discontinuities, commonly called a viscosity approach. Numerically, this was realized by locally introducing degrees of freedom around the discontinuities by means of an adaptive wavelet method in an a-posteriori fashion.

In the present paper, we apply this method to systems of conservation laws, specifically, Euler’s equations for gas dynamics. Moreover, as the viscosity stabilization produces some Gibbs phenomena, we discuss different postprocessing techniques known from data and image processing together with a number of numerical comparisons.

Key words: Systems of conservation laws, viscosity method, adaptive wavelets, postprocessing.

AMS subject classification (MSC2010): 65M12.

1 Introduction

Systems of conservation laws constitute a large class of nonlinear hyperbolic partial differential equations (PDEs) for which the development of correct, stable and efficient numerical schemes is even more important since many theoretical results about existence and/or uniqueness of solutions are still not yet sufficiently resolved. Due to the nonlinearity of the PDE, the solution will develop isolated discontinuities of different types after finite time even for smooth initial data. A well-studied system of equations are the one-dimensional compressible Euler equations of gas dynamics since one can construct exact solutions for them, see Example 1.1 below. They are a model for the time evolution of a perfect fluid in the absence of dissipative effects. Mathematically, they constitute a 3 × 3 system of partial differential equations in space $x \in [0,1]$ and time $t \geq 0$ of the form

\begin{align}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) &= 0, \\
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p) &= 0, \\
\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} (u(E + p)) &= 0,
\end{align}

(1.1)

which describes the time evolution of the density of the fluid $\rho$, the velocity $u \in \mathbb{R}$, and the energy density $E$ per unit length. Moreover, $p$ denotes the pressure. The equations in (1.1) are expressed in conservation form because the mass, the momentum and the energy are conserved.

∗mheindl@uni-bonn.de
†Corresponding author; kunoth@math.uni-paderborn.de, Institut für Mathematik, Universität Paderborn, Warburger Str. 100, 33098 Paderborn, Germany, www2.math.uni-paderborn.de/ags/kunoth
Since this is a system of three equations in four variables, one closes the system with the state equation
\[ p = (\gamma - 1)(E - \frac{1}{2}\rho u^2), \tag{1.2} \]
see, e.g., [Sm,We]. Here \( \gamma > 1 \) denotes the adiabatic gas constant; for an ideal diatomic gas like oxygen or nitrogen, one has \( \gamma = 1.4 \). We abbreviate the system (1.1) as
\[ \frac{\partial}{\partial t} U + \frac{\partial}{\partial x} F(U) = 0. \tag{1.3} \]
using \( U := (\rho, u, p)^T \) for the vector of unknowns.

**Example 1.1 (Sod-Shock-Tube Problem)** The sod-shock-tube problem is an example of a Riemann problem which means that it consists of a conservation law, together with piecewise constant initial data having a single discontinuity. It will be illustrative to explain briefly the type of discontinuities the solution of such systems may exhibit and how one can construct exact solutions for (1.1) for the standard test case of a sod-shock-tube problem, see [So,To]. We consider a tube which is closed at both ends. The tube is separated by a vertical diaphragm into two regions. In the left region, there is a gas at high pressure while the gas in the right region is of small pressure. Both gases are initially at rest. We choose for \( U \) at \( t = 0 \) the initial conditions
\[ U_L = \begin{pmatrix} 1 \\ 0 \\ 2.5 \end{pmatrix} \text{ for } 0 \leq x < 0.5; \quad U_R = \begin{pmatrix} 0.125 \\ 0 \\ 0.25 \end{pmatrix} \text{ for } 0.5 \leq x \leq 1. \tag{1.4} \]
This means that density and pressure are discontinuous across the diaphragm. At time \( t = 0 \), the diaphragm bursts and the gases redistribute within the tube. The bursting diaphragm produces a series of pressure waves. Each of these increases the speed of sound \( a := \sqrt{\gamma p}/\rho \) behind them. They compress into a shock wave propagating from left to right through the driven, low-pressure gas. This shock wave increases the temperature and pressure of the low-pressure gas and induces a flow in the direction of the shock wave but at lower velocity than the leading wave. Simultaneously, a rarefaction wave travels back into the high-pressure gas. Across the interface which separates both gases, a limited degree of mixing occurs, and this interface follows the leading wave at a lower velocity. One can detect four regions as illustrated in the top left of Figure 1 for which one can classify the solution: a rarefaction wave, a contact discontinuity, or a shock discontinuity. A contact discontinuity occurs since pressure \( p \) and velocity \( u \) are continuous but the density \( \rho \) and the energy \( E \) are discontinuous. Moreover, a shock wave propagates through the gas which is characterized by the fact that all quantities \( \rho, u, p, E \) are discontinuous across the shock front. Reflection boundary conditions at \( x = 0 \) and at \( x = 1 \) then produce interesting behaviour of the solution, namely, discontinuities bouncing from left to right and interacting with each other.

With the initial values (1.4), one can construct the solution of (1.1) for all times as follows. Let \( U_i := (\rho_i, u_i, p_i) \) denote the variables in the \( i \)-th region, \( i = 1, \ldots, 4 \). Because of the initial conditions (1.4), one knows \( U_1 \) and \( U_4 \); the contact discontinuity initially separates the two gases. With these values, one can determine the values of \( U \) in the whole domain as follows. The contact discontinuity separates the two gases and travels with velocity \( W \). The term \( \frac{p_i}{p_1} \) can implicitly be determined by using the speed of sound \( a_i \) and gas constants \( \gamma_i \) relative to each region indexed by \( i \) as
\[ \frac{p_4}{p_1} = \frac{p_2}{p_1} \left( 1 - \frac{\frac{\gamma_4 - 1}{\gamma_4} \left( \frac{\gamma_4}{\gamma_1} \right) \left( \frac{p_2}{p_1} - 1 \right)}{\sqrt{2\gamma_1(2\gamma_1 + (\gamma_1 + 1)\left( \frac{p_2}{p_1} - 1 \right))}} \right)^{-\frac{2\gamma_4}{\gamma_4 - 1}}. \]
From this, one obtains

\[ u_3 = u_p, \quad u_2 = u_p, \quad p_3 = p_2, \quad p_2 = p_1 \frac{p_2}{p_1}, \]
\[ \rho_3 = \rho_4 \left( \frac{p_2}{p_1} \right)^{\frac{1}{\gamma_4}}, \quad \rho_2 = \rho_1 \frac{1 + \left( \frac{\gamma_4 - 1}{2} \right) \left( \frac{p_2}{p_1} \right)}{\frac{\gamma_4 - 1}{\gamma - 1} + \frac{2}{\gamma_4}}, \]
\[ u_p = a_1 \left( \frac{p_2}{p_1} - 1 \right) \left( \frac{2}{\gamma_4 + 1} \right)^{\frac{1}{\gamma_4}}, \quad W = a_1 \sqrt{\left( \frac{\gamma_4 - 1}{2} \right) \left( \frac{p_2}{p_1} - 1 \right) + 1}. \]

The left and right end of the rarefaction wave travels with velocities \(-a_4\) and \(u_3 - a_3\), respectively. Within this wave, one has

\[ u = \left( \frac{2}{\gamma_4 + 1} \right) \left( a_4 + \frac{x}{t} \right), \quad \rho = \rho_4 \left( 1 - \left( \frac{\gamma_4 - 1}{2} \right) \left( \frac{u}{a_4} \right) \right)^{\frac{2}{\gamma_4 - 1}}, \]
\[ p = p_4 \left( 1 - \left( \frac{\gamma_4 - 1}{2} \right) \left( \frac{u}{a_4} \right) \right)^{\frac{2}{\gamma_4 - 1}}. \]

For an illustration, we display in Figure 1 the exact solution \(U = (\rho, u, p)^T\) at time \(t = 0.15\). □

![Figure 1: Exact solution of the sod-shock-tube problem at time \(t = 0.15\).](image)

From a numerical point of view, the computation of solutions with discontinuities is difficult; numerous schemes based on finite differences or finite volumes have been proposed, see, e.g., [Kr, LV, To]. Such problems appear already for Burgers’ equation which is a simplified one-dimensional model for inviscid fluid flow consisting of just one equation.

Our wavelet viscosity method we want to advocate here is the following. Discretization schemes for the Burgers’ equation based on weak formulations which are standard for elliptic PDEs cause strong numerical instabilities, see, e.g., [Ca, CGK]. Such methods can be stabilized by adding an artificial diffusion term to the PDE. This, however, has the effect that problem-inherent discontinuities are smeared out and that one has to sacrifice in accuracy, even in regions where the solution is smooth. Much better results can be attained by so-called spectral viscosity methods in which diffusion is added only to the highest frequencies in a spectral setting; see [Td] and several follow-up papers. This idea method was carried-over in [CG1] to a multiscale finite
element formalism in terms of hierarchical finite element bases [Y]: artificial diffusion was added only at the highest hierarchical levels, and convergence to the entropy solution was proved for the scalar univariate case. A discussion about the efficient implementation of the hierarchical finite element viscosity method and examples in two space dimensions have been presented in [CG2], and it was demonstrated numerically in [CG1, CG2] that regularization can be restricted to the highest hierarchical level.

These results were obtained employing hierarchical finite element bases with respect to a uniform grid. In [CGK], we constructed a converging wavelet-based method to numerically solve Burgers’ equation. We were able to apply a simple smoothing on the solution based on elementary properties of a wavelet-specific detection of singularities. Most importantly, we extended the scheme to the adaptive case. In fact, adaptive schemes which are capable of concentrating degrees of freedom in the neighborhood of discontinuities offer high potential for efficient numerical simulations. Thus, after having studied the quality attainable in the wavelet formulation with respect to a uniform grid, we focused on its efficiency by proposing an adaptive algorithm that concentrates the degrees of freedom only in the neighborhood of discontinuities. Specifically, diffusion is added only in the vicinity of the discontinuities and the concentration of degrees of freedom follows the discontinuities as it evolves with time.

In the present paper, we extend this idea to systems of one-dimensional conservation laws as introduced at the beginning of this introduction. The remainder of this paper is structured as follows. In the next section, we collect a few theoretical properties about the class of (semi-orthogonal) spline-wavelets employed in the sequel. Section 3 presents our adaptive wavelet viscosity scheme. In Section 4, three methods for post-processing are introduced and their performances compared for a linear advection problem. Section 5 presents corresponding numerical results for the sod-shock-tube problem from Example 1.1.

2 Wavelet Basics

In order to describe our wavelet viscosity method, we need wavelet bases with properties appropriate for the efficient numerical solution of PDEs, see, e.g., [Da1, K] and references therein. Typically for weak formulations, the solution belongs with respect to space to a Sobolev space $H \subseteq L^2(\Omega)$ living on a bounded domain $\Omega \subset \mathbb{R}^d$. In view of the system (1.1), we only need to consider the univariate case $d = 1$ and take $\Omega = (0, 1)$.

Wavelets are indexed by a parameter $\lambda = (j, k)$ encoding the resolution level or scale $|\lambda| := j$ and the spatial location $k$. In view of the finite domain $(0, 1)$, there is a coarsest level $j_0$. Let $\mathbb{I}$ denote the infinite set of all possible indices. A wavelet basis is a collection of functions

$$\Psi := \{\psi_\lambda : \lambda \in \mathbb{I}\} \subset H$$

with the following properties:

(R) $\Psi$ constitutes a Riesz basis for $H$, i.e., every $v \in H$ has a unique expansion in terms of $\Psi$,

$$v = \sum_{\lambda \in \mathbb{I}} v_\lambda \psi_\lambda =: v^T \Psi, \quad v := (v_\lambda)_{\lambda \in \mathbb{I}},$$

and its expansion coefficients satisfy a norm equivalence relation. This means that there exist constants $0 < c_H \leq C_H < \infty$ such that

$$c_H \|v\|_{\ell_2(\mathbb{I})} \leq \|v^T \Psi\|_H \leq C_H \|v\|_{\ell_2(\mathbb{I})}, \quad v \in \ell_2(\mathbb{I}),$$
holds. In other words, wavelet expansions induce isomorphisms between the function space $H$ and the sequence space $\ell_2(\mathbb{I})$ which we abbreviate as

$$\|v\|_{\ell_2(\mathbb{I})} \sim \|v^T\Psi\|_H, \quad v \in \ell_2(\mathbb{I}).$$  \hspace{1cm} (2.4)

We use the relation $a \sim b$ to express $a \lesssim b$ and $a \gtrsim b$, meaning that $a$ can be estimated from above and below by a constant multiple of $b$, independent of all parameters on which $a$ or $b$ may depend.

(L) The functions $\psi_\lambda$ are locally supported, i.e., for each $\lambda \in \mathbb{I}$ one has

$$\text{diam} (\text{supp} \psi_\lambda) \sim 2^{-|\lambda|}.$$  \hspace{1cm} (2.5)

(CP) There exists an integer $\tilde{m}$ such that

$$\langle v, \psi_\lambda \rangle \lesssim 2^{-|\lambda|(d/2+\tilde{m})} \left| \frac{\partial^{\tilde{m}} v}{\partial x^{\tilde{m}}} \right|_{L_\infty(\text{supp} \psi_\lambda)},$$  \hspace{1cm} (2.6)

where $\langle \cdot, \cdot \rangle$ is the dual pairing between $H$ and its topological dual $H'$. This means that integrating a function against a wavelet is like taking an $\tilde{m}$th order difference which annihilates the smooth part of $v$. This feature called cancellation property comes into play for the evaluation of nonlinearities. It also entails a direct approximation estimate for a finite-dimensional subspace of $H$ that is spanned by all wavelet functions up to a highest refinement level $J$.

Properties (R), (L), and (CP) allow us to prove strong theoretical statements such as (asymptotically) optimal condition number estimates for linear elliptic operators, or convergence results for adaptive methods for linear and nonlinear variational problems [CDD1, CDD2, Da2, K]. Simultaneously, they allow one to work computationally with piecewise polynomials. Concrete constructions of biorthogonal wavelet bases based on B–Splines can be found in [K]. Here we employ the piecewise linear boundary–adapted B–spline pre–wavelets from [SDS] that are a special case of the pre–wavelets on the interval constructed in [CQ]. These bases have the additional property that they are

(O) semi–orthogonal with respect to $L_2(0,1)$, i.e., for $|\lambda| \neq |\mu|$, one has

$$\int_0^1 \psi_\lambda(x) \psi_\mu(x) \, dx = 0.$$  \hspace{1cm} (2.7)

Although this property cannot directly be exploited here for problems in one space dimension, it is advantageous in the multidimensional case where tensor products of bases are taken and, as a consequence, entries of the appearing mass matrices corresponding to wavelets on different refinement levels vanish.

3 An Adaptive Wavelet Viscosity Scheme

We briefly recall the spatially adaptive wavelet scheme developed in [Ca, CGK] which is a slight modification of the adaptive algorithm proposed in [JMP]. Our scheme involves a local frequency-selective addition of artificial viscosity. We make the ansatz

$$U(x,t) = \sum_{\lambda \in \mathbb{I}} d_\lambda(t) \psi_\lambda(x),$$  \hspace{1cm} (3.1)
where $\Psi$ is the linear spline–wavelet basis of the Sobolev space $H^1(0,1)$ satisfying (R), (L), (CP) and (O). Since $U \in \mathbb{R}^3$ for the problem (1.1), the expansion coefficients $d_\lambda$ in (3.1) are also vectors in $\mathbb{R}^3$. The approximation of the solution $U(m \delta t, \cdot)$ at time $m \delta t$ is given by $U^m = \sum_{\lambda \in \Lambda^m} d^m_\lambda \psi_\lambda$.

The configuration $\Lambda^m$ is updated by a dynamically adaptive scheme and the time evolution is computed by a second-order Adams-Bashforth scheme. The indices are selected in the refinement step in two sequential stages. The configuration at the next time step may require wavelets of a higher level. Thus, firstly, one adds all children of the currently chosen wavelet indices. Recall that a wavelet indexed by $(j,k)$ is called a child of a wavelet indexed by $(j',k')$ if $j = j' + 1$ and $k \in \{2k', 2k' + 1\}$. In the second stage, this index set is further extended by also including the horizontal neighbors of the already selected wavelet indices. The operator $Q^{j_{\text{cut}}}$ appearing in (3.3) below kills oscillations on scales smaller than $j_{\text{cut}}$, i.e.,

$$Q^{j_{\text{cut}}} \left( \sum_{\lambda \in I} d_\lambda \psi_\lambda (x) \right) = \sum_{\lambda \in I, |\lambda| \geq j_{\text{cut}}} d_\lambda \psi_\lambda (x).$$

Moreover, $\nu$ in (3.3) denotes the artificial viscosity. The following algorithm also includes a coarsening step based on thresholding small wavelet coefficients after (3.3) in order to avoid unnecessary complexity.

**Algorithm 3.1 [Adaptive Wavelet Galerkin Viscosity Scheme]**

Fix a threshold $\varepsilon(j)$ depending on the resolution level. Given an initial solution $U^0$ expanded in wavelets relative to some index set $\tilde{\Lambda}^0$, define $\Lambda^0$ by thresholding $\tilde{\Lambda}^0$, i.e.,

$$\Lambda^0 := \{ \lambda \in \tilde{\Lambda}^0 : \|d_\lambda\| > \varepsilon(0) \},$$

where $\| \cdot \|$ denotes the maximum norm in Euclidean space $\mathbb{R}^3$.

Then, for each $m = 0, \ldots$, perform the following steps:

1. **Refinement**
   
   (a) Vertical expansion: construct a set $\Lambda^m_v$ containing the wavelet indices in $\Lambda^m$ and all their children.
   
   (b) Horizontal expansion: form
   
   $$\tilde{\Lambda}^{m+1} := \{ \lambda = (j,k) \text{ such that } (j,k') \in \Lambda^m_v, k' \in \{k-2,k-1,k,k+1,k+2\} \}.$$

2. **Compute**
   
   $U^{m+1} = \sum_{\lambda \in \tilde{\Lambda}^{m+1}} d^{m+1}_\lambda \psi_\lambda$ as solution of
   
   $$\left( \frac{1}{\tau} (U^{m+1} - U^m) + \frac{1}{2} \frac{\partial}{\partial x} (U^*_m)^2, \psi_\lambda \right) = -\nu \left( \frac{\partial}{\partial x} Q^{j_{\text{cut}}} (U^*_m), \frac{\partial}{\partial x} \psi_\lambda \right) \text{ for } \lambda \in \tilde{\Lambda}^{m+1},$$

   with $U^*_m = \frac{3}{2} U^m - \frac{1}{2} U^{m-1}$ (with $U^*_m = U^m$ for $m = 0$).

3. **Coarsening**: compute $\Lambda^{m+1}$ by thresholding $\tilde{\Lambda}^{m+1}$ to obtain
   
   $$\Lambda^{m+1} := \{ \lambda \in \tilde{\Lambda}^{m+1} : \|d_\lambda\| > \varepsilon(|\lambda|) \}.$$

The evaluation of the nonlinearity in the left hand side of (3.3) is performed in a wavelet-based least squares scheme explained in detail in [Ca, CGK].

To illustrate the performance of this algorithm (and the effect of post processing considered later), we consider a simple example of a linear transport equation.
Example 3.2 (Advection Problem) Consider the one-dimensional equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0, \quad 0 < x < 1, \quad u(x, 0) = u_0(x) \text{ periodic},$$

with initial values

$$u_0(x) = \begin{cases} 
\frac{1}{6}(G_1(x, \beta, z - \delta) + (G_1(x, \beta, z + \delta) + 4G_1(x, \beta, z)), & 0.1 \leq x \leq 0.2, \\
1, & 0.3 \leq x \leq 0.4, \\
1 - |10(x - 0.1)|, & 0.5 \leq x \leq 0.6, \\
\frac{1}{6}(G_2(x, \alpha, a - \delta) + (G_1(x, \alpha, a + \delta) + 4F_2(x, \alpha, a)), & 0.7 \leq x \leq 0.8, \\
0, & \text{otherwise.} 
\end{cases}$$

Here the functions $G_1$ and $G_2$ are defined as

$$G_1(x, \beta, z) = e^{-\beta(x-z)^2}$$
$$G_2(x, \alpha, a) = \sqrt{\max(1 - \alpha^2(x-a)^2), 0}$$

where $a = 0.5$, $z = -0.7$, $\delta = 0.005$, $\alpha = 10$ and $\beta = \frac{\log 2}{36 \delta^2}$. The solution of this problem is the combination of a Gaussian wave, a rectangular signal, a triangular wave and half of an ellipse. The exact solution can be obtained by translating the initial values with the same velocity. We see in Figure 2 the numerical solution in the right column without post-processing explained later, and in the left column the distribution of the wavelet coefficients, exhibiting their spatial distribution over the $x$-axis. The vertical axis shows the refinement level $j$ with increasing value from bottom to top. Darker colors of the boxes refer to higher absolute and, therefore, more important values of the corresponding wavelet coefficients. The following parameters have been used. The maximal resolution level allowed here was set to $J_{\text{max}} = 10$ for which, however, no coefficients appear because of the thresholding in (3.4). The artificial viscosity acts only on the elements of the highest level. We see that by comparing the first with the second row that halving the thresholding parameter $\varepsilon$ increases the amount of active wavelet coefficients from 113 to 162, and the Gibbs effects appear stronger localized.

![Figure 2: Wavelet coefficients and numerical solution without post-processing for parameters $J_{\text{max}} = 10$, $\nu = 2^{-J_{\text{max}}+1}$ and thresholds $\varepsilon(j) = 10^{-6} \cdot 2^{-j}$ (top) and $\varepsilon(j) = 10^{-7} \cdot 2^{-j}$ (bottom).](image)
4 Post Processing

As we have seen in Figure 2, the adaptive wavelet Galerkin viscosity scheme very nicely locates the discontinuities but suffers in their vicinity from Gibbs phenomena, i.e., strong oscillations. Such Gibbs artefacts are not a result of the wavelet approach but also appear for spectral or finite element viscosity schemes. Extending the studies in [Ca, CGK], we want to investigate next different methods to smooth the solutions locally and reduce these oscillations.

The first method which we denote as post processing by Díez operates as follows: identify the spatial areas in which Gibbs phenomena are active by inspecting the decay of the wavelet coefficients, eliminate these coefficients and recompute the remaining ones, see [Ca, CGK] for a precise description and performance studies. This is a simple but effective and inexpensive method of low complexity since it only involves a local reconstruction using few degrees of freedom.

Other schemes have been designed within the framework of image processing and denoising and are much more expensive. The iterative scheme by Durand and Froment [DF] combines wavelet denoising and the total variation diminishing scheme [ROF]. [GSZ] presented another iterative method based on nonlinear complex-valued diffusion which can be interpreted as a shock filter.

We see in Figures 3 and 4 as two examples the performance of two of these methods for the linear advection equation from Example 3.2. We see that, in comparison to the solution without post-processing, both methods work very well when compared to the exact solution with minimal differences, and a decrease of the thresholding has no visible effect. Note, however, that the Díez post processing is somewhat cheaper, see [Hei] for computing times.

![Figure 3](image3.png)

Figure 3: Example 3.2 using post processing by Díez (left) and Durand-Froment (right) with parameters $J_{\text{max}} = 10$, $\nu = 2^{-J_{\text{max}}+1}$ and threshold $\varepsilon(j) = 10^{-6} \cdot 2^{-j}$.

![Figure 4](image4.png)

Figure 4: Example 3.2 using post processing by Díez (left) and Durand-Froment (right) with $\varepsilon(j) = 10^{-7} \cdot 2^{-j}$. 
5 Numerical Examples

We show in Figure 5 first the exact solution of the sod-shock-tube problem from Example 1.1 and the approximate solution computed with Algorithm 3.1. We have added here the values of the mach number $M := |u|/a$ and the entropy which is often also studied for the Euler equations. The next Figures 6 and 7 show the solutions without post processing in the left column and with the Díez post processing with respect to the last two levels and the Durand-Froment post processing on the right, respectively. We see that (on this scale) the Díez post processing has much less effect than for the linear advection problem but also that the Durand-Froment scheme introduces some additional oscillations for the entropy away from the discontinuities. Finally, the corresponding comparison with the more sophisticated and expensive GSZ scheme [GSZ] in Figure 8 does not exhibit better results than the Díez post processing.

Many more examples including Lax-shock-tube and Mach-3-problems together with computation times can be found in [Hei].

Figure 5: Sod-shock-tube problem: exact solution (left column), numerical solution (without post processing) at time $t = 0.15$ (right column).
6 Conclusion and Outlook

We have employed in this paper an adaptive wavelet-Galerkin viscosity method to a system of nonlinear conservation laws, the one-dimensional Euler equations of gas dynamics. The viscosity was applied locally on a fine resolution level just for stabilization.

Our method localizes very well the discontinuities but also introduces local oscillations around them, like other viscosity schemes based on spectral or finite elements. To reduce these effects, we have applied different post processing techniques and found that the simple one by Díez works as well or better than the more sophisticated ones by Durand-Froment or GSZ.

A somewhat more systematic approach to overcome these problems would be to include local viscosity based on estimating the entropy as in [GPP] which works very well for different discretizations for conservation laws based on finite or spectral elements. In fact, our viscosity parameter has the same value for each location. Computing the local entropy in a residual-based way within our adaptive wavelet method offers in our opinion great potential as this may avoid the necessity of post-processing.
Figure 7: Sod-shock-tube problem: numerical solution without post processing (left column), numerical solution with post processing by Durand-Froment (right) at time $t = 0.15$; thresholding parameter $\varepsilon = 10^{-6}$ and $n = 5000$ iterations.
Figure 8: Sod-shock-tube problem: numerical solution without post processing (left column), numerical solution with post processing by GSZ (right) at time $t = 0.15$; thresholding parameter $\varepsilon = 10^{-6}$ and $n = 50$ iterations.
References


