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Restrained Weakly Connected Independent Domination in the Join of Graphs

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Abstract

In this paper, we explore the concept of restrained weakly connected independent domination in graphs. In particular, we characterized the restrained weakly connected independent dominating sets in the join of graphs and obtain the restrained weakly connected independent domination numbers. A connected graph is constructed with a given weakly connected independent domination number, restrained weakly connected independent domination number, and maximum weakly connected independent domination number.

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1 Introduction and Preliminary Results

Let $G = (V(G), E(G))$ be a simple connected graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in$

$E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* of X is $N(X) = \bigcup_{v \in X} N(v)$ and the *closed neighborhood* of X is $N[X] = X \cup N(X)$. A subset S of $V(G)$ is an *independent set* if for every $x, y \in S$, $xy \notin E(G)$. The *independence number* $\beta(G)$ of G is the largest cardinality of an independent set of G. A subset S of $V(G)$ is called *weakly connected* if the subgraph $\langle S \rangle_w = (N_G[S], E_W)$ weakly induced by S, is connected, where E_W is the set of all edges with at least one vertex in S.

A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set of G which is independent (resp., weakly connected) is called an *independent* (resp., *weakly connected*) *dominating set* of G. The *independent* (resp. *weakly connected*) *domination number* of G, denoted by $i(G)$ (resp., $\gamma_w(G)$), is the smallest cardinality of an independent (resp., weakly connected) dominating set of G. An independent dominating set of G which is weakly connected is called a *weakly connected independent dominating set* of G. The *weakly connected independent domination number* of G, denoted by $i_w(G)$, is the smallest cardinality of a weakly connected independent dominating set of G. Similarly, the *upper weakly connected independent domination number* of G, denoted by $\beta_w(G)$, is the largest cardinality of a weakly connected independent dominating set of G.

A dominating set S is called a *restrained dominating set* of G if for every $u \in V(G) \backslash S$, there exists $w \in V(G) \backslash S$ such that $uw \in E(G)$. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the smallest cardinality of a secure dominating set of G. A set S is called a *restrained weakly connected independent dominating set* of G if S is a weakly connected independent dominating set of G and for every $u \in V(G) \backslash S$, there exists $w \in V(G) \backslash S$ such that $uw \in E(G)$. The *restrained weakly connected independent domination number* of G, denoted by $i_{rw}(G)$, is the smallest cardinality of a restrained weakly connected dominating set of G.

The concept of weakly connected independent domination is discussed in [2] [3, and [4]. Another domination parameter is the restrained domination which was discussed in [1] and [5]. A combination of these two concepts give rise to a new variant of domination called restrained weakly connected independent domination.

Remark 1.1 *Let G be a graph of order n. Then $1 \leq \gamma(G) \leq i_w(G) \leq i_{rw}(G)$.*

Theorem 1.2 *Let a, b, and c be positive integers such that $1 \leq a \leq b$ and $c = a + b - 1$. Then there exists a connected graph G such that $i_w(G) = a$, $i_{rw}(G) = b$ and $\beta_w(G) = c$.*

Proof: Consider the path $P = [u_1, v_1, u_2, v_2..., u_{2a-2}, v_{2a-2}, u_{2a-1}, u_{2a}]$. Let G be a graph obtained from P by adding the edges u_iu_{i+1} for $i = 1, 2, ..., a + 1$ and adding the edges u_1w_j, for $j = 0, 1, 2, ..., b - a$ (see Figure 1). Then

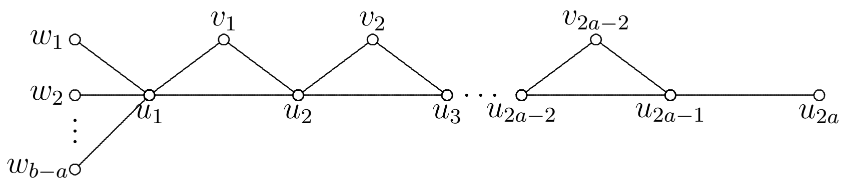

Figure 1: A graph G with $i_w(G) \leq i_{rw}(G)$

$\{u_1, u_3, ..., u_{2a-1}\}$ is a weakly connected independent dominating set of G, $\{u_2, u_4, ..., u_{2a}\} \cup \{w_1, w_2, ..., w_{b-a}\}$ is a restrained weakly connected independent dominating set of G, and $\{w_1, w_2, ..., w_{b-a}\} \cup \{v_1, v_2, ..., v_{2a-1}, u_{2a}\}$ is a maximum weakly connected independent dominating set of G. Hence, $i_w(G) = a$, $i_{rw}(G) = b$ and $\beta_w(G) = c$. □

Corollary 1.3 *The difference $i_{rw} - i_w$ can be made arbitrarily large.*

2 Results

The *join* of two graphs G and H, denoted by $G+H$, is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 2.1 *Let G be a graph of order $n \geq 2$. Then $i_{rw}(K_1 + G) = 1$ if and only if G has no isolated vertex.*

Proof: Suppose $i_{rw}(K_1+G) = 1$. Let $S = \{v\}$ be a restrained weakly connected independent dominating set of $K_1 + G$. Suppose $v \in V(K_1)$. Since S is a restrained dominating set, for each $x \in V(K_1 + G)\backslash S = V(G)$, there exists $y \in V(G)$ such that $xy \in E(G)$. Hence, G has no isolated vertex. Suppose $v \in V(G)$. Since S is a dominating set, for each $u \in V(G)\backslash\{v\}$, $uv \in E(G)$. Therefore, G has no isolated vertex.

Conversely, suppose G has no isolated vertex. Let $K_1 = \langle\{v\}\rangle$ and set $S = \{v\}$. Then S is a weakly connected independent dominating set of K_1+G. Let $u \in V(K_1 + G)\backslash S$. Then $u \in V(G)$. Thus, $u \in V(\langle C\rangle)$, where C is a nontrivial component of G. Hence, there exists $w \in V(\langle C\rangle) \subseteq V(K_1 + G)\backslash S$ such that $uw \in E(K_1 + G)$. Therefore, S is a restrained weakly connected independent dominating set of $K_1 + G$. □

Theorem 2.2 *Let G be a graph of order $n \geq 3$. Then $i_{rw}(K_1 + G) = 2$ if and only if $G = \langle C \rangle \cup \langle \{v\} \rangle$, where C is a nontrivial component of G with $\gamma(\langle C \rangle) = 1$.*

Proof: Suppose $i_{rw}(K_1 + G) = 2$. Let $S = \{u, v\}$ be a restrained weakly connected independent dominating set of $K_1 + G$. Since S is independent, $S \subseteq V(G)$ and $uv \notin E(G)$. Since $i_{rw}(K_1 + G) = 1$, G must have an isolated vertex by Theorem 2.1. But u and v are not both isolated vertices since S is a restrained dominating set. Hence, G has one isolated vertex, say v. The vertex u is a dominating vertex of the other component of G. Hence, $G = \langle C \rangle \cup \langle \{v\} \rangle$, where C is a nontrivial component of G with $\gamma(\langle C \rangle) = 1$.

Suppose $G = \langle C \rangle \cup \langle \{v\} \rangle$, where C is a nontrivial component of G with $\gamma(\langle C \rangle) = 1$. Then $i_{rw}(K_1 + G) \neq 1$. Let $S = \{v, w\}$, where w is a dominating vertex of $\langle C \rangle$. Clearly S is a restrained weakly connected independent dominating set of $K_1 + G$. Therefore, $i_{rw}(K_1 + G) = 2$. □

The next theorem can be found in [2].

Theorem 2.3 *Let G and H be graphs. Then $S \subseteq V(G + H)$ is a weakly connected independent dominating set of $G + H$ if and only if either S is an independent dominating set of G or S is an independent dominating set of H.*

A similar result characterizes the restrained weakly connected independent dominating set of $G + H$.

Theorem 2.4 *Let G and H be nontrivial and nonempty graphs. Then $S \subseteq V(G+H)$ is a restrained weakly connected independent dominating set of $G+H$ if and only if S is an independent dominating set of G or S is a independent dominating set of H.*

Proof: If $S \subseteq V(G + H)$ is a restrained weakly connected independent dominating set of $G + H$, then S is an independent dominating set of G or S is a independent dominating set of H by Theorem 2.3.

Conversely, suppose S is an independent dominating set of G. By Theorem 2.3, S is a weakly connected independent dominating set of $G + H$. Let $x \in V(G + H)\backslash S$. If $x \in V(G)\backslash S$, then $xy \in E(G + H)$ for all $y \in V(H) \subseteq V(G + H)\backslash S$. Suppose $x \in V(H)$. Since G is nontrivial, there exists $z \in V(G)\backslash S \subseteq V(G+H)\backslash S$ such that $xz \in E(G+H)$. Hence, S is a restrained weakly connected independent dominating set of $G + H$. Similarly, if S is a independent dominating set of H, then S is a restrained weakly connected independent dominating set of $G + H$. □

Corollary 2.5 *Let G and H be nontrivial and nonempty graphs. Then*

$$i_{rw}(G+H) = \min\{i(G), i(H)\}.$$

Proof: Assume that $i(G) \leq i(H)$. Let S be a minimum restrained weakly connected independent dominating set of $G + H$. Suppose that S is not a minimum independent dominating set of G. Then there exists an independent dominating set S^* of G such that $|S^*| < |S|$. By Theorem 2.4, S^* is a restrained weakly connected independent dominating set of $G+H$. This contradicts the fact that S is a minimum restrained weakly connected independent dominating set of $G+H$. Thus, S is a minimum independent dominating set of G. Hence, $i_{rw}(G+H) = |S| = i(G)$. Therefore, $i_{rw}(G+H) = \min\{i(G), i(H)\}$. □

Theorem 2.6 *Let G be a graph of order $m \geq 3$ and let $n \geq 3$ be an integer. If G has no isolated vertex, then*

$$i_{rw}(G+\overline{K_n}) = \min\{i(G), n\}.$$

Proof: Let S be a minimum restrained weakly connected independent dominating set of $G+\overline{K_n}$. Then either $S \subseteq V(G)$ or $S \subseteq V(\overline{K_n})$ since S is independent. Thus either S is a minimum independent dominating set of G or $S = V(\overline{K_n})$. Hence, $i_{rw}(G+\overline{K_n}) = \min\{i(G), n\}$. □

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