The abelianization of the level 2 mapping class group

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Abstract

In this paper, we determine the abelianization of the level d mapping class group for $d = 2$ and odd d. We also extend the homomorphism of the Torelli group defined by Heap to a homomorphism of the level 2 mapping class group.

Contents

1 Introduction

Let $g \geq 1$, $r = 0, 1$, and $d > 0$. We denote by $\Sigma_{g,r}$ a closed oriented connected surface of genus g with r boundary components. We denote by $\mathrm{Diff}_+(\Sigma_{g,r}, \partial\Sigma_{g,r})$ the group of orientation-preserving diffeomorphisms of $\Sigma_{g,r}$ which fix the boundary pointwise. The mapping class group of $\Sigma_{g,r}$ is defined by $\mathcal{M}_{g,r} := \pi_0 \mathrm{Diff}_+(\Sigma_{g,r}, \partial\Sigma_{g,r})$. Fix the symplectic basis $\{A_i, B_i\}_{i=1}^g$ of the first homology group $H_1(\Sigma_{g,r}; \mathbf{Z})$. Then the natural action of $\mathcal{M}_{g,r}$ on this group gives rise to the classical representation $\rho : \mathcal{M}_{g,r} \to \mathrm{Sp}(2g; \mathbf{Z})$ onto the integral symplectic group. The kernel $\mathcal{I}_{g,r}$ of this representation is called the Torelli group.

The level d mapping class group $\mathcal{M}_{g,r}[d] \subset \mathcal{M}_{g,r}$ is defined by the kernel of the mod d reduction $\mathcal{M}_{g,r} \to \mathrm{Sp}(2g; \mathbf{Z}_d)$ of ρ. The level d congruence subgroup $\Gamma_g[d]$ of the symplectic group is defined by the kernel of mod d reduction map $\mathrm{Ker}(\mathrm{Sp}(2g; \mathbf{Z}) \to \mathrm{Sp}(2g; \mathbf{Z}_d))$. This is equal to the image of $\mathcal{M}_{g,r}[d]$ under ρ. The group $\mathcal{M}_g[d]$ arises as the orbifold fundamental group of the moduli space of nonsingular curves of genus g with level d structure. In particular, for $d \geq 3$, the level d mapping class groups are torsion-free, and the abelianizations of the level d mapping class groups are equal to the first homology groups of the corresponding moduli spaces.

In this paper, we determine the abelianization of this group $\mathcal{M}_{g,r}[d]$ and $\Gamma_g[d]$, for $d = 2$ and odd d when $g \geq 3$. This is an analogous result in Satoh[22] and Lee-Szczarba[17] for the abelianizations of the level d congruence subgroups of $\mathrm{Aut}\, F_n$ and $\mathrm{GL}(n; \mathbf{Z})$. To determine the abelianization $H_1(\mathcal{M}_{g,r}[2]; \mathbf{Z})$, we construct an injective homomorphism $\beta_\sigma : \mathcal{M}_{g,1}[2] \to \mathrm{Map}(H_1(\Sigma_g; \mathbf{Z}_2), \mathbf{Z}_8)$. This function is defined using the Rochlin functions of mapping tori. We will show that this is an extension of a homomorphism of the Torelli group defined by Heap[6]. To determine the abelianization $H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$ for odd d, we construct the Johnson homomorphism of modulo d on $\mathcal{M}_{g,r}[d]$.

Historically, McCarthy[18] proved that the first rational homology group of a finite index subgroup of $\mathcal{M}_{g,r}$ which includes the Torelli group vanishes for $r = 0$. More generally, Hain[5] proved that this group vanishes for any $r \geq 0$.

Theorem 1.1 (McCarthy, Hain)**.** *Let $\mathcal{M}$ be a finite index subgroup of $\mathcal{M}_{g,r}$ that includes the Torelli group, where $g \geq 3$, $r \geq 0$. Then*

$$H_1(\mathcal{M}; \mathbf{Q}) = 0.$$

Farb raised the problem to compute the abelianization, that is the first integral homology group, of the group $\mathcal{M}_{g,r}[d]$ in Farb[4] Problem 5.23 p.43. Recently, Putman[20] also determined the abelianization of the level d congruence subgroup of the symplectic group and the level d mapping class group for odd d when $g \geq 3$. See also [21].

This paper is organized as follows. In section 2, we determine the commutator subgroup of the level d congruence subgroup of the symplectic group for every integer $d \geq 2$. This is mainly relies on the work of Mennicke[19] and Bass-Milnor-Serre[1] on congruence subgroups of the symplectic group. We also obtain the abelianization of $\Gamma_g[d]$ (Corollary 2.2). Let $\mathrm{spin}(M)$ be the set of spin structures of an oriented manifold M with trivial second Stiefel-Whitney class. In section 3, we will construct the injective homomorphism $\beta_\sigma : \mathcal{M}_{g,1}[2] \to \mathrm{Map}(H_1(\Sigma_g; \mathbf{Z}_2), \mathbf{Z}_8)$ for $\sigma \in \mathrm{spin}(\Sigma_g)$. We will determine the abelianization of the level 2 mapping class group using this homomorphism. Let n be a positive integer. Denote the Rochlin function by $R(M, \ ) : \mathrm{spin}(M) \to \mathbf{Z}_{16}$ for a $4n-1$-manifold M. For $\sigma \in \mathrm{spin}(M)$, $R(M, \sigma)$ is defined as the signature of a compact $4n$-manifold which spin bounds (M, σ). See for example Turaev[23]. We will define the homomorphism $\beta_\sigma(\varphi)$ using the difference $R(M, \sigma) - R(M, \sigma')$ for a mapping torus $M = M_\varphi$ of $\varphi \in \mathcal{M}_{g,1}[2]$. Turaev[23] proved that it can be written as the pin$^-$ bordism class of a surface embedded in the mapping torus. We can compute β_σ by examining this pin$^-$ bordism class.

The main theorem in this paper proved in Section 4 is illustrated as follows. For $\{x_i\}_{i=1}^n \subset H_1(\Sigma_{g,1}; \mathbf{Z}_2)$, define $I : H_1(\Sigma_{g,1};\mathbf{Z}_2)^n \to \mathbf{Z}_2$ by

$$I(x_1,x_2,\cdots,x_n) := \sum_{1\leq i<j\leq n} (x_i \cdot x_j) \bmod 2,$$

where $x_i \cdot x_j$ is the intersection number of x_i with x_j. We denote by $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ the free $\mathbf{Z}_8$-module generated by all formal symbol $[X]$ for $X \in H_1(\Sigma_{g,1})$. Define $\Delta_0^n : H_1(\Sigma_{g,1};\mathbf{Z})^n \to \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ by

$$\Delta_0^n(x_1,x_2,\cdots,x_n) = \sum_{i=1}^{n}[x_i] + \sum_{1\leq i<j\leq n}(-1)^{I(x_i,x_j)}[x_i+x_j] + \sum_{1\leq i<j<k\leq n}(-1)^{I(x_i,x_j,x_k)}[x_i+x_j+x_k]$$
$$+\cdots+(-1)^{I(x_1,x_2,\cdots,x_n)}[x_1+x_2+x_3+\cdots+x_n] \in \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)].$$

Theorem 1.2. *Let $g \geq 3$. Denote by $L_{g,1} \subset \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]$ the submodule generated by*

$$[0],\ 4\Delta_0^2(x_1,x_2),\ 2\Delta_0^3(x_1,x_2,x_3),\ \Delta_0^n(x_1,x_2,\cdots,x_n) \in \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)],$$

for $n \geq 3$ and $\{x_i\}_{i=1}^n \subset H_1(\Sigma_{g,1};\mathbf{Z}_2)$. Then, we have

$$\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L_{g,1} \cong H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}),$$

as an $\mathcal{M}_{g,1}$-module.

We also determine the abelianization of the level 2 mapping class group of a closed surface in subsection 4.4. In section 5, we determine the abelianization of the level d mapping class group for odd d. The main tool is the Johnson homomorphism of modulo d on the level d mapping class group. This derives from the extension of the Johnson homomorphism defined by Kawazumi[14]. For $H := H_1(\Sigma_{g,r};\mathbf{Z})$, denote by Λ^3H/H the cokernel of the homomophism

$$\begin{array}{ccc} H & \to & \Lambda^3 H \\ x & \mapsto & \sum_{i=1}^{g}(A_i\wedge B_i)\wedge x. \end{array}$$

Then, the abelianization of the level d mapping class group is written as:

Theorem 1.3. *For $g \geq 3$ and odd integer $d \geq 3$,*

$$\begin{aligned} H_1(\mathcal{M}_g[d];\mathbf{Z}) &= (\Lambda^3H/H\otimes\mathbf{Z}_d)\oplus H_1(\Gamma_g[d];\mathbf{Z}) \\ &= \mathbf{Z}_d^{(4g^3-g)/3}, \\ H_1(\mathcal{M}_{g,1}[d];\mathbf{Z}) &= (\Lambda^3H\otimes\mathbf{Z}_d)\oplus H_1(\Gamma_g[d];\mathbf{Z}) \\ &= \mathbf{Z}_d^{(4g^3+5g)/3}. \end{aligned}$$

2 The abelianization of the level d congruence subgroup of symplectic group

In this section, we determine the abelianization of the level d congruence subgroup $\Gamma_g[d]$ of the symplectic group $\mathrm{Sp}(2g;\mathbf{Z})$. We denote the identity matrix by I. A matrix $A \in \Gamma_g[d]$ can be written as $A = I + dA'$ with an integral $2g \times 2g$ matrix A'. Denote the matrix

$$A' = \begin{pmatrix} p(A) & q(A) \\ r(A) & s(A) \end{pmatrix},$$

where $p(A), q(A), r(A)$, and $s(A)$ are $g \times g$ matrices. We also denote the (i,j)-element of a matrix u by u_{ij}. For an even integer d, define the subgroup $\Gamma_g[d, 2d]$ of the symplectic group by

$$\Gamma_g[d, 2d] := \{A \in \Gamma_g[d] \mid q_{ii}(A) = r_{ii}(A) \equiv 0 \bmod 2 \text{ for } i = 1, 2 \cdots, g\}.$$

This subgroup was proved to be the normal subgroup of $\mathrm{Sp}(2g;\mathbf{Z})$ in Igusa[8] Lemma 1.(i).

We will prove in this section:

Proposition 2.1. *Let $g \geq 2$. For an odd integer d,*

$$\Gamma_g[d^2] = [\Gamma_g[d], \Gamma_g[d]].$$

For an even integer d,

$$\Gamma_g[d^2, 2d^2] = [\Gamma_g[d], \Gamma_g[d]].$$

Before proving Proposition 2.1, we calculate the abelianization of the congruence subgroup $\Gamma_g[d]$ using this proposition. First, we compute the module $\Gamma_g[d]/\Gamma_g[d^2] \cong H_1(\Gamma_g[d];\mathbf{Z})$ when d is an odd integer. For $A := I + dA', B = I + dB' \in \Gamma_g[d]$, we have

$$AB = I + d(A' + B') \bmod d^2. \tag{1}$$

Hence, we can define the surjective homomorphism $m : \Gamma_g[d] \to \mathbf{Z}_d^{2g^2+g}$ by

$$m(A) := (\{p_{ij}(A)\}_{1\leq i\leq g, 1\leq j\leq g}, \{q_{ij}(A)\}_{1\leq i\leq j\leq g}, \{r_{ij}(A)\}_{1\leq i\leq j\leq g}) \bmod d.$$

This is the restriction of the homomorphism of the level d congruence subgroup of $\mathrm{SL}(2g;\mathbf{Z})$ defined by Lee and Szczarba[17]. From the symplectic condition, we obtain $p(A) + {}^t s(A) \equiv 0, q(A) \equiv {}^t q(A)$, and $r(A) \equiv {}^t r(A)$ $\bmod d$. Then, we have the exact sequence

$$1 \longrightarrow \Gamma_g[d^2] \longrightarrow \Gamma_g[d] \xrightarrow{m} \mathbf{Z}_d^{2g^2+g} \longrightarrow 1. \tag{2}$$

This shows that $H_1(\Gamma_g[d];\mathbf{Z}) \cong \mathbf{Z}_d^{2g^2+g}$.

Next, we consider the case when d is even. We compute the group $\Gamma_g[d]/\Gamma_g[d^2, 2d^2] \cong H_1(\Gamma_g[d];\mathbf{Z})$. By the exact sequence (2), we have another exact sequence

$$0 \longrightarrow \frac{\Gamma_g[d^2]}{\Gamma_g[d^2, 2d^2]} \longrightarrow \frac{\Gamma_g[d]}{\Gamma_g[d^2, 2d^2]} \xrightarrow{m} \mathbf{Z}_d^{2g^2+g} \longrightarrow 0. \tag{3}$$

For a matrix

$$A = I + d^2 \begin{pmatrix} p(A) & q(A) \\ r(A) & s(A) \end{pmatrix} \in \Gamma_g[d^2],$$

define the surjective homomorphism $m'_1 : \Gamma_g[d^2] \to \mathbf{Z}_2^{2g}$ by

$$m'_1(A) := (\{q_{ii}(A)\}_{i=1}^g, \{r_{ii}(A)\}_{i=1}^g) \bmod 2.$$

The kernel is equal to $\Gamma_g[d^2, 2d^2]$. Hence, this induces the isomorphism $\Gamma_g[d^2]/\Gamma_g[d^2, 2d^2] \cong \mathbf{Z}_2^{2g}$. The exact sequence (3) is consequently wrriten as

$$0 \longrightarrow \mathbf{Z}_2^{2g} \longrightarrow \frac{\Gamma_g[d]}{\Gamma_g[d^2, 2d^2]} \xrightarrow{\ m\ } \mathbf{Z}_d^{2g^2+g} \longrightarrow 0. \tag{4}$$

For a homology class $y \in H_1(\Sigma_{g,r}; \mathbf{Z})$, define the transvection $T_y \in \mathrm{Sp}(2g; \mathbf{Z})$ by $T_y(x) := x + (y \cdot x)y$. Then by the exact sequence (4), we see that $\operatorname{Ker} m$ is generated by the elements $T_{A_i}^{d^2}, T_{B_i}^{d^2}$, where $i = 1, 2, \cdots, g$. Since $q_{ii}(T_{A_i}^d) = 1$ and $r_{ii}(T_{B_i}^d) = 1$, the order of $T_{A_i}^d, T_{B_i}^d \in \Gamma_g[d]/\Gamma_g[d^2, 2d^2]$ are $2d$. Hence we have:

Corollary 2.2. *For $g \geq 2$,*

$$H_1(\Gamma_g[d]; \mathbf{Z}) = \begin{cases} \mathbf{Z}_d^{2g^2+g} & \text{if } d \text{ is odd,} \\ \mathbf{Z}_d^{2g^2-g} \oplus \mathbf{Z}_{2d}^{2g} & \text{if } d \text{ is even.} \end{cases}$$

2.1 Proof of Proposition 2.1

In this subsection, we prove Proposition 2.1.

By the equation (1), we have

$$[\Gamma_g[d], \Gamma_g[d]] \subset \Gamma_g[d^2]$$

for every $d \geq 2$. In particular, if d is even, it is shown that

$$[\Gamma_g[d], \Gamma_g[d]] \subset \Gamma_g[d^2, 2d^2]$$

in Igusa[8] Lemma 1.(ii). Hence, it suffices to prove

$$\Gamma_g[d^2] \subset [\Gamma_g[d], \Gamma_g[d]], \text{ for } d \text{ odd, and}$$
$$\Gamma_g[d^2, 2d^2] \subset [\Gamma_g[d], \Gamma_g[d]], \text{ for } d \text{ even.}$$

First, we show that $2d[T_{A_1}^d] = 0$ for every d. A straightforward computation shows the following lemma.

Lemma 2.3. *For $g \geq 2$ and $d \geq 1$,*

$$T_{a_1A_1+b_1B_1+a_2A_2}^d = (T_{A_2}^d)^{(a_1b_1+1)a_2^2}(T_{B_1+A_2}^d T_{A_2}^{-d} t_{B_1}^{-d})^{b_1a_2}(T_{A_1+A_2}^d t_{A_1}^{-d} T_{A_2}^{-d})^{a_1a_2} T_{a_1A_1+b_1B_1}^d.$$

If we put $a_1 = 1, a_2 = -1, b_1 = 0$, we obtain

$$[T_{A_1+A_2}^d] + [T_{A_1-A_2}^d] = 2[T_{A_1}^d] + 2[T_{A_2}^d].$$

Let $x, y \in H_1(\Sigma_{g,r}; \mathbf{Z})$ be elements such that $x \cdot y = 0$ and $\{x, y\}$ can be extended to form a basis of $H_1(\Sigma_{g,r}; \mathbf{Z})$). Then, there exists $\varphi \in \mathcal{M}_{g,r}$ which satisfies $\varphi_*(x) = A_1$, and $\varphi_*(y) = A_2$. This shows that:

Lemma 2.4. *If $x, y \in H_1(\Sigma_{g,r}; \mathbf{Z})$ satisfy $x \cdot y = 0$, and $\{x, y\}$ can be extended to form a basis of $H_1(\Sigma_{g,r}; \mathbf{Z})$, then we have*

$$[T_{x+y}^d] + [T_{x-y}^d] = 2[T_x^d] + 2[T_y^d].$$

Remark 2.5. *For $i = 1, 2, 3, 4$, let $D_i \subset S^2$ be mutually disjoint disks. By the assumption of x, y in Lemma 2.4, we can choose an embedding $i : S^2 - \amalg_{i=1}^4 D_i \to \Sigma_{g,r}$ such that $[i(\partial D_1)] = -[i(\partial D_2)] = x$, and $[i(\partial D_3)] = -[i(\partial D_4)] = y$. The Lantern relation of this embedding*

$$T_{x+y}T_{x-y} = T_x^2 T_y^2 \in \mathrm{Sp}(2g; \mathbf{Z})$$

also shows the above relation.

Put $x = kA_1 + A_2$, $y = A_1$ in the equation of Lemma 2.4, and take the summation over $k = 1, 2, \cdots, d-1$. Then, we have

$$2d[T_{A_1}^d] = 0 \in H_1(\Gamma_g[d]; \mathbf{Z}). \tag{5}$$

Next, we show that $d[T_{A_1}^d] = 0$ when d is odd. By the equation (5) and Lemma 2.4, we have

$$d[T_{x+2ky}^d] = d[T_{x+2(k+1)y}^d]$$

for $k \in \mathbf{Z}$. If d is odd, we obtain

$$d[T_x^d] = d[T_{x+y}^d] = d[T_y^d]. \tag{6}$$

If we put $a_1 = b_1 = 2$, and $a_2 = 1$ in Lemma 2.3, we have

$$[T_{2A_1+2B_1+A_2}^d] = 5[T_{A_2}]^d + 2([T_{B_1+A_2}^d] - [T_{A_2}^d] - [T_{B_1}^d]) + 2([T_{A_1+A_2}^d] - [T_{A_1}^d] - [T_{A_2}^d]) + [T_{2A_1+2B_1}^d]. \tag{7}$$

If we apply the equation (6) to d times the equation (7), we have

$$d[T_{A_1}^d] = 0 \in H_1(\Gamma_g[d]; \mathbf{Z}). \tag{8}$$

We need the theorem proved by Mennicke[19], which is essential in this proof.

Theorem 2.6 (Mennicke)**.** *Let $g \geq 2$ and $d > 0$. If Q is a normal subgroup of $\mathrm{Sp}(2g; \mathbf{Z})$ which contains $T_{A_1}^d$, then*

$$\Gamma_g[d] \subset Q.$$

By the equation (8), we have $T_{A_1}^{d^2} \in [\Gamma_g[d], \Gamma_g[d]]$ when d is odd. By the equation (5), we also have $T_{A_1}^{2d^2} \in [\Gamma_g[d], \Gamma_g[d]]$ when d is even. Hence we obtain

$$\Gamma_g[2d^2] \subset [\Gamma_g[d], \Gamma_g[d]] \text{ if } d \text{ is even, and}$$
$$\Gamma_g[d^2] \subset [\Gamma_g[d], \Gamma_g[d]] \text{ if } d \text{ is odd.}$$

Thus, we have proved the case when d is odd. To prove the case when d is even, it suffices to show:

Lemma 2.7. *If d is even,*

$$\Gamma_g[d^2, 2d^2] \subset [\Gamma_g[d], \Gamma_g[d]].$$

Proof. We have already known that $\Gamma_g[2d^2] \subset [\Gamma_g[d], \Gamma_g[d]]$. Hence, we examine the quotient group $\Gamma_g[d^2, 2d^2]/\Gamma_g[2d^2]$. The symplectic group $\mathrm{Sp}(2g; \mathbf{Z})$ acts on $\Gamma_g[d^2, 2d^2]/\Gamma_g[2d^2]$ by the conjugation action. For $1 \leq i, j \leq 2g$, denote the $2g \times 2g$ matrix $e_{i\,j}$ which has 1 in the (i, j)-element, and 0 in the other elements. First, we prove that $\Gamma_g[d^2, 2d^2]/\Gamma_g[2d^2]$ is generated by $I + d^2(e_{1\,g+2} + e_{2\,g+1})$, $I + d^2(e_{1\,1} - e_{g+1\,g+1})$ as a $\mathrm{Sp}(2g; \mathbf{Z})$-module.

Similar to the homomorphism m, for a matrix

$$A = I + d^2 \begin{pmatrix} p(A) & q(A) \\ r(A) & s(A) \end{pmatrix} \in \Gamma_g[d^2, 2d^2],$$

we define the surjective homomorphism $m_2' : \Gamma_g[d^2, 2d^2] \to \mathbf{Z}_2^{2g^2-g}$ by

$$m_2' := (\{p_{ij}(A)\}_{1\le i\le g, 1\le j\le g}, \{q_{ij}(A)\}_{1\le i<j\le g}, \{r_{ij}(A)\}_{1\le i<j\le g}) \mod 2.$$

Then, it is easy to see that $\operatorname{Ker} m_2' = \Gamma_g[2d^2]$, and m_2' induces the isomorphism $\Gamma_g[d^2, 2d^2]/\Gamma_g[2d^2] \cong \mathbf{Z}_2^{2g^2-g}$. For i, j such that $1 \le i, j \le g$, $i \ne j$, there are elements of $\mathrm{Sp}(2g; \mathbf{Z})$ which map 4-tuple of homology classes (A_1, A_2, B_1, B_2) to

$$(A_i, A_j, B_i, B_j), (B_i, B_j, -A_i, -A_j), \text{ and } (A_i, -B_j, B_i, A_j),$$

respectively. By the conjugation action, these elements send $I + d^2(e_{1\,g+2} + e_{2\,g+1})$ to

$$I + d^2(e_{i\,g+j} + e_{j\,g+i}), I + d^2(e_{g+i\,j} + e_{g+j\,i}), \text{ and } I + d^2(e_{i\,j} - e_{g+j\,g+i}) \in \Gamma_g[d^2, 2d^2]/\Gamma_g[2d^2],$$

respectively. Denote the Kronecker delta by δ_{ij}. Then we have

$$p_{kl}(I + d^2(e_{i\,g+j} + e_{j\,g+i})) = q_{kl}(I + d^2(e_{g+i\,j} + e_{g+j\,i})) = r_{kl}(I + d^2(e_{i\,j} - e_{g+j\,g+i})) = \delta_{ik}\delta_{jl}. \tag{9}$$

In the same way, there is an element of $\mathrm{Sp}(2g; \mathbf{Z})$ which map the pair (A_1, B_1) to (A_i, B_i). This element sends $I + d^2(e_{1\,1} - e_{g+1\,g+1})$ to $I + d^2(e_{i\,i} - e_{g+i\,g+i})$. Note that

$$p_{kk}(I + d^2(e_{i\,i} + e_{g+i\,g+i})) = \delta_{ik}. \tag{10}$$

Then we see that from the equations (9) and (10), $\Gamma_g[d^2, 2d^2]/\Gamma_g[2d^2] \cong \mathbf{Z}_2^{2g^2-g}$ is generated by the elements $I + d^2(e_{1\,g+2} + e_{2\,g+1})$, $I + d^2(e_{1\,1} - e_{g+1\,g+1})$ as a $\mathrm{Sp}(2g; \mathbf{Z})$-module.

Next, we will show that

$$I + d^2(e_{1\,1} - e_{g+1\,g+1}), I + d^2(e_{1\,g+2} + e_{2\,g+1}) \in [\Gamma_g[d], \Gamma_g[d]]. \tag{11}$$

For $A = I + dA', B = I + dB' \in \Gamma_g[d]$, we have

$$ABA^{-1}B^{-1} \equiv I + d^2(A'B' - B'A') \mod d^3.$$

If we put $A' = e_{1\,g+1}, B' = e_{g+1\,1}$, and $A' = e_{1\,2} - e_{g+2\,g+1}, B' = e_{2\,g+2}$, we get (11).

The fact (11) shows

$$\Gamma_g[d^2, 2d^2] \subset [\Gamma_g[d], \Gamma_g[d]].$$

This proves the lemma. □

Hence, we complete the proof of Proposition 2.1.

3 The abelianization of the level 2 mapping class group

In this section, we will define a family of homomorphisms

$$\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \Omega_2^{pin^-} \cong \mathbf{Z}_8,$$

for $\sigma \in \mathrm{spin}(\Sigma_g)$ and $x \in H_1(\Sigma_g;\mathbf{Z}_2)$ (Lemma 3.2). This family determines the abelianization of the level 2 mapping class group. The homomorphism $\beta_{\sigma,x}$ is proved to be an extension of the homomorphism $\omega_{\sigma,y}$ defined by Heap [6] to the level 2 mapping class group (Subsection 3.5). We will calculate the values of this homomorphism on generators of the level 2 mapping class group using the Brown invariant (Proposition 3.8).

3.1 Spin structures of mapping tori

Fix a closed disk neighborhood $N(c_0)$ of a point c_0 in Σ_g. The mapping class group $\pi_0 \operatorname{Diff}_+(\Sigma_g, N(c_0))$ is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g which fix the neighborhood $N(c_0)$ pointwise. By restricting each diffeomorphism to $\Sigma_g - \operatorname{Int} N(c_0)$, the group $\pi_0 \operatorname{Diff}_+(\Sigma_g, N(c_0))$ is isomorphic to $\mathcal{M}_{g,1}$. Hence, we identify these two groups. We also identify the kernel $\operatorname{Ker}(\pi_0 \operatorname{Diff}_+(\Sigma_g, N(c_0)) \to \operatorname{Sp}(2g;\mathbf{Z}_2))$ of the mod 2 reduction of ρ with $\mathcal{M}_{g,1}[2]$.

For $\varphi = [f] \in \mathcal{M}_{g,1}$, denote the mapping torus of φ by $M := M_\varphi := \Sigma_g \times [0,1]/\sim$, where the equivalence relation is given by $(f(x),0) \sim (x,1)$. In this subsection, we define a map $\theta : \mathrm{spin}(\Sigma_g) \to \mathrm{spin}(M_\varphi)$ for $\varphi \in \mathcal{M}_{g,1}[2]$.

First, we define the spin structure of an oriented vector bundle. Let $E \to V$ be a smooth oriented vector bundle of rank n on a smooth manifold V. We denote by $P(E)$ the oriented frame bundle associated to this bundle. When the Stiefel-Whitney class w_2 of E vanishes, we define the spin structure of E by a right inverse homomorphism of the natural homomorphism $H_1(P(E);\mathbf{Z}_2) \to H_1(V;\mathbf{Z}_2)$. Denote by $\mathrm{spin}(E)$ the set of spin structure of E. Since $P(E)$ is a principal $GL_+(n)$ bundle and w_2 vanishes, the Serre spectral sequence shows that

$$0 \longrightarrow H^1(V;\mathbf{Z}_2) \longrightarrow H^1(P(E);\mathbf{Z}_2) \longrightarrow \mathbf{Z}_2 \longrightarrow 0$$

is exact. Define the injective map

$$\mathrm{spin}(E) \to H^1(P(E);\mathbf{Z}_2)$$

by $\sigma \mapsto v$, where v is the unique nontrivial element in $\operatorname{Ker}\sigma$. The element $v \in H^1(P(E);\mathbf{Z}_2)$ restricts to an non-trivial element in each fiber of $P(E) \to V$. This is also equivalent to consider the double cover of the orthonormal frame bundle associated to the bundle E with a fiber metric. In detail, for example, see Lee-Miller-Weintraub[16] Section 1.1. For an oriented smooth n-manifold V, we define the spin structure of V by the spin structure of the tangent bundle TV. We denote simply by $\mathrm{spin}(V) := \mathrm{spin}(TV)$ the set of spin structure on V. Note that a spin structure of V is equivalent to a spin structure of $V \times (-\epsilon,\epsilon)^k$, for $\epsilon > 0$ and $k > 0$.

Next, we define the injective map $\theta : \mathrm{spin}(\Sigma_g) \to \mathrm{spin}(M_\varphi)$. Fix a spin structure on Σ_g. Since $\varphi \in \mathcal{M}_{g,1}[2]$ acts on $H_1(\Sigma_g;\mathbf{Z}_2)$ trivially, the Wang exact sequence is written as

$$0 \longrightarrow H_1(\Sigma_g;\mathbf{Z}_2) \longrightarrow H_1(M;\mathbf{Z}_2) \longrightarrow H_1(S^1;\mathbf{Z}_2) \longrightarrow 0.$$

The inclusion map $l : N(c_0) \times S^1 \to M$ gives the splitting

$$H_1(M;\mathbf{Z}_2) = H_1(\Sigma_g;\mathbf{Z}_2) \oplus H_1(S^1;\mathbf{Z}_2).$$

In order to define the spin structure on M, we will construct homomorphisms from each direct summand to $H_1(P(M);\mathbf{Z}_2)$. For $N(c_0) \times S^1 \subset M$, define the framing $\hat{l} : S^1 \to P(N(c_0) \times S^1)$ by $\hat{l}(t) = (v_0 \cos 2\pi t +$

$v_1 \sin 2\pi t, v_1 \cos 2\pi t - v_0 \sin 2\pi t, v_{S^1}(t))$, where $\{v_0, v_1\}$ is a frame of $T_{c_0}N(c_0)$, and $v_{S^1}(t) \in T_t S^1$ is a nonzero tangent vector. This framing induces the homomorphism

$$H_1(S^1; \mathbf{Z}_2) \xrightarrow{\hat{\iota}} H_1(P(N(c_0) \times S^1); \mathbf{Z}_2) \xrightarrow{\text{inc}_*} H_1(P(M); \mathbf{Z}_2),$$

where inc_* is the homomorphism induced by the inclusion map.

Next, consider the natural smooth map $P(\Sigma_g \times (-\epsilon, \epsilon)) \to P(M)$ induced by the inclusion of a tubular neighborhood $\Sigma_g \times (-\epsilon, \epsilon) \subset M$ for small ϵ. Using the spin structure on Σ_g, we have the homomorphism

$$H_1(\Sigma_g; \mathbf{Z}_2) \xrightarrow{\sigma} H_1(P(\Sigma_g \times (-\epsilon, \epsilon)); \mathbf{Z}_2) \xrightarrow{\text{inc}_*} H_1(P(M); \mathbf{Z}_2). \quad (12)$$

Thus, we have constructed the homomorphism $H_1(M; \mathbf{Z}_2) \to H_1(P(M); \mathbf{Z}_2)$. In this way, we obtain the map $\theta : \text{spin}(\Sigma_g) \to \text{spin}(M)$.

3.2 A spin manifold bounded by Mapping tori

Let $P_0 := S^2 - \amalg_{i=1}^3 \text{Int}\, D_i$ denote a pair of pants, where $\{D_i\}$ are mutually disjoint disks and $\text{Int}\, D_i$ is interior of D_i in S^2. Pick the paths $\alpha, \beta, \gamma \in \pi_1(P_0, x_0)$ going once round boundary components as in Figure 1. Denote by $\text{Diff}_+(\Sigma_g, N(c_0))[2]$ the kernel of the representation of $\text{Diff}_+(\Sigma_g, N(c_0))$ on $H_1(\Sigma_g; \mathbf{Z}_2)$. Consider Σ_g bundles with its structure group $\text{Diff}_+(\Sigma_g, N(c_0))[2]$. For $\varphi, \psi \in \mathcal{M}_{g,1}[2]$, there exists a Σ_g bundle $p : W = W_{\varphi,\psi} \to P_0$ such that the topological monodromy $\pi_1(P_0, x_0) \to \mathcal{M}_{g,1}[2]$ sends α, β, and $\gamma \in \pi_1(P_0, x_0)$ to φ, ψ, and $(\varphi\psi)^{-1} \in \mathcal{M}_{g,1}[2]$, respectively. This bundle is unique up to diffeomorphism. Note that the boundary ∂W is diffeomorphic to the disjoint sum $M_\varphi \amalg M_\psi \amalg M_{(\varphi\psi)^{-1}}$.

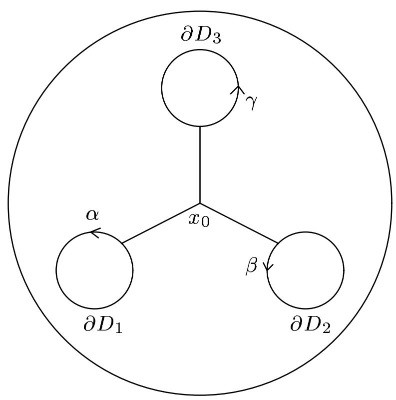

Figure 1: loops in a pair of pants

In this subsection, we define a spin structure of W. We show that the induced spin structure on each boundary component of W is equal to that of the mapping torus defined in the last subsection.

Since $\varphi, \psi \in \mathcal{M}_{g,1}[2]$ act on $H_1(\Sigma_g; \mathbf{Z}_2)$ trivially, we have the splitting

$$H_1(W; \mathbf{Z}_2) = H_1(\Sigma_g; \mathbf{Z}_2) \oplus H_1(P_0; \mathbf{Z}_2)$$

by the inclusion map $N(c_0) \times P_0 \to W$. In order to define the spin structure on W, we will construct homomorphisms from each direct summand to $H_1(P(W); \mathbf{Z}_2)$. By the local triviality of the bundle $W \to P_0$, we have a neighborhood $\Sigma_g \times (-\epsilon, \epsilon)^2 \subset W$ of the fiber on $x_0 \in P_0$. Define the homomorphism

$$H_1(\Sigma_g; \mathbf{Z}_2) \xrightarrow{\sigma} H_1(P(\Sigma_g \times (-\epsilon, \epsilon)^2); \mathbf{Z}_2) \xrightarrow{\text{inc}_*} H_1(P(W); \mathbf{Z}_2). \quad (13)$$

Next, we will construct the homomorphism $H_1(P_0;\mathbf{Z}_2) \to H_1(P(W);\mathbf{Z}_2)$. In the disk $D^2 = \{(x,y) \in \mathbf{R}^2 \mid x^2 + y^2 \le 1\}$, choose two mutually disjoint disks $D_1, D_2 \subset \operatorname{Int} D^2$. Choose an orthonormal frame $\{v_0', v_1'\}$ of $\mathbf{R}^2$. Let $s : D^2 - D_1 - D_2 \to P(D^2 - D_1 - D_2) = (D^2 - D_1 - D_2) \times \mathbf{R}^2$ be the trivial framing defined by $s(x) = (x, v_0', v_1')$. By identifying P_0 with $D^2 - D_1 - D_2$, we have the map $\hat{l}' : P_0 \to P(N(c_0) \times P_0)$ by $\hat{l}'(x) = (v_0, v_1, s(x))$. This map and the inclusion $N(c_0) \times P_0 \to W$ induce the homomorphism

$$H_1(P_0;\mathbf{Z}_2) \xrightarrow{\hat{l}'} H_1(P(N(c_0) \times P_0);\mathbf{Z}_2) \xrightarrow{\mathrm{inc}_*} H_1(P(W);\mathbf{Z}_2). \tag{14}$$

Define the spin structure of W by the homomorphisms (13) and (14).

Note that the homomorphism (13) is equal to the composite of (12) and the inclusion homomorphism $H_1(P(M \times [0,\epsilon));\mathbf{Z}_2) \to H_1(P(W);\mathbf{Z}_2)$, where $M \times [0,\epsilon) \subset W$ is a collar neighborhood. We also see that the diagram

$$\begin{array}{ccc} H_1(\partial D_i;\mathbf{Z}_2) & \longrightarrow & H_1(P_0;\mathbf{Z}_2) \\ \downarrow & & \downarrow \\ H_1(P(N(c_0) \times S^1);\mathbf{Z}_2) & \longrightarrow & H_1(P(N(c_0) \times P_0);\mathbf{Z}_2) \end{array}$$

commutes. Hence, the manifold W is spin bounded by M_φ, M_ψ, and $M_{(\varphi\psi)^{-1}}$ which were defined in the last subsection.

3.3 The homomorphism $\beta_{\sigma,x}$ on the level 2 mapping class group

In this subsection, we will construct a homomorphism which determines the abelianization of the group $\mathcal{M}_{g,1}[2]$, using the Rochlin functions of mapping tori.

First we review the simply transitive action of $H_1(\Sigma_g;\mathbf{Z}_2)$ on $\operatorname{spin}(\Sigma_g)$. Identify $H_1(\Sigma_g;\mathbf{Z}_2)$ with $H^1(\Sigma_g;\mathbf{Z}_2)$ by the Poincaré duality. By the Serre spectral sequence, we have the exact sequence

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow H_1(P(\Sigma_g);\mathbf{Z}_2) \longrightarrow H_1(\Sigma_g;\mathbf{Z}_2) \longrightarrow 1.$$

For $x \in H^1(\Sigma_g;\mathbf{Z}_2) = \operatorname{Hom}(H_1(\Sigma_g;\mathbf{Z}_2),\mathbf{Z}_2)$, we denote again by $x : H_1(\Sigma_g;\mathbf{Z}_2) \to H_1(P(\Sigma_g);\mathbf{Z}_2)$ the composite of $x : H_1(\Sigma_g;\mathbf{Z}_2) \to \mathbf{Z}_2$ and the inclusion $\mathbf{Z}_2 \subset H_1(P(\Sigma_g);\mathbf{Z}_2)$. Hence, for $\sigma \in \operatorname{spin}(\Sigma_g)$, we have another spin structure $\sigma + x : H_1(\Sigma_g;\mathbf{Z}_2) \to H_1(P(\Sigma_g);\mathbf{Z}_2)$. In this way, $H_1(\Sigma_g;\mathbf{Z}_2)$ acts on $\operatorname{spin}(\Sigma_g)$.

Every spin 3-manifold is known to bound a spin 4-manifold. For $\varphi \in \mathcal{M}_{g,1}[2]$, choose a compact oriented spin manifold V which is spin bounded by the mapping torus $M = M_\varphi$. Then the Rochlin function of (M,σ) is defined by

$$R(M,\sigma) := \operatorname{Sign} V \bmod 16.$$

This is well-defined by Rochlin's theorem, and is called the Rochlin function.

Definition 3.1. *For $x \in H_1(\Sigma_g;\mathbf{Z}_2)$ and $\sigma \in \operatorname{spin}(\Sigma_g)$, define the map*

$$\beta_{\sigma,x} : \mathcal{M}_{g,1}[2] \to \Big(\frac{1}{2}\mathbf{Z}\Big)/8\mathbf{Z}$$

by $\beta_{\sigma,x}(\varphi) := (R(M_\varphi,\theta(\sigma)) - R(M_\varphi),\theta(\sigma+x))/2 \mod 8$.

Here, $\theta : \operatorname{spin}(\Sigma_g) \to \operatorname{spin}(M_\varphi)$ is the map defined in Subsection 3.2.

Lemma 3.2. *$\beta_{\sigma,x}$ is a homomorphism.*

Proof. As we saw in Subsection 3.2, for $\varphi, \psi \in \mathcal{M}_{g,1}[2]$, the spin 4-manifold $W_{\varphi,\psi}$ is spin bounded by the mapping tori $M_\varphi \amalg M_\psi \amalg M_{(\varphi\psi)^{-1}}$. Therefore, we have

$$R(M_\varphi, \theta(\sigma)) + R(M_\psi, \theta(\sigma)) - R(M_{\varphi\psi}, \theta(\sigma)) \equiv \operatorname{Sign} W_{\varphi,\psi},$$
$$R(M_\varphi, \theta(\sigma+x)) + R(M_\psi, \theta(\sigma+x)) - R(M_{\varphi\psi}, \theta(\sigma+x)) \equiv \operatorname{Sign} W_{\varphi,\psi} \mod 16.$$

Hence, we have $\beta_{\sigma,x}(\varphi\psi) = \beta_{\sigma,x}(\varphi) + \beta_{\sigma,x}(\psi)$. □

As we will show in Subsection 3.4, the image $\operatorname{Im}\beta_{\sigma,x}$ is in $\mathbf{Z}_8$. Denote by $\operatorname{Map}(H_1(\Sigma_g;\mathbf{Z}_2);\mathbf{Z}_8)$ the free $\mathbf{Z}_8$-module consisting of all maps $H_1(\Sigma_g;\mathbf{Z}_2) \to \mathbf{Z}_8$. We can define the homomorphism $\beta_\sigma : \mathcal{M}_{g,1}[2] \to \operatorname{Map}(H_1(\Sigma_g;\mathbf{Z}_2);\mathbf{Z}_8)$ by $\beta_\sigma(\varphi)(x) = \beta_{\sigma,x}(\varphi)$.

3.4 Brown invariant

The $\mathrm{Pin}^-(n)$ group is a central extension of the $O(n)$ bundle. Let F be a (not necessarily orientable) closed surface. A pin$^-$ structure of F is defined by the principal $\mathrm{Pin}^-(2)$ bundle associated to the tangent bundle TF, where $\mathrm{Pin}^-(2)$ acts on $\mathbf{R}^2$ via its covering projection to $O(2)$. Brown defined the invariant of a closed surface F with a pin$^-$ structure, called the Brown invariant. In this subsection, We review this invariant and its relation to the Rochlin functions stated by Turaev[23].

Definition 3.3. *Let F be a (not necessarily orientable) closed surface. If a function $\hat{q} : H_1(F;\mathbf{Z}_2) \to \mathbf{Z}_4$ satisfies $\hat{q}(x+y) = \hat{q}(x) + \hat{q}(y) + 2x \cdot y$, we call $\hat{q}$ the quadratic enhancement.*

On a closed surface F, a pin$^-$ structure α induces a quadratic enhancement $\hat{q}_\alpha$ as follows. Denote the determinant line bundle of the tangent bundle by $\det F$. Then $E := TF \oplus \det F$ has a canonical orientation. The set of pin$^-$ structures of F is known to corresponds bijectively to the set of spin structures of E. In detail, see Kirby-Taylor [15]. For an element $v \in H_1(F;\mathbf{Z}_2)$, choose a simple closed curve $K \subset F$ which represents v. Denote the normal bundle of $TK \subset TF|_K$ and $TF \subset E|_F$ by $N(F/K)$ and $N(E/F)$, respectively. Then the restriction $E|_K$ can be written as $E|_K = TK \oplus N(F/K) \oplus N(E/F)|_K$. If we fix the orientation of K, the bundle $E' = N(F/K) \oplus N(E/F)|_K$ gets also oriented.

Choose a non-zero section $s_K : K \to TK$. We call a framing $s : K \to P(E')$ is odd if and only if the induced homomorphism $s \oplus s_K : H_1(K;\mathbf{Z}_2) \to H_1(P(E);\mathbf{Z}_2)$ is not equal to the homomorphism $H_1(K;\mathbf{Z}_2) \to H_1(P(E);\mathbf{Z}_2)$ induced by the spin structure of E.

Definition 3.4. *Choose an odd framing on K. Using it, count the number of right half twists that the odd framing of K makes in a complete traverse with F. We denote this number by $\hat{q}_\alpha(v)$. This induces the map $\hat{q}_\alpha : H_1(F;\mathbf{Z}_2) \to \mathbf{Z}_4$. We call it the quadratic enhancement of a pin$^-$ structure α.*

This number does not depend on the choice of the representative of a homology class, the orientation of K, and the odd framing of $N(F/K) \oplus N(E/F)$. In detail, see Kirby-Taylor [15] section 3. In particular, if F is an orientable surface, this number is equal to twice the quadratic function induced by the spin structure on F.

Definition 3.5. *Let F be a closed surface with its pin$^-$ strucure α. Then, the Brown invarant $B_\alpha \in \mathbf{Z}_8$ of α is defined by the equation*

$$\sqrt{|H_1(F;\mathbf{Z}_2)|}\exp(2\pi\sqrt{-1}B_\alpha/8) = \sum_{x \in H_1(F;\mathbf{Z}_2)} \exp(2\pi\sqrt{-1}\hat{q}_\alpha(x)/4).$$

Consider a closed surface F which represents $s \in H_2(M;\mathbf{Z}_2)$. Then, the surface F has canonical pin$^-$ structure induced by the spin structure of the tubular neighborhood of F. Furthermore, the pin$^-$ bordism class of F does not depend on the representative of $s \in H_2(M;\mathbf{Z}_2)$ (Kirby-Taylor [15] (4.8)). Denote the pin$^-$ bordism group by $\Omega_*^{pin^-}$. It is known that the Brown invariant gives the isomorphism $\Omega_2^{pin^-} \cong \mathbf{Z}_8$ (Kirby-Taylor[15] Lemma 3.6). For $\sigma, \sigma' \in \mathrm{spin}(M)$, Turaev[23] showed that the difference $R(M,\sigma) - R(M,\sigma')$ is written by the Brown invariant of the pin$^-$ structure of an embedded surface.

Lemma 3.6 (Turaev [23] Lemma 2.3). *Let M be a closed manifold with its Stiefel-Whitney class $w_2 = 0$. Denote the closed surface $F \subset M$ which represents the Poincaré dual of $x \in H^1(M;\mathbf{Z}_2)$. For a spin structure σ of M, denote the induced pin$^-$ structure α of F. Then we have*

$$R(M,\sigma) - R(M,\sigma + x) = 2B_\alpha.$$

Apply the lemma to the case when M is a mapping torus. Then, we obtain $\beta_{\sigma,x}(\varphi) = B_\alpha \in \mathbf{Z}_8$, for $\varphi \in \mathcal{M}_{g,1}[2]$ and $x \in H_1(\Sigma_g;\mathbf{Z}_2)$.

3.5 Heap's homomorphism

In this subsection, we review the homomorphism $\omega_{\sigma,y} : \mathcal{I}_{g,1} \to \mathbf{Z}_2$ defined by Heap[6], and show that the homomorphism $\beta_{\sigma,x}$ defined in Subsection 3.3 is the extension of $\omega_{\sigma,y}$ to the level 2 mapping class group.

First we define a spin manifold $M' = M'_\varphi$. For $\sigma \in \mathrm{spin}(\Sigma_g)$ and $\varphi \in \mathcal{M}_{g,1}[2]$, endow the spin structure $\theta(\sigma)$ on the mapping torus $M := M_\varphi$. Denote by $M' = (M - N(c_0) \times S^1) \cup (\partial N(c_0) \times D^2)$ the manifold obtained by the elementary surgery on $N(c_0) \times S^1 \subset M$. We can choose the spin structure of $\partial N(c_0) \times D^2$ so that it induces in the boundary $\partial N(c_0) \times S^1$ the spin structure induced by $\theta(\sigma)$. Hence, the elementary surgery is compatible with the spin structure, and M' has the induced spin structure.

Next, we define Heap's homomorphism $\omega_{\sigma,y}$. For a group G, denote the spin bordism group of $K(G,1)$-space by $\Omega_*^{spin}(G) := \Omega_*^{spin}(K(G,1))$. Note that if $\varphi \in \mathcal{I}_{g,1}$, we have $H_1(\Sigma_g;\mathbf{Z}) \cong H_1(M';\mathbf{Z})$. Hence, we have the canonical homomorphism $\pi_1(M') \to H_1(\Sigma_g;\mathbf{Z})$. Let $f : M' \to K(H_1(\Sigma_g;\mathbf{Z}),1)$ be a continuous map corresponding to this homomorphism. This map induces the homomorphism

$$\eta_{\sigma,2} : \mathcal{I}_{g,1} \to \Omega_3^{spin}(H_1(\Sigma_g;\mathbf{Z})),$$

which maps $\varphi \in \mathcal{I}_{g,1}$ to $[(f, M')]$. In the same fashion, if $\varphi \in \mathcal{M}_{g,1}[2]$, we have $H_1(\Sigma_g;\mathbf{Z}_2) \cong H_1(M';\mathbf{Z}_2)$. Let

$$\eta_{\sigma,2}[2] : \mathcal{M}_{g,1}[2] \to \Omega_3^{spin}(H_1(\Sigma_g;\mathbf{Z}_2))$$

be the homomorphism induced by $\pi_1(M') \to H_1(\Sigma_g;\mathbf{Z}_2)$.

For $y \in H^1(\Sigma_g;\mathbf{Z}) = \mathrm{Hom}(H_1(\Sigma_g;\mathbf{Z}),\mathbf{Z})$, we have the commutative diagram

$$\begin{array}{ccc} H_1(\Sigma_g;\mathbf{Z}) & \xrightarrow{\;y\;} & \mathbf{Z} \\ {\scriptstyle \bmod 2}\downarrow & & \downarrow{\scriptstyle \bmod 2} \\ H_1(\Sigma_g;\mathbf{Z}_2) & \xrightarrow{y \bmod 2} & \mathbf{Z}_2. \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccccc} \mathcal{I}_{g,1} & \xrightarrow{\eta_{\sigma,2}} & \Omega_3^{spin}(H_1(\Sigma_g;\mathbf{Z})) & \xrightarrow{y_*} & \Omega_3^{spin}(\mathbf{Z}) \cong \mathbf{Z}_2 \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_{g,1}[2] & \xrightarrow{\eta_{\sigma,2}[2]} & \Omega_3^{spin}(H_1(\Sigma_g;\mathbf{Z}_2)) & \xrightarrow{(y \bmod 2)_*} & \Omega_3^{spin}(\mathbf{Z}_2) \cong \mathbf{Z}_8. \end{array}$$

Then, Heap's homomorphism

$$\omega_{\sigma,y} : \mathcal{I}_{g,1} \to \mathbf{Z}_2$$

is defined by $\omega_{\sigma,y} = y_*\eta_{\sigma,2}$ for $y \in H_1(\Sigma_g;\mathbf{Z})$.

Lemma 3.7. *For $\sigma \in \mathrm{spin}(\Sigma_g)$, $y \in H_1(\Sigma_g;\mathbf{Z})$ and $\psi \in \mathcal{I}_{g,1}$,*

$$\beta_{\sigma, y \bmod 2}(\psi) = 4\omega_{\sigma,y}(\psi) \in \mathbf{Z}_8.$$

Proof. First, we explain the isomorphisms $\Omega_3^{spin}(\mathbf{Z}) \cong \mathbf{Z}_2$ and $\Omega_3^{spin}(\mathbf{Z}_2) \cong \mathbf{Z}_8$ in more detail.

For an $[(f, M')] \in \Omega_3^{spin}(S^1)$, choose a closed oriented surface $F_y \subset M'$ which represents the Poincaré dual of the pullback $y := f^*c \in H^1(M;\mathbf{Z})$ of a generator $c \in H^1(S^1;\mathbf{Z})$. Then F_y has the spin structure σ_y induced by $\sigma \in \mathrm{spin}(M)$. For an oriented compact spin surface F, denote by $\mathrm{Arf}(\sigma)$ the Arf invariant of $\sigma \in \mathrm{spin}(F)$. By the Atiyah-Hirzebruch spectral sequence, the homomorphism

$$\Omega_3^{spin}(\mathbf{Z}) \cong \Omega_2^{spin} \cong \mathbf{Z}_2$$

defined by $[(f, M')] \mapsto \mathrm{Arf}(\sigma_y)$ is isomorphic. This homomorphism does not depend on the choice of the generator c.

Similarly, for $[(f, M')] \in \Omega_3^{spin}(\mathbf{Z}_2)$, choose a closed surface $F_x \subset M'$ which represents the Poincaré dual of $x := f^*w_1 \in H^1(M';\mathbf{Z}_2)$ of the Stiefel-Whitney class $\omega_1 \in H^1(\mathbf{RP}^\infty;\mathbf{Z}_2)$. Then, $F_x \subset M'$ has the pin^- structure α_x induced from the spin structure of M'. Then, there is an well-known isomorphism

$$\Omega_3^{spin}(\mathbf{Z}_2) \to \Omega_2^{pin^-}$$

given by $[(M', f)] \mapsto [F_x, \alpha_x]$. Under the isomorphism

$$\Omega_2^{pin^-} \cong \mathbf{Z}_8,$$

$[F_x, \alpha_x]$ maps to B_{α_x}.

Next, we prove $\beta_{\sigma,x}(\varphi) = x_*\eta_{\sigma,2}[2](\varphi)$ for $\varphi \in \mathcal{M}_{g,1}[2]$ and $x \in H_1(\Sigma_g;\mathbf{Z}_2)$. Consider x as an element of $H^1(M_\varphi;\mathbf{Z}_2)$ under the inclusion $H_1(\Sigma_g;\mathbf{Z}_2) \cong H^1(\Sigma_g;\mathbf{Z}_2) \cong H^1(M_\psi;\mathbf{Z}_2)$. We can choose a surface $F_x \subset M - (N(c_0) \times S^1)$ which represents the Poincaré dual of x with a pin^- structure α_x. Then, we have

$$\beta_{\sigma,x}(\varphi) = B_{\alpha_x} = x_*\eta_{\sigma,2}[2](\varphi).$$

By the definition of the Brown invariant, we see that the homomorphism $\Omega_3^{spin}(\mathbf{Z}) \to \Omega_3^{spin}(\mathbf{Z}_2) \cong \mathbf{Z}_8$ is written by 4 times the Arf invariant of a spin structure of the surface F_y. By the commutative diagram, we have $\beta_{\sigma, y \bmod 2}(\psi) = 4\omega_{\sigma,y}(\psi) \in \mathbf{Z}_8$ for $\psi \in \mathcal{I}_{g,1}$ and $y \in H_1(\Sigma_g;\mathbf{Z})$. □

3.6 The value of $\beta_{\sigma,x}$

Humphries ([7] p.314 Proposition 2.1) shows that the level 2 mapping class group $\mathcal{M}_{g,r}[2]$ is generated by the square of the Dehn twists along all non-separating simple closed curve when $g \geq 3$. We will compute the value of the homomorphism β_σ defined in Subsection 3.3 on the generators of $\mathcal{M}_{g,1}[2]$, using the Brown invariant. For $x \in H_1(\Sigma_g;\mathbf{Z}_2)$, define the map $i_x : H_1(\Sigma_g;\mathbf{Z}_2) \to \mathbf{Z}_8$ by

$$i_x(y) = \begin{cases} 1 & \text{if } x \cdot y \equiv 1 \bmod 2, \\ 0 & \text{if } x \cdot y \equiv 0 \bmod 2. \end{cases}$$

Note that this is not a homomorphism. For $\sigma \in \mathrm{spin}(\Sigma_g)$, denote by $q_\sigma : H_1(\Sigma_g;\mathbf{Z}_2) \to \mathbf{Z}_2$ the quadratic function of σ.

Proposition 3.8. *Let C be a non-separating simple closed curve in $\Sigma_g - N(c_0)$. Then we have*

$$\beta_\sigma(t_C^2) = (-1)^{q_\sigma(C)} i_{[C]} \in \mathrm{Map}(H_1(\Sigma_g;\mathbf{Z}_2), \mathbf{Z}_8).$$

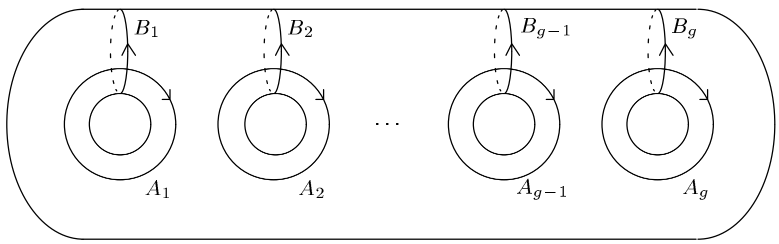

Figure 2: the symplectic basis

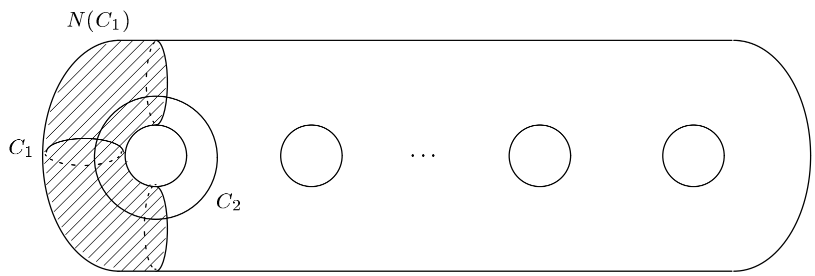

Figure 3: the neighborhood $N(C_1)$

Proof. We denote the symplectic basis $\{A_i, B_i\}_{i=1}^g$ represented by the simple closed curves in Figure 2. Choose the oriented simple closed curves C_1 and C_2 as described in Figure 3. For any non-separating simple closed curve C, if we choose a mapping class $\varphi \in \mathcal{M}_{g,1}$ such that $\varphi(C_1) = C$, we have

$$\begin{aligned}\beta_{\sigma,x}(t_C^2) &= \beta_{\sigma,x}(\varphi t_{C_1}^2 \varphi^{-1}) \\ &= (R(M_{\varphi t_{C_1}^2 \varphi^{-1}}, \theta(\sigma)) - R(M_{\varphi t_{C_1}^2 \varphi^{-1}}, \theta(\sigma + x)))/2 \\ &= (R(M_{t_{C_1}^2}, \theta(\varphi^* \sigma)) - R(M_{t_{C_1}^2}, \theta(\varphi^* \sigma + \varphi_*^{-1}(x))))/2 \\ &= \beta_{\varphi^*\sigma, \varphi_*^{-1}(x)}(t_{C_1}^2).\end{aligned}$$

Hence It suffices to show that $\beta_\sigma(t_{C_1}^2) = (-1)^{q_\sigma(C_1)} i_{[C_1]}$. Let $M := M_{t_{C_1}^2}$ denote a mapping torus.

First, we calculate the value $\beta_{\sigma,A_1+B_1}(t_{C_1}^2)$. Consider the compact 3-manifold $M_1 := N(C_1) \times I/\sim \subset M$. Choose the compact surface $F_1 \subset M_1$ as shown in Figure 4. For the arc $r = C_2 \cap (\Sigma_g - \operatorname{Int} N(C_1))$ as in Figure 3, denote another subsurface $F_2 := r \times S^1 \subset M$. Then, the surface $F := F_1 \cup F_2$ represents the Poincaré dual of the homology class $A_1 + B_1 \in H_1(\Sigma_g; \mathbf{Z}_2) = H^1(\Sigma_g; \mathbf{Z}_2) \subset H^1(M; \mathbf{Z}_2)$. Let α denote the pin$^-$ structure of F induced by the spin structure of M. By Lemma 3.6 proved by Turaev, the value $\beta_{\sigma,A_1+B_1}(t_{C_1}^2)$ equals to the Brown invariant B_α. Hence we investigate the quadratic enhancement $\hat{q}_\alpha : H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_4$ of the pin$^-$ structure of F. Pick the generator x, y, z of $H_1(F; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ as in Figure 4. We may assume $x = A_1, y = B_1, z = [S^1] \in H_1(M; \mathbf{Z})$. Then, we have

$$\hat{q}_\alpha(x) = -1 + 2q_\sigma(A_1),\ \hat{q}_\alpha(y) = 1 + 2q_\sigma(B_1),\ \hat{q}_\alpha(z) = 0,$$
$$\hat{q}_\alpha(x+y) = 2 + 2q_\sigma(A_1) + 2q_\sigma(B_1),\ \hat{q}_\alpha(y+z) = 1 + 2q_\sigma(B_1),\ \hat{q}_\alpha(z+x) = 1 + 2q_\sigma(A_1),$$
$$\hat{q}_\alpha(x+y+z) = 2q_\sigma(A_1) + 2q_\sigma(B_1).$$

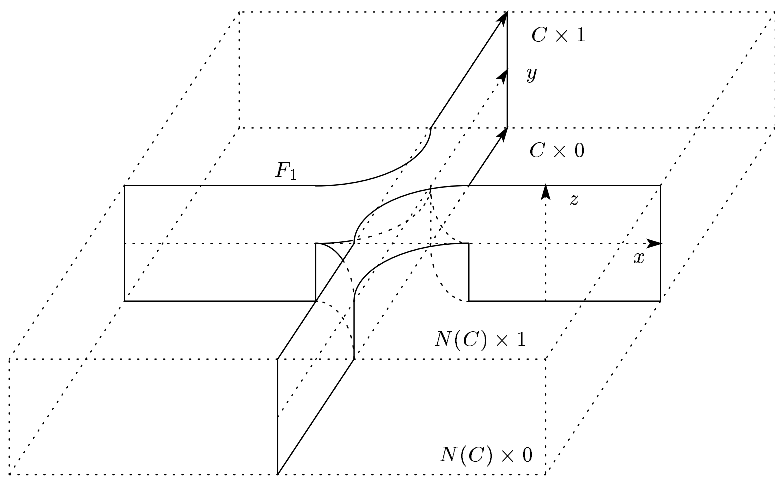

Figure 4: the surface $F_1 \subset M_1$

Hence the Brown invariant B_α satisfies

$$\sqrt{|H_1(F;\mathbf{Z}_2)|}\exp(2\pi\sqrt{-1}B_\alpha/8) = \sum_{x\in H_1(F;\mathbf{Z}_2)} \exp(2\pi\sqrt{-1}\hat{q}_\alpha(x)/4)$$
$$= 2\exp(2\pi\sqrt{-1}(2q_\sigma(B_1)+1)/4)+2.$$

Hence we have $\beta_{\sigma,A_1+B_1}(t^2_{C_1}) = B_\alpha = (-1)^{q_\sigma(B_1)}$.

Next, we show that $\beta_{\sigma,x}(t^2_{C_1}) = 0$ for $x \in \langle B_1, A_2, B_2 \cdots, A_g, B_g\rangle$. Choose the simple closed curve $C_x \subset \Sigma_g - N(C_1)$ which represents $x \in H_1(\Sigma_g;\mathbf{Z}_2)$. Denote the subsurface $F' := C_x \times S^1 \subset M$. This subsurface represents the Poincaré dual of $x \in H_1(\Sigma_g;\mathbf{Z}_2) = H^1(\Sigma_g;\mathbf{Z}_2) \subset H^1(M;\mathbf{Z}_2)$ in M. Choose a generator $x' := [C_x \times 0]$, $y' := [c \times S^1]$ of $H_1(F';\mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, where $c \in C_x$ is a point. Then, F' is orientable and the spin structure of F' is induced by that of M. We denote this spin structure by σ'. Since spin group naturally injects into pin$^-$ group, we can consider σ' as the pin$^-$ structure of F'. Then, the quadratic enhancement $\hat{q}_{\sigma'}$ is equal to twice the quadratic function $q_{\sigma'}$. Hence we have

$$\hat{q}_{\sigma'}(x') = 2q_{\sigma'}(x) = 2q_\sigma(x),\ \hat{q}_{\sigma'}(y') = 0.$$

This shows that $\beta_{\sigma,x}(t^2_{C_1}) = B_{\sigma'} = 0$.

Finally, we prove $\beta_{\sigma,A_1+x}(t^2_{C_1}) = \beta_{\sigma,A_1}(t^2_{C_1})$. We have

$$\beta_{\sigma,A_1+x}(t^2_{C_1}) = (R(M,\sigma) - R(M,\sigma+A_1+x))/2$$
$$= (R(M,\sigma) - R(M,\sigma+A_1))/2 + (R(M,\sigma+A_1) - R(M,\sigma+A_1+x))/2$$
$$= \beta_{\sigma,A_1}(t^2_{C_1}) + \beta_{\sigma+A_1,x}(t^2_{C_1}).$$

Since we have $\beta_{\sigma,A_1}(t^2_{C_1}) = 0$, it follows that $\beta_{\sigma,A_1+x}(t^2_{C_1}) = \beta_{\sigma,A_1}(t^2_{C_1})$.

Thus, for all $x \in H_1(\Sigma_g;\mathbf{Z}_2)$, we have

$$\beta_{\sigma,x}(t^2_{C_1}) = (-1)^{q_\sigma(C_1)} i_{[C_1]}(x).$$

□

4 Proof of Theorem 1.2

We calculate the order of the homology group $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$ in Subsections 4.2 and 4.3. Using these results, we will prove Theorem 1.2. We also determine the abelianization of the level 2 mapping class group for closed surfaces.

4.1 A homomorphism $\Phi : \mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z})$

For a module $K = \mathbf{Z}, \mathbf{Z}_d$, we denote by $H_1(\Sigma_{g,r};K)^{pri}$ the set of primitive elements in $H_1(\Sigma_{g,r};K)$. Let

$$S_d := H_1(\Sigma_{g,r};\mathbf{Z}_d)^{pri}/\{\pm 1\}.$$

In this subsection, we define the homomorphism $\Phi : \mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z})$. In particular, this homomorphism is surjective when $d = 2$.

The level d mapping class group acts on the set of isotopy classes of non-separating simple closed curves. We will prove that S_d corresponds bijectively to the orbit space of this action. Note that any element of $H_1(\Sigma_{g,r};\mathbf{Z})^{pri}$ is known to be represented by a simple closed curve.

Lemma 4.1. *The* $\bmod d$ *reduction homomorphism* $H_1(\Sigma_{g,r};\mathbf{Z})^{pri} \to H_1(\Sigma_{g,r};\mathbf{Z}_d)^{pri}$ *is surjective.*

Proof. For $v_d \in H_1(\Sigma_{g,r};\mathbf{Z}_d)^{pri}$, choose $v \in H_1(\Sigma_{g,r};\mathbf{Z})$ which satisfies $v \bmod d = v_d \in H_1(\Sigma_{g,r};\mathbf{Z}_d)^{pri}$. If v is not primitive, there exists an integer $k \geq 2$ and a primitive element $w \in H_1(\Sigma_{g,r};\mathbf{Z})^{pri}$ such that $v = kw$. Since v_d is primitive, k and d are coprime. Then, there exist integers $k', d' \in \mathbf{Z}$ such that $kk' + dd' = 1$. Choose $w' \in H_1(\Sigma_{g,r};\mathbf{Z})^{pri}$ such that $w \cdot w' = 1$. We have

$$(v + dw') \cdot (-d'w + k'w') = kk' + dd' = 1.$$

Hence, $v + dw' \in H_1(\Sigma_{g,r};\mathbf{Z})$ is primitive and $v + dw' \bmod d = v_d \in H_1(\Sigma_{g,r};\mathbf{Z}_d)$. □

Lemma 4.2. *Let* $C_1, C_1' \subset \Sigma_{g,r}$ *be non-separating simple closed curves such that* $[C_1] = [C_1'] \in H_1(\Sigma_{g,r};\mathbf{Z}_d)/\{\pm 1\}$. *Then, there exists a mapping class* $[f] \in \mathcal{M}_{g,r}[d]$ *such that* $f(C_1) = C_1'$.

Proof. Fix orientations of C_1 and C_1' so that $[C_1] = [C_1'] \in H_1(\Sigma_{g,r};\mathbf{Z}_d)$. Denote $u := ([C_1'] - [C_1])/d \in H_1(\Sigma_{g,r};\mathbf{Z})$. Choose the simple closed curve C_2 which intersects C_1 transversely at one point. Since $[C_1'] \in H_1(\Sigma_{g,r};\mathbf{Z})$ is primitive, there exists $v \in H_1(\Sigma_{g,r};\mathbf{Z})$ which satisfies $[C_1'] \cdot v = -u \cdot [C_2]$. If we put $\alpha_2' := [C_2] + dv$, we have

$$\begin{aligned}
[C_1'] \cdot \alpha_2' &= [C_1'] \cdot ([C_2] + dv) \\
&= (du + [C_1]) \cdot [C_2] + d[C_1'] \cdot v \\
&= du \cdot [C_2] + 1 + d[C_1'] \cdot v \\
&= 1.
\end{aligned}$$

In particular, the element α_2' is primitive. Hence there exists C_2' such that $[C_2'] = \alpha_2'$, and intersect C_1' transversely in one point.

Choose a diffeomorphism $f : \Sigma_{g,r} \to \Sigma_{g,r}$ which satsifies $f(C_1) = C_1'$, $f(C_2) = C_2'$, $f|_{\partial\Sigma_{g,r}} = id_{\partial\Sigma_{g,r}}$. Denote by $\{Y_i\}_{i=2}^{2g-2}$ the homology class of $H_1(\Sigma_{g,r};\mathbf{Z})$ such that $\{[C_1],[C_2]\} \cup \{Y_i\}_{i=1}^{2g-2}$ makes the symplectic basis. Since we have $f_*([C_i]) \equiv [C_i] \bmod d$ for $i = 1, 2$, The symplectic action of f on $H_1(\Sigma_{g,r};\mathbf{Z}_d)$ induces the action on $\bigoplus_{i=1}^{2g-2} \mathbf{Z}_d Y_i$. For an closed tubular neighborhood $N(C_i)$ of C_i, denote the surface $F := \Sigma_{g,r} - (\cup \operatorname{Int} N(C_i))$. Here, the action of the mapping class group $\mathcal{M}_{g-1,r+1}$ of F on $H_1(F;\mathbf{Z}_d)/\operatorname{Im}(H_1(\partial F;\mathbf{Z}_d) \to H_1(F;\mathbf{Z}_d))$ induces the surjective homomorphism $\mathcal{M}_{g-1,r+1} \to \mathrm{Sp}(2g-2;\mathbf{Z}) \to \mathrm{Sp}(2g-2;\mathbf{Z}_d)$. Hence there exists $g \in \mathrm{Diff}(F, \partial F)$ such that

$$g_*(Y_i) = f_*^{-1}(Y_i) \in H_1(F;\mathbf{Z}_d).$$

It is easy to see that $[f(g \cup id_{\amalg N(C_i)})] \in \mathcal{M}_{g,r}[d]$ is the desired mapping class. □

By Lemma 4.1, every element of S_d is represented by a simple closed curve. By Lemma 4.2, S_d corresponds to the orbit space of the action of $\mathcal{M}_{g,r}[d]$ on isotopy classes of non-separating simple closed curves.

Now, we define the homomorphism $\mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z})$. Denote by $t_C \in \mathcal{M}_{g,r}$ the Dehn twist along a simple closed curve $C \subset \Sigma_{g,r}$. By Lemmas 4.1 and 4.2, we can define the map $\Phi : S_d \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z})$ by $\Phi([C]) = \langle [C] \rangle := [t_C^d]$. Extend this map to a homomorphism of $\mathbf{Z}$-module

$$\Phi_d : \mathbf{Z}[S_d] \to H_1(\mathcal{M}_{g,r}[d];\mathbf{Z}).$$

We consider the case when $d = 2$. Then, we have $S_2 = H_1(\Sigma_{g,r};\mathbf{Z}_2) - 0$. Define $\Phi_2([0]) := 0$ and extend Φ_2 to

$$\Phi := \Phi_2 : \mathbf{Z}[H_1(\Sigma_{g,r};\mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z}).$$

Lemma 4.3. *The homomorphism Φ is surjective, and factors through $\mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)]$.*

Proof. Humphries[7] proved that the level 2 mapping class group is generated by Dehn twists along non-separating curves. Hence, Φ is surjective. Denote by $H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]}$ the coinvariant of the action of $\mathcal{M}_{g,r}[2]$ on $H_1(\mathcal{I}_{g,r};\mathbf{Z})$. Consider the exact sequence

$$H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]} \longrightarrow H_1(\mathcal{M}_{g,r}[2];\mathbf{Z}) \longrightarrow H_1(\Gamma_g[2];\mathbf{Z}) \longrightarrow 0.$$

The coinvariant $H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]}$ is proved to be a $\mathbf{Z}_2$-module in Johnson[13] Theorems 1 and 4, and we proved that $H_1(\Gamma_g[2];\mathbf{Z})$ is a $\mathbf{Z}_4$-module in Section 2. Hence $H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ is a $\mathbf{Z}_8$-module. This shows that Φ factors through the module $\mathbf{Z}_8[H_1(\Sigma_{g,r};\mathbf{Z}_2)]$. □

4.2 Upper bound of the order $|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})|$

In this subsection, we examine the kernel of the inclusion homomorphism

$$H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]} \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z}),$$

and give an upper bound of the order of $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$.

First, we review the $\mathbf{Z}_2$-module $B^3_{g,r}$ defined by Johnson[12]. We consider the commutative polynomial ring R with coefficient $\mathbf{Z}_2$ in formal symbol $\bar{x}$ for $x \in H_1(\Sigma_{g,r};\mathbf{Z})$. Denote by J the ideal of this polynomial ring generated by

$$\overline{x+y} - (\bar{x} + \bar{y} + x \cdot y), \quad \bar{x}^2 - \bar{x},$$

for $x, y \in H \otimes \mathbf{Z}_2$,

Denote by R_n the module consisting of polynomials whose degrees are less than or equal to n. Define the module B^n by

$$B^n = \frac{R_n}{J \cap R_n},$$

and denote

$$B^3_{g,1} := B^3.$$

Let $A_i, B_i\}_{i=1}^g$ denote a symplectic basis defined in Proposition 3.8. For the element $\alpha = \Sigma_{i=1}^g \bar{A}_i \bar{B}_i \in B^2$, define the homomorphism $B^1 \to B^3_{g,1}$ by $x \mapsto x\alpha$. Denote its cokernel by B^3_g. Johnson determined the $\mathrm{Sp}(2g;\mathbf{Z})$-module structure of $H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]}$.

Theorem 4.4 (Johnson[13] Theorem 1, Theorem 4)**.**

$$H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]} \cong B^3_{g,r}.$$

Next, we examine the kernel of $\iota : B^3_{g,r} \cong H_1(\mathcal{I}_{g,r};\mathbf{Z})_{\mathcal{M}_{g,r}[2]} \to H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$.

Lemma 4.5. *For $r = 0, 1$,*

$$1 \in \operatorname{Ker}\iota.$$

Proof. As in Figure 5, choose the simple closed curves C_1, C_2, D_1 so that $[C_1] = B_1, [C_2] = A_1$. For $X \in H_1(\Sigma_{g,r};\mathbf{Z}_2)$, we denote simply $\langle X\rangle := \Phi(X)$. Then, by Lemma 12a in Johnson[12] and the chain relation, we

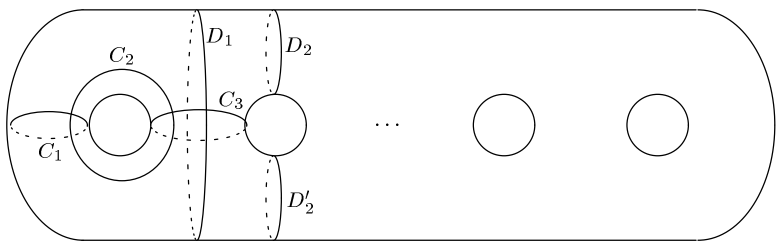

Figure 5: the curves

have

$$\begin{aligned}\iota(\overline{A}_1\overline{B}_1) &= [t_{D_1}]\\ &= [(t_{C_1}t_{C_2})^6]\\ &= 2[t^2_{C_1}] + 2[t^2_{C_2}] + 2[t^{-1}_{C_1}t^2_{C_2}t_{C_1}]\\ &= 2\langle A_1\rangle + 2\langle B_1\rangle + 2\langle A_1 + B_1\rangle . \end{aligned} \tag{15}$$

If we choose $\varphi \in \mathcal{M}_{g,r}$ such that $\varphi(A_1) = A_1$, $\varphi(B_1) = B_1 + B_2$, we have

$$\iota(\overline{A}_1(\overline{B_1 + B_2})) = 2\langle A_1\rangle + 2\langle B_1 + B_2\rangle + 2\langle A_1 + B_1 + B_2\rangle . \tag{16}$$

In the same fashion, by Lemma 12b in Johnson[12] and the chain relation, we have

$$\begin{aligned}\iota(\overline{A}_1\overline{B}_1(\overline{B_2} + 1)) &= [t_{D_2}t^{-1}_{D'_2}]\\ &= [(t_{C_1}t_{C_2}t_{C_3})^4] - \langle B_2\rangle\\ &= \langle B_1\rangle + \langle A_1\rangle + \langle B_1 + B_2\rangle + \langle A_1 + B_1\rangle + \langle A_1 + B_1 + B_2\rangle + \langle A_1 + B_2\rangle - \langle B_2\rangle .\end{aligned}$$

Since $\iota(2\overline{A}_1\overline{B}_1(\overline{B_2} + 1)) = 0$, we have

$$2\langle A_1 + B_1 + B_2\rangle = -2(\langle B_1\rangle + \langle A_1\rangle + \langle B_1 + B_2\rangle + \langle A_1 + B_1\rangle + \langle A_1 + B_2\rangle - \langle B_2\rangle).$$

Put this into the equation (16), then we have

$$\begin{aligned}\iota(\overline{A}_1(\overline{B_1 + B_2})) &= 2\langle A_1\rangle + 2\langle B_1 + B_2\rangle\\ &\quad - 2(\langle A_1\rangle + \langle B_1\rangle - \langle B_2\rangle + \langle A_1 + B_1\rangle + \langle A_1 + B_2\rangle + \langle B_1 + B_2\rangle)\\ &= -2\langle B_1\rangle + 2\langle B_2\rangle - 2\langle A_1 + B_1\rangle - 2\langle A_1 + B_2\rangle .\end{aligned} \tag{17}$$

By the equation (15) and (17),

$$\begin{aligned}\iota(\overline{A}_1\overline{B}_2) &= \iota(\overline{A}_1(\overline{B_1+B_2}) + \overline{A}_1\overline{B}_1) \\ &= 2\langle A_1\rangle + 2\langle B_2\rangle - 2\langle A_1+B_2\rangle.\end{aligned}$$

If we choose $\varphi \in \mathcal{M}_{g,r}$ so that $\varphi_*(A_1) = A_1$, $\varphi_*(B_2) = A_1 + B_2$, we have

$$\iota(\overline{A}_1(\overline{A}_1 + \overline{B}_2)) = 2\langle A_1\rangle + 2\langle A_1+B_2\rangle - 2\langle B_2\rangle.$$

Hence we obtain

$$\begin{aligned}\iota(\overline{A}_1) &= \iota(\overline{A}_1(\overline{A}_1 + \overline{B}_2) - \overline{A}_1\overline{B_2}) \\ &= 4\langle A_1\rangle.\end{aligned}$$

As we stated in the last subsection, $H_1(\mathcal{M}_{g,r}[2];\mathbf{Z})$ is a $\mathbf{Z}_8$-module. Hence we have $8\langle A_1\rangle = 8\langle B_1\rangle = 0$. Therefore, we see that

$$\begin{aligned}\iota(1) &= \iota(\overline{A_1+B_1} - \overline{A}_1 - \overline{B}_1) \\ &= 4(\langle A_1+B_1\rangle - \langle A_1\rangle - \langle B_1\rangle) \\ &= 4(\langle A_1+B_1\rangle + \langle A_1\rangle + \langle B_1\rangle) \\ &= \iota(2\overline{A}_1\overline{B}_1) = 0 \in H_1(\mathcal{M}_{g,r}[2];\mathbf{Z}).\end{aligned}$$

□

By this lemma, we obtain the upper bound

$$|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})| \leq |B^3_{g,1}/\langle 1\rangle|\,|H_1(\Gamma_g[2];\mathbf{Z})|.$$

4.3 Lower bound of the order $|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})|$

In this subsection, we give a lower bound of the order of $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$

$$|H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})| \geq |\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}|.$$

Using this result we determine the abelianization $H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$.

For $\sigma \in \mathrm{spin}(\Sigma_g)$ and $\{x_j\}_{j=1}^n \subset H_1(\Sigma_{g,1};\mathbf{Z}_2)$, define $\Delta_\sigma(x_1, x_2, \cdots, x_n)$ by

$$\begin{aligned}\Delta_\sigma(x_1,x_2,\cdots,x_n) := \sum_{j=1}^n (-1)^{q_\sigma(x_j)}[x_j] + \sum_{1\leq j<k\leq n}(-1)^{q_\sigma(x_j+x_k)}[x_j+x_k] \\ + \cdots + (-1)^{q_\sigma(x_1+x_2+\cdots+x_n)}[x_1+x_2+\cdots+x_n] \in \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)].\end{aligned}$$

Lemma 4.6.

$$\begin{aligned}\Delta_\sigma(x_1,x_2,\cdots,x_{n-1},x_n,x_{n+1}) =& \Delta_\sigma(x_1,x_2,\cdots,x_{n-1},x_n+x_{n+1}) + \Delta_\sigma(x_1,x_2,\cdots,x_{n-1},x_n) \\ &+ \Delta_\sigma(x_1,x_2,\cdots,x_{n-1},x_{n+1}) - 2\Delta_\sigma(x_1,x_2,\cdots,x_{n-1}).\end{aligned}$$

Proof. For $X \in H_1(\Sigma_{g,r};\mathbf{Z}_2)$, denote

$$\begin{aligned}\Delta^X_\sigma(x_1,x_2,\cdots,x_n) := \sum_{j=1}^n (-1)^{q_\sigma(x_j+X)}[x_j+X] + \sum_{1\leq j<k\leq n}(-1)^{q_\sigma(x_j+x_k+X)}[x_j+x_k+X] \\ + \cdots + (-1)^{q_\sigma(x_1+x_2+\cdots+x_n+X)}[x_1+x_2+\cdots+x_n+X].\end{aligned}$$

By the definition of Δ_σ, we have

$$\Delta_\sigma(x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1}) = \Delta_\sigma(x_1, x_2, \cdots, x_{n-1}, x_n) + \Delta_\sigma^{x_{n+1}}(x_1, x_2, \cdots, x_{n-1}, x_n) \tag{18}$$

Similarly, we see that

$$\begin{aligned}\Delta_\sigma^{x_{n+1}}(x_1, x_2, \cdots, x_{n-1}, x_n) =& \Delta_\sigma^{x_{n+1}}(x_1, x_2, \cdots, x_{n-1}) + \Delta_\sigma^{x_n+x_{n+1}}(x_1, x_2, \cdots, x_{n-1}) \\ =& \Delta_\sigma(x_1, x_2, \cdots, x_{n-1}, x_{n+1}) - \Delta_\sigma(x_1, x_2, \cdots, x_{n-1}) \\ &+ \Delta_\sigma(x_1, x_2, \cdots, x_{n-1}, x_n + x_{n+1}) - \Delta_\sigma(x_1, x_2, \cdots, x_{n-1})\end{aligned}$$

Put this into the equation (18), then we obtain what we intended to prove. □

Lemma 4.7. *For* $\{x_j\}_{j=1}^n \subset H_1(\Sigma_{g,1}; \mathbf{Z}_2)$ *and* $x \in H_1(\Sigma_g; \mathbf{Z}_2)$,

$$\beta_\sigma \Phi(\Delta_\sigma^n(x_1, x_2, \cdots, x_n))(x) = 2^{n-1} \prod_{j=1}^n i_{x_j}(x).$$

Proof. In Proposition 3.8, we proved that $\beta_\sigma \Phi((-1)^{q_\sigma(x)} x_1)(x) = i_{x_1}(x)$. Assume that for $n-1$ the equation holds. By the Lemma 4.6, we have

$$\begin{aligned}\beta_\sigma \Phi(\Delta_\sigma^n(x_1, x_2, \cdots, x_n, x_{n+1}))(x) &= 2^{n-1} \prod_{j=1}^{n-1} i_{x_j}(x)(i_{x_n+x_{n+1}}(x) + i_{x_n}(x) + i_{x_{n+1}}(x) - 1) \\ &= 2^n \prod_{j=1}^{n+1} i_{x_j}(x).\end{aligned}$$

This proves the lemma. □

Denote the homology classes X_n by $X_{2j-1} := A_j$, and $X_{2j} = B_j$ for $j = 1, 2, \cdots, g$. For convenience, we denote $X_{n+2g} = X_n$ for $n = 1, 2, \cdots 2g$. Define the surjective homomorphism

$$\Psi : \mathrm{Map}(H_1(\Sigma_g; \mathbf{Z}_2); \mathbf{Z}_8) \to \mathbf{Z}_8^{2g} \oplus \mathbf{Z}_8^{\binom{2g}{2}} \oplus \mathbf{Z}_8^{\binom{2g}{3}}$$

by

$$\Psi(f) := (\{f(X_{i_1})\}_{i_1=1}^{2g}, \{f(X_{i_1} + X_{i_2})\}_{1 \le i_1 \le i_2 \le 2g}, \{f(X_{i_1} + X_{i_2} + X_{i_3})\}_{1 \le i_1 \le i_2 \le i_3 \le 2g}).$$

Lemma 4.8.

$$\mathrm{Im}(\Psi \beta_\sigma) = \mathbf{Z}_8^{2g} \oplus 2\mathbf{Z}_8^{\binom{2g}{2}} \oplus 4\mathbf{Z}_8^{\binom{2g}{3}}.$$

Proof. We examine the value of $\Psi\beta_\sigma$ on $\Phi(\Delta_\sigma(X_{i_1}, X_{i_2}, \cdots, X_{i_n})) \in H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z})$, using Lemma 4.7.

For $f = \beta_\sigma \Phi(\Delta_\sigma(X_{i_1}))$ where $1 \le i_1 \le 2g$, we have

$$f(X_l) = \begin{cases} 1, & \text{if } X_l = X_{i_1+g}, \\ 0, & \text{otherwise,} \end{cases} \qquad f(X_l + X_m) = f(X_l + X_m + X_n) = 0.$$

For $f = \beta_\sigma \Phi(\Delta_\sigma(X_{i_1}, X_{i_2}))$ where $1 \le i_1 < i_2 \le 2g$,

$$f(X_l) = 0,$$

$$f(X_l + X_m) = \begin{cases} 2, & \text{if } \{X_l, X_m\} = \{X_{i_1+g}, X_{i_2+g}\}, \\ 0, & \text{otherwise,} \end{cases} \qquad f(X_l + X_m + X_n) = 0.$$

For $f = \beta_\sigma \Phi(\Delta_\sigma(X_{i_1}, X_{i_2}, X_{i_3}))$ where $1 \leq i_1 < i_2 < i_3 \leq 2g$,

$$f(X_l) = f(X_l + X_m) = 0,$$

$$f(X_l + X_m + X_n) = \begin{cases} 4, \text{ if } \{X_l, X_m, X_n\} = \{X_{i_1+g}, X_{i_2+g}, X_{i_3+g}\}, \\ 0, \text{ otherwise.} \end{cases}$$

We also have $\beta_\sigma \Phi(\Delta_\sigma(X_{i_1}, X_{i_2}, \cdots, X_{i_n})) = 0$ for $n \geq 4$.

Since Φ is surjective, we have determined the image of the homomorphism $\Psi\beta_\sigma$. □

By this lemma, we obtain the lower bound

$$|H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z})| \geq |\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}|.$$

Now, we determine the abelianization $H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z})$ as a $\mathbf{Z}$-module.

Proposition 4.9. *For $g \geq 3$,*

$$H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z}) \cong \mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}.$$

Proof. Denote by $\langle 1 \rangle$ the cyclic group generated by $1 \in B^3_{g,r}$. We have

$$|\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}| \leq |H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z})| \leq |B^3_{g,1}/\langle 1 \rangle||H_1(\Gamma_g[2]; \mathbf{Z})|.$$

By the definition of $B^3_{g,1}$, it is easy to see that

$$|B^3_{g,1}/\langle 1 \rangle||H_1(\Gamma_g[2]; \mathbf{Z})| = |\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}|.$$

By comparing the order of groups, we see that the surjective homomorphism

$$\Psi\beta_\sigma : H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z}) \to \mathbf{Z}_8^{2g} \oplus 2\mathbf{Z}_8^{\binom{2g}{2}} \oplus 4\mathbf{Z}_8^{\binom{2g}{3}}$$

is isomorphic. □

Remark 4.10. *In particular, we have* $\operatorname{Ker} \iota = \langle 1 \rangle$ *when* $r = 1$.

Now, we prove Theorem 1.2.

proof of Theorem 1.2. We compute the kernel of the homomorphism $\Phi : \mathbf{Z}_8[H_1(\Sigma_{g,1}; \mathbf{Z}_2)] \to H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z})$.

Since β_σ is injective, $\operatorname{Ker} \beta_\sigma \Phi = \operatorname{Ker} \Phi$. Hence, by Lemma 4.7 we have

$$4\Delta^2_\sigma(x_1, x_2), 2\Delta^3_\sigma(x_1, x_2, x_3), \Delta^n_\sigma(x_1, x_2, \cdots, x_n) \in \operatorname{Ker} \Phi,$$

for $n \geq 3$ and $\{x_i\}_{i=1}^n \subset H_1(\Sigma_{g,1}; \mathbf{Z})$. By Lemma 4.6, it is easy to see that

$$4\Delta^2_\sigma(x_1, x_2), 2\Delta^3_\sigma(x_1, x_2, x_3), \Delta^n_\sigma(x_1, x_2, \cdots, x_n)$$

is generated by

$$4\Delta^2_\sigma(X_{i_1}, X_{i_2}), 2\Delta^3_\sigma(X_{i_1}, X_{i_2}, X_{i_3}), \Delta^n_\sigma(X_{i_1}, X_{i_2}, \cdots, X_{i_n}),$$

where $\{X_i\}_{i=1}^{2g} \subset H_1(\Sigma_{g,1}; \mathbf{Z})$ is the symplectic basis. Hence, the submodule $L_{g,1}$ is generated by these elements. An easy calculation shows that

$$|\mathbf{Z}_8[H_1(\Sigma_{g,1}; \mathbf{Z}_2)]/L_{g,1}| = |\mathbf{Z}_8^{2g} \oplus \mathbf{Z}_4^{\binom{2g}{2}} \oplus \mathbf{Z}_2^{\binom{2g}{3}}| = |H_1(\mathcal{M}_{g,1}[2]; \mathbf{Z})|.$$

Therefore, the surjective homomorphism

$$\Phi : \mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L_{g,1} \to H_1(\mathcal{M}_{g,1}[2];\mathbf{Z})$$

is isomorphic.

If we choose the spin structure σ_0 such that its quadratic function q_{σ_0} satisfies $q_{\sigma_0}(X_i)=0$, we have

$$q_{\sigma_0}(x_1+x_2+\cdots+x_n) = \sum_{1\le i<j\le n}(x_i\cdot x_j) \bmod 2 = I(x_1,x_2,\cdots,x_n).$$

Hence we have $\Delta_0 = \Delta_{\sigma_0}$, and Theorem 1.2 is proved. □

4.4 The abelianization of the level 2 mapping class group of a closed surface

In this subsection, we determine the abelianization of the level 2 mapping class group of a closed surface Σ_g. It is well-known that the homomorphism

$$\mathcal{M}_{g,1}[2] \to \mathcal{M}_g[2]$$

is surjective. As stated in Johnson [13] Section 6, the kernel $\mathrm{Ker}(H_1(\mathcal{I}_{g,1};\mathbf{Z})_{\mathcal{M}_{g,1}[2]} \to H_1(\mathcal{I}_g;\mathbf{Z})_{\mathcal{M}_g[2]})$ is generated by

$$\sum_{i=1}^g \overline{A}_i\overline{B}_i,\ \sum_{i=1}^g \overline{A}_i\overline{B}_i\overline{X} \in B^3_{g,1} \quad \text{for } X = A_1,B_1,\cdots,A_g,B_g.$$

Hence, $\mathrm{Ker}(H_1(\mathcal{M}_{g,1}[2];\mathbf{Z}) \to H_1(\mathcal{M}_g[2];\mathbf{Z}))$ is generated by the image of these elements under ι. Therefore, $H_1(\mathcal{M}_g[2];\mathbf{Z})$ is isomorphic to the quotient of $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L_{g,1}$ by the image of these elements under ι.

We write $\iota(\overline{A}_i\overline{B}_i)$, $\iota(\overline{A}_i\overline{B}_i\overline{X})$ as elements of $\mathbf{Z}_8[H_1(\Sigma_{g,1};\mathbf{Z}_2)]/L_{g,1}$. As we saw in Lemma 4.5, we have

$$\iota(\overline{A}_1\overline{B}_1) = 2\Phi\Delta_0^2(A_1,B_1) + 4\langle A_1\rangle + 4\langle B_1\rangle,$$

and

$$\begin{aligned}\iota(\overline{A}_1\overline{B}_1(\overline{B_2+1})) &= \langle B_1\rangle + \langle A_1\rangle + \langle B_1+B_2\rangle + \langle A_1+B_1\rangle + \langle A_1+B_1+B_2\rangle + \langle A_1+B_2\rangle - \langle B_2\rangle \\ &= -\Phi\Delta_0^3(A_1,B_1,B_2) + 2\Phi\Delta_0^2(A_1,B_2) + 2\Phi\Delta_0^2(B_1,B_2) - 4\langle B_2\rangle \\ &= \Phi\Delta_0^3(A_1,B_1,B_2) + 2\Phi\Delta_0^2(A_1,B_2) + 2\Phi\Delta_0^2(B_1,B_2) + 4\langle B_2\rangle.\end{aligned}$$

Hence for $X = A_1,B_1,\cdots,A_g,B_g$, we have

$$\begin{aligned}\iota(\overline{A}_i\overline{B}_i) &= \Phi\{2\Delta_0^2(A_i,B_i) + 4[A_i] + 4[B_i]\},\\ \iota(\overline{A}_i\overline{B}_i\overline{X}) &= \Phi\{\Delta_0^3(A_i,B_i,X) + 2\Delta_0^2(A_i,X) + 2\Delta_0^2(A_i,B_i) + 2\Delta_0^2(B_i,X) + 4[A_i] + 4[B_i] + 4[X]\}.\end{aligned}$$

Proposition 4.11. *Let $g\ge 3$. Denote by L_g the submodule of $\mathbf{Z}_8[H_1(\Sigma_g;\mathbf{Z}_2)]$ generated by*

$$[0],\ 4\Delta_0^2(x_1,x_2),\ 2\Delta_0^3(x_1,x_2,x_3),\ \Delta^n(x_1,x_2,\cdots,x_n),$$
$$\sum_{i=1}^g\{2\Delta_0^2(A_i,B_i) + 4[A_i] + 4[B_i]\},$$
$$\sum_{i=1}^g\{\Delta_0^3(A_i,B_i,X) + 2\Delta_0^2(A_i,X) + 2\Delta_0^2(B_i,X) + 4[X]\},$$

for $\{x_i\}_{i=1}^n \subset H_1(\Sigma_g;\mathbf{Z}_2)$ and $X = A_1,B_1,\cdots,A_g,B_g$. Then, we have

$$\mathbf{Z}_8[H_1(\Sigma_g;\mathbf{Z}_2)]/L_g \cong H_1(\mathcal{M}_g[2];\mathbf{Z}).$$

5 The abelianization of the level d mapping class group for odd d

In this section, we prove Theorem 1.3. The exact sequence

$$1 \to \mathcal{I}_{g,r} \to \mathcal{M}_{g,r}[d] \to \Gamma_g[d] \to 1$$

plays an important role. By the Lyndon-Hochschild-Serre spectral sequence, we have the exact sequence

$$H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z}) \to H_1(\Gamma_g[d]; \mathbf{Z}) \to 0.$$

5.1 Mod d reduction of inclusion homomorphism

Lemma 5.1. *Let $g \geq 3$. The homomorphism $H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$ factors through $H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \otimes \mathbf{Z}_d$.*

Proof. For any pair of simple closed curves C_1, C_1' which bounds a subsurface of genus 1 in $\Sigma_{g,1}$, the mapping class $t_{C_1} t_{C_1'}^{-1}$ is in Torelli group $\mathcal{I}_{g,1}$. Johnson[9] showed that $\mathcal{I}_{g,1}$ is generated by all pairs of twists $t_{C_1} t_{C_1'}^{-1}$, for $g \geq 3$ and such an bounding pair C_1, C_1'. In particular, $\mathcal{I}_g$ is also generated by pairs of twists as above. Johnson ([11] Lemma 11) also shows that any pair of simple closed curves C_2, C_2' which bounds a subsurface in $\Sigma_{g,r}$ satisfies $(t_{C_2} t_{C_2'}^{-1})^d \in [\mathcal{M}_{g,r}[d], \mathcal{I}_{g,r}]$.

Therefore for $\varphi \in \mathcal{I}_{g,r}$, we have $[\varphi^d] = 0 \in H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z})$ for $r = 0, 1$. This proves the lemma. □

We have already determines the abelianization of level d congruence subgroup of the symplectic group in Section 2. We will construct the splitting of

$$H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \otimes \mathbf{Z}_d \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z}) \to H_1(\Gamma_g[d]; \mathbf{Z}) \to 0$$

for $r = 0, 1$, and prove Theorem 1.3 in the next subsection.

5.2 Johnson homomorphism mod d

In this subsection, we state that the mod d reduction of the Johnson homomorphism can be defined on the level d mapping class group.

For $n \geq 2$, we denote by F_n the free group of rank n, and by $H := F_n/[F_n, F_n]$ the abelianization of F_n. Let $\mathrm{Aut}(F_n)$ be the automorphism group of the free group F_n. Then, $\mathrm{Aut}\, F_n$ acts on H. For a commutative ring R with unit element, denote the tensor algebra of $H \otimes R$ by

$$\hat{T} := \prod_{m=0}^{\infty} H^{\otimes m} \otimes R.$$

We denote $\hat{T}_i := \prod_{m \geq i} H^{\otimes i} \otimes R$ for $i \geq 1$.

Definition 5.2. *The map $\theta : F_n \to 1 + \hat{T}_1$ is called R-valued Magnus expansion of F_n if $\theta : F_n \to 1 + \hat{T}_1$ is a group homomorphism, and for any $\gamma \in F_n$, θ satisfies*

$$\theta(\gamma) \equiv 1 + [\gamma] \ (\mathrm{mod}\, \hat{T}_2).$$

In detail, see Kawazumi[14] Section 1 and Bourbaki[3] Ch.2, §5, no.4, 5. In the following, we put $R := \mathbf{Z}_d$ for an odd integer d. We denote by $\theta_m : F_n \to H^{\otimes m} \otimes \mathbf{Z}_d$ the m-th component of θ. Denote the kernel

$$\Gamma_2^d := \mathrm{Ker}(F_n \to H \otimes \mathbf{Z}_d),$$

then the restriction of θ_2 to $\Gamma_2^d \to H^{\otimes 2} \otimes \mathbf{Z}_d$ is a homomorphism. For $a, b \in F_n$, denote by $A, B \in H_1(F_n;\mathbf{Z})$ the homology classes. Then, we have

$$\theta_2(aba^{-1}b^{-1}) = A \otimes B - B \otimes A,$$
$$\theta_2(a^d) = \frac{d(d-1)}{2} A \otimes A = 0.$$

Hence we obtain

$$\theta_2(\Gamma_2^d) = \Lambda^2 H \otimes \mathbf{Z}_d.$$

From the above calculation, we see that $\theta_2|_{\Gamma_2^d}$ is $\mathrm{Aut}\, F_n$-equivariant. Define the level d IA-automorphism group by $IA_n[d] := \mathrm{Ker}(\mathrm{Aut}\, F_n \to GL(n;\mathbf{Z}_d))$. For $H^* := \mathrm{Hom}(H, \mathbf{Z})$, define the mod d Johnson homomorphism by

$$\begin{array}{rccl} \tau_d : & IA_n[d] & \to & \mathrm{Hom}(H, \Lambda^2 H \otimes \mathbf{Z}_d) \quad \cong H^* \otimes \Lambda^2 H \otimes \mathbf{Z}_d. \\ & \varphi & \mapsto & ([x] \to \theta_2(x^{-1}\varphi(x))) \end{array}$$

Then, we see that τ_d is an $\mathrm{Aut}(F_n)$-equivariant homomorphism, as in Johnson [10] Lemmas 2C and 2D, Kawazumi [14] section 3.

Next, we state that we can define the mod d Johnson homomorphism on the level d mapping class group. Choose symplectic generators $\{a_i, b_i\}_{i=1}^g$ of $\pi_1(\Sigma_{g,1}, *)$ $(* \in \partial\Sigma_{g,1})$ which represent the symplectic basis $\{A_i, B_i\}$. Then we have the isomorphism $\pi_1(\Sigma_{g,1}, *) \cong F_{2g}$, and $H \cong H_1(\Sigma_{g,1};\mathbf{Z})$. The action of $\mathcal{M}_{g,1}[d]$ on the fundamental group of the surface induces the homomorphism $\mathcal{M}_{g,1}[d] \to IA_n[d]$. Hence we have the homomorphism

$$\tau_d : \mathcal{M}_{g,1}[d] \to H^* \otimes \Lambda^2 H \otimes \mathbf{Z}_d \cong H \otimes \Lambda^2 H \otimes \mathbf{Z}_d$$

which is independent of the choice of the generators of $\pi_1(\Sigma_{g,1})$. Note that by the Poincaré duality, we have

$$H^* \otimes \Lambda^2 H \otimes \mathbf{Z}_d \cong H \otimes \Lambda^2 H \otimes \mathbf{Z}_d.$$

It is easy to see that the restriction of τ_d to $\mathcal{I}_{g,1}$ is equal to the mod d reduction of the Johnson homomorphism. Now, we calculate the image of the Johnson homomorphism on the level d mapping class group.

Lemma 5.3. *For $g \geq 3$,*

$$\tau_d(\mathcal{M}_{g,1}[d]) \subset \Lambda^3 H \otimes \mathbf{Z}_d.$$

Proof. By the Theorem 2.6, $\mathcal{M}_{g,1}[d]$ is generated by the d times Dehn twists along all non-separating curves and the Torelli group $\mathcal{I}_{g,1}$. For the simple closed curve C_1 as shown in Figure 5, we have

$$\tau_d(t_{C_1}^d) = \frac{d(d-1)}{2} B_1 \otimes B_1 \otimes B_1 = 0,$$

because d is odd. Since $\tau_d|_{\mathcal{I}_{g,1}}$ is equal to the mod d reduction of the Johnson homomorphism, we also have $\tau_d(\mathcal{I}_{g,1}) \subset \Lambda^3 H \otimes \mathbf{Z}_d$. □

Next, We will define the Johnson homomorphism τ_d for closed surfaces.

Lemma 5.4. *We consider $\Sigma_{g,1}$ as a subsurface of Σ_g. By gluing each mapping class on $\Sigma_{g,1}$ with identity on the disk, we have the surjective homomorphism $\mathcal{M}_{g,1}[d] \to \mathcal{M}_g[d]$. Then, for $g \geq 3$, the homomorphism*

$$\tau_d : \mathcal{M}_g[d] \to \Lambda^3 H / H \otimes \mathbf{Z}_d$$

is well-defined.

Proof. It is known that $\mathrm{Ker}(\mathcal{M}_{g,1}[d] \to \mathcal{M}_g[d])$ is generated by twisting pair $T_C T_{C'}^{-1}$ and separating twist $T_{\partial\Sigma_{g,1}}$, where (C, C') be a pair which bounds subsurface of genus $g-1$ (see Birman[2] pp156-160). By the result of Johnson [10] Lemmas 4A and 4B, we have $\tau_d(T_{\partial\Sigma_{g,1}}) = 0$, and $\tau_d(T_C T_{C'}^{-1}) \in H \subset \Lambda^3 H$. Since $H \subset \Lambda^3 H$ is a $Sp(2g; \mathbf{Z})$-invariant subspace, we see that τ_d of the closed surface is well-defined. □

We prove Theorem 1.3 using the homomorphism defined as above.

proof of Theorem 1.3. Consider the homomorphism

$$\tau_d : \mathcal{M}_{g,1}[d] \to \Lambda^3 H \otimes \mathbf{Z}_d,$$
$$\tau_d : \mathcal{M}_g[d] \to \Lambda^3 H/H \otimes \mathbf{Z}_d,$$

defined in Lemma 5.1. By the structure of the abelianization determined in Johnson[13] Theorems 3 and 6, τ_d induces the isomorphism

$$H_1(\mathcal{I}_{g,1}; \mathbf{Z}) \otimes \mathbf{Z}_d \cong \Lambda^3 H \otimes \mathbf{Z}_d,$$
$$H_1(\mathcal{I}_g; \mathbf{Z}) \otimes \mathbf{Z}_d \cong \Lambda^3 H/H \otimes \mathbf{Z}_d$$

when d is odd. Hence, we have the splitting of the exact sequence

$$H_1(\mathcal{I}_{g,r}; \mathbf{Z}) \otimes \mathbf{Z}_d \to H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z}) \to H_1(\Gamma_g[d]; \mathbf{Z}) \to 0 \ (r = 0, 1),$$

by the homomorphism τ_d. This shows that

$$H_1(\mathcal{M}_{g,r}[d]; \mathbf{Z}) = \begin{cases} \Lambda^3 H \oplus H_1(\Gamma_g[d]; \mathbf{Z}), & \text{when } r = 1 \\ \Lambda^3 H/H \oplus H_1(\Gamma_g[d]; \mathbf{Z}), & \text{when } r = 0 \end{cases}$$

This proves the theorem. □

References

[1] H. Bass, J. Milnor, and J.P. Serre, *Solution of the congruence subgroup problem for* SL_n *(*$n \geq 3$*) and* Sp_{2n} *(*$n \geq 2$*)*, Publications Mathématiques de L'IHÉS **33** (1967), no. 1, 59–137.

[2] J.S. Birman, *Braids, Links, and Mapping Class Groups.(AM-82)*, Princeton University Press, 1975.

[3] N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, 1972.

[4] B. Farb, *Some Problems on Mapping Class Groups and Moduli Space*, Problems on Mapping Class Groups And Related Topics **74** (2006).

[5] R. Hain, *Torelli groups and Geometry of Moduli Spaces of Curves*, Current Topics in Complex Algebraic Geometry (CH Clemens and J. Kollar, eds.) MSRI publications **28** (1995), 97–143.

[6] A. Heap, *Bordism Invariants of the Mapping Class Group*, Arxiv preprint math.GT/0502587 (2005).

[7] S.P. Humphries, *Normal closures of powers of Dehn twists in mapping class groups*, Glasgow J. Math **34** (1992), 313–317.

[8] J. Igusa, *On the Graded Ring of Theta-Constants*, American Journal of Mathematics **86** (1964), no. 1, 219–246.

[9] D. Johnson, *Homeomorphisms of a Surface which Act Trivially on Homology*, Proceedings of the American Mathematical Society **75** (1979), no. 1, 119–125.

[10] ———, *An abelian quotient of the mapping class group $\mathcal{I}_g$*, Mathematische Annalen **249** (1980), no. 3, 225–242.

[11] ———, *Conjugacy relations in subgroups of the mapping class group and a group-theoretic description of the Rochlin invariant*, Mathematische Annalen **249** (1980), no. 3, 243–263.

[12] ———, *Quadratic Forms and the Birman-Craggs Homomorphisms*, Transactions of the American Mathematical Society **261** (1980), no. 1, 235–254.

[13] ———, *The structure of the Torelli Group III: The abelianization of $\mathcal{I}_g$*, Topology **24** (1985), no. 2, 127–144.

[14] N. Kawazumi, *Cohomological Aspects of Magnus Expansions*, Arxiv preprint math.GT/0505497 (2005).

[15] R.C. Kirby and L.R. Taylor, *Pin structures on low-dimensional manifolds*, Geometry of low-dimensional manifolds **2**, 177–242.

[16] R. Lee, E. Miller, and S. Weintraub, *The Rochlin invariant, theta functions and the holonomy of some determinant line bundle*, J. reine angew. Math **392** (1988), 187–218.

[17] R. Lee and R.H. Szczarba, *On the homology and cohomology of congruence subgroups*, Inventiones Mathematicae **33** (1976), no. 1, 15–53.

[18] J.D. McCarthy, *On the first cohomology group of cofinite subgroups in surface mapping class groups*, Topology **40** (2000), no. 2, 401–418.

[19] J. Mennicke, *Zur Theorie der Siegelschen Modulgruppe*, Mathematische Annalen **159** (1965), no. 2, 115–129.

[20] A. Putman, *The abelianization of the level L mapping class group*, arXiv:math.GT/0803.0539 (2007).

[21] M. Sato, *On symmetric mapping class groups*, Master's thesis, University of Tokyo, (2007) (Japanese).

[22] T. Satoh, *The abelianization of the congruence IA-automorphism group of a free group*, Mathematical Proceedings of the Cambridge Philosophical Society **142** (2007), no. 02, 239–248.
Corrigendum ibid. **143** (2007), no. 1 255–256.

[23] V.G. Turaev, *Cohomology rings, linking forms and invariants of spin structures of three-dimensional manifolds*, Math. USSR Sbornik **48** (1984), no. 1, 65–79.

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