International Mathematical Forum, Vol. 13, 2018, no. 7, 315 - 322
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/imf.2018.8322

Eigenvalues and Eigenfunctions of $-\Delta$ for an $\mathbb{R}^2$ Unit Ball and Related Theorems

Humberto Serrano D.

Universidad Distrital Francisco Jose de Caldas, Colombia

Abstract

The present work computes the eigenvalues and eigenfunctions of the Laplacian operator, $-\Delta$, for a particular situation where $\Omega = B_1(0) \subset \mathbb{R}^2$. To this end, separation of variables was used together with the bounded solutions of Bessel's differential equations. This problem set may arise, for example, when solving a wave equation representing the vibration associated to an elastic circular membrane for $N = 2$ and $\Omega = B_1(0)$. A set of theorems related to the present analysis are shown at the end of the paper.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with a boundary denoted by $\partial\Omega$. The authors consider the problem of finding a function $u : \overline{\Omega} \times [0, \infty) \to \mathbb{R}$, such that

$$\begin{aligned} u_{tt} &\equiv \Delta u \text{ in } \Omega \times (0, \infty), \quad u \equiv 0 \text{ in } \partial\Omega \times (0, \infty), \\ u(x, 0) &= u_0(x) \text{ in } \Omega, \quad u_t(x, 0) = v_0(x) \text{ in } \Omega, \end{aligned} \tag{1}$$

where $\Delta = \sum_{i=1}^{n} \partial^2 u/\partial x_i^2$ denotes the Laplacian operator over the variables x_i, $i = 1, 2, \cdots, N$ and t represents time, $u_0(x)$ and $v_0(x)$ are given functions. The operator $(u_{tt} - \Delta)$ is denoted by $\Box$ and is referred to as the D'Alembertian. An equation like (1) is a typical example of a hyperbolic-type PDE. For $N = 1$ and $\Omega = (0, 1)$, the problem represented by (1) models the vibrations of a string with no external forces considered.
When $N = 2$, the problem associated to (1) models the small vibrations of

an elastic membrane that is fixed to a circular frame of the form $\Omega \subset \mathbb{R}^2$ see [2]. The condition $u = 0$ in $\partial\Omega$ explicitly indicates that the membrane has been fixed in $\partial\Omega$ for $t > 0$. The initial motion and speed are given by $u_0(x)$ and $v_0(x)$, respectively. Here, (1) is considered for the vibration analysis of a circular and elastic membrane, with no external forces considered, where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $t \geq 0$ (see [4], [3])

$$\begin{aligned} &\Box u(x, y, t) = 0 \text{ in } \Omega \times (0, \infty), \quad u(x, y, t) \equiv 0 \text{ in } \partial\Omega \times (0, \infty), \\ &u(x, y, 0) = 0 \text{ in } \Omega, \quad u_t(x, y, 0) = 0 \text{ in } \Omega, \end{aligned} \tag{2}$$

where the function $u \in C^2(\overline{\Omega} \times [0, \infty))$ that satisfies (2) represents the vertical motion of the membrane at a particular point (x, y) and instant t. Suppose that $u(x, y, t) = v(x, y)T(t)$. By substituting in (2) the following pair of equations is obtained

$$T''(t) + \lambda T(t) = 0 \text{ and } \Delta v + \lambda v = 0,$$

where $\lambda > 0$.
This entails the search for the corresponding eigenvalues and eigenfunctions for $t > 0$ in the following problem set

$$-\Delta v = \lambda v \text{ in } \Omega, \quad v \not\equiv 0 \text{ in } \Omega, \quad v \equiv 0 \text{ in } \partial\Omega. \tag{3}$$

Eigenvalue problems are concerned with fundamental modes of vibration in a given physical system.
By using Green's identities (see [7], [1]), it is possible to prove that

1. if $v \not\equiv 0$ is to eigenfunction of the problem (3), associated to eigenvalue λ, then
$$\lambda = \frac{\displaystyle\iint_\Omega |\nabla v(x, y)|^2 dxdy}{\displaystyle\iint_\Omega v^2(x, y) dxdy}. \tag{4}$$

2. if v_1 and v_2 are eigenfunctions of the problem (3), associated to different eigenvalues λ and $\tilde{\lambda}$, respectively, then
$$\langle v_1, v_2 \rangle = \iint_\Omega v_1(x, y) v_2(x, y) dxdy = 0. \tag{5}$$

2 Main Result

If $\Omega = B_1(0) = \{(x, y) : x^2 + y^2 < 1\}$, $v \in C^2(\overline{\Omega})$, $v \not\equiv 0$, $-\Delta v = \lambda v$ in Ω , $v/\partial\Omega \equiv 0$, $v(r, \theta) = v(r\cos\theta, r\sin\theta) = f(r)g(\theta)$, where, $0 \leq \theta \leq 2\pi$,

$0 \le r \le 1$, then the problem (3) has a solution given by

$$v(r,\theta) \equiv J_m(\alpha_{m,k}r)(A\cos(m\theta) + B\sin(m\theta)) \tag{6}$$
$$\equiv J_m(\sqrt{\lambda}r)(A\cos(m\theta) + B\sin(m\theta)), \tag{7}$$

for two positive integers m and k, $\lambda = \alpha_{m,k}^2$ (see [4]). Furthermore, every function of the form (15) is an eigenfunction associated to the eigenvalue $\lambda = \alpha_{m,k}^2$. Based on the previous hypothesis, we have

$$\Delta v = \frac{1}{r}(rf'(r))'g(\theta) + \frac{1}{r^2}f(r)g''(\theta) \equiv -\lambda v = -\lambda f(r)g(\theta), \quad f(1) = 0.$$

Then

$$\frac{r(rf'(r))'}{f(r)} + \lambda r^2 = -\frac{g''(\theta)}{g(\theta)} = c, \tag{8}$$

where c is a constant.
From the previous identity we deduce that $g''(\theta) + cg(\theta) = 0$, for $0 \le \theta \le 2\pi$, consequently $g(\theta + 2\pi) = g(\theta)$, if and only if $c > 0$. In [2] the author prove that $g(\theta) = A\cos\sqrt{c}\theta + B\sin\sqrt{c}\theta$, $A^2 + B^2 \ne 0$. Due to the periodical nature of the function $g(\theta)$, there exists an integer $m \ge 0$, such that $\sqrt{c}2\pi = 2m\pi$, $\sqrt{c} = m$, $c = m^2$. By substituting the value of the constant c in (8) by m^2, then

$$g''(\theta) + m^2 g(\theta) = 0, \text{ for } 0 \le \theta \le 2\pi \tag{9}$$
$$r^2 f''(r) + rf'(r) + (\lambda r^2 - m^2)f(r) \equiv 0, \quad f(1) = 0 \tag{10}$$

Since $\lambda > 0$, let $\alpha = \sqrt{\lambda}$, $x = \alpha r$, $r = x/\alpha$; $y(x) = f(x/\alpha)$, $f(r) = y(\alpha r)$, $f'(x/\alpha) = \alpha y'(x)$, $f''(x/\alpha) = \alpha^2 y''(x)$. By substituting in (10), then

$$f''(x/\alpha) + (1/r)f'(x/\alpha) + (\alpha^2 - m^2/r^2)f(x/\alpha) \equiv 0, \tag{11}$$
$$y''(x) + (1/x)y'(x) + (1 - m^2/x^2)y(x) \equiv 0, \tag{12}$$
$$x^2 y''(x) + xy'(x) + (x^2 - m^2)y(x) \equiv 0 \tag{13}$$

(13) is the so called Bessel's ordinary differential equation of order m. When solving this equation using the Frobenius method, the bounded solution obtained is as follows:

$$y(x) = J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!}\left(\frac{x}{2}\right)^{2k+m}. \tag{14}$$

In [3], [4] the non-negative roots of (14) are proved to form a countable set of values represented by $\alpha_{m,k}$, such that

$$0 < \alpha_{m,1} < \alpha_{m,2} < \cdots < \alpha_{m,k} < \cdots$$

, each $J_m(x)$ has infinite number of positive zeros, and they are not regularly spaced. Therefore, $f(r) = y(\alpha r) = J_m(\alpha r)$. Based on the previous analysis, it is possible to conclude that $v(r,\theta) = J_m(\alpha.r)(A\cos(m\theta) + B\sin(m\theta))$, for some non-negative integer m. Since $0 = f(1) = y(\alpha) = J_m(\alpha)$, the existence of a positive integer k is guaranteed, such that $\alpha = \alpha_{m,k}$ (see [4], [2]); this is to say that

$$v(r,\theta) = J_m(\alpha_{m,k}r)(A\cos(m\theta) + B\sin(m\theta)) \tag{15}$$

As a particular case, we have that $v_1(r,\theta) = J_m(\alpha_{m,k}r)\cos(m\theta)$, $v_2(r,\theta) = J_m(\alpha_{m,k}r)\sin(m\theta)$ are the corresponding eigenfunctions of the eigenvalue $\lambda = \alpha^2_{m,k}$. Since $J_m(\alpha_{m,k}r)$ is a solution to (10), the function (15) is an eigenfunction associated to $\lambda = \alpha^2_{m,k}$ for some non-negative integer m.

3 Related Theorems

The following results characterize the eigenvalues and eigenfunctions of $-\Delta$

Theorem 1. *Let $\Omega = B_1(0) = \{(x,y) : x^2 + y^2 < 1\}$, $v \in C^2(\Omega) \cap C(\overline{\Omega})$, $v \not\equiv 0$, and let $\lambda \in \mathbb{R}$ be an eigenvalue of the problem*

$$-\Delta v = \lambda v \text{ in } \Omega, \quad v \not\equiv 0, \quad v/\partial\Omega \equiv 0 \tag{16}$$

, then there are two non-negative integers m and k, such that $\lambda = \alpha^2_{m,k}$.

Theorem 2. *Let $\Omega = B_1(0) = \{(x,y) : x^2 + y^2 < 1\}$, if $-\Delta v = \lambda v$ in Ω , $v \equiv 0$ in $\partial\Omega$ and $\lambda = \alpha^2_{m,k}$ for two integers $m \geq 0$ and $k \geq 1$, then*

$$v(r,\theta) \equiv J_m(\alpha_{m,k}r)(A\cos(m\theta) + B\sin(m\theta))$$

, where A and B are constants.

Proof (of Theorem 1). Suppose that $v \in C^2(\Omega) \cap C(\overline{\Omega})$, v satisfies (16) , $v \not\equiv 0$ and $\lambda \neq \alpha^2_{m,k}$, for any integer $m \geq 0$, and $k \geq 1$. Since $v(r,\theta)$ is given by (15) and

$$v_1(r,\theta) = J_m(\alpha_{m,k}r)\cos(m\theta) \text{ and } v_2(r,\theta) = J_m(\alpha_{m,k}r)\sin(m\theta),$$

are eigenfunctions associates to the eigenvalue $\lambda = \alpha^2_{m,k}$ then by (5) we obtain

$$\int_0^{2\pi}\int_0^1 v(r,\theta)J_m(\alpha_{m,k}r)\cos(m\theta) r\,dr\,d\theta = 0,$$
$$\int_0^1\int_0^{2\pi} v(r,\theta)J_m(\alpha_{m,k}r)\sin(m\theta) r\,dr\,d\theta = 0.$$

For all $0 \leq r \leq 1$ (r fixed), see [2], we expand the function $v(r,\theta)$ in a trigonometric Fourier series with respect to the variable θ

$$v(r,\theta) = \sum_{m=0}^{\infty} [a_m(r)\cos(m\theta) + b_m(r)\sin(m\theta)], \tag{17}$$

where

$$a_m(r) = \frac{1}{\pi}\int_0^{2\pi} v(r,\theta)\cos(m\theta)d\theta, m = 0,1,2,\ldots, \tag{18}$$

$$b_m(r) = \frac{1}{\pi}\int_0^{2\pi} v(r,\theta)\sin(m\theta)d\theta, m = 1,2,\ldots \tag{19}$$

$v(1,\theta) = 0$, then $a_m(1) = \int_0^{2\pi} v(1,\theta)\cos(m\theta)d\theta = 0$, therefore $a_m(r)$ satisfies the equation (10) and $a_m(r)$ is bounded.

$$a_m(r) = \frac{1}{\pi}\int_0^{2\pi} v(r,\theta)\cos(m\theta)d\theta \tag{20}$$

$$= \frac{1}{\pi}f(r)\int_0^{2\pi} g(\theta)\cos(m\theta)d\theta \tag{21}$$

$$= c_m y(\alpha r) = c_m J_m(\alpha_{m,k} r), \tag{22}$$

for a positive integer k, where c_m are the Fourier coefficients of the function $g(\theta)$ for $0 \leq \theta \leq 2\pi$. The previous result means that $\lambda = \alpha_{m,k}^2$, for a positive integer k, and there exists $c_m \in \mathbb{R}$, such that $a_m(r) = c_m J_m(\alpha_{m,k} r)$.
Since $v(r,\theta)$ is given by (15) and $v_1(r,\theta) = J_m(\alpha_{m,k} r)\cos m\theta$ is eigenfunction associated to the eigenvalue $\lambda = \alpha_{m,k}^2$, then (see [2])

$$\begin{aligned} 0 &= \int_0^1 \int_0^{2\pi} v(r,\theta) J_m(\alpha_{m,k} r)\cos(m\theta) r dr d\theta \\ &= \int_0^1 \left[J_m(\alpha_{m,k} r) r \int_0^{2\pi} v(r,\theta)\cos(m\theta)d\theta \right] dr \\ &= \int_0^1 c_m J_m^2(\alpha_{m,k}) r dr, \end{aligned}$$

implies that $c_m = 0$ and so $a_m(r) = 0$. By using the same argument, it is possible to prove that $b_m(r) = 0$. Then it can be concluded that

$$\int_0^{2\pi} v(r,\theta)\cos(m\theta)d\theta = \int_0^{2\pi} v(r,\theta)\sin(m\theta)d\theta = 0 \text{ for } 0 \leq r \leq 1.$$

By continuity, $v \equiv 0$ in $B_1(0)$.

Alternatively we can expand each of the functions $a_m(r)$ y $b_m(r)$ in Fourier series with respect to the system $\{J_m(\alpha_{m,k}r)\}$. The result is

$$a_m(r) = \sum_{k=1}^{\infty} A_{m,k} J_m(\alpha_{m,k}r) \tag{23}$$

$$b_m(r) = \sum_{k=1}^{\infty} B_{m,k} J_m(\alpha_{m,k}r) \tag{24}$$

where

$$A_{m,k} = \frac{\displaystyle\int_0^1 r a_m(r) J_m(\alpha_{m,k}r)dr}{\displaystyle\int_0^1 r[J_m(\alpha_{m,k}r)]^2 dr}, \quad m = 0, 1, \ldots, k = 1, 2, \ldots \tag{25}$$

$$B_{m,k} = \frac{\displaystyle\int_0^1 r b_m(r) J_m(\alpha_{m,k}r)dr}{\displaystyle\int_0^1 r[J_m(\alpha_{m,k}r)]^2 dr}, \quad m = 0, 1, \ldots, k = 1, 2, \ldots \tag{26}$$

If $m \geq 0$

$$\int_0^1 r[J_m(\alpha_{m,k}r)]^2 dr = \frac{1}{2}[J_{m+1}(\alpha_{m,k})]^2. \tag{27}$$

By substitution of (18) into (25)

$$A_{m,k} = \frac{\displaystyle\int_0^1 r \left[\int_0^{2\pi} v(r,\theta)\cos(m\theta)d\theta\right] J_m(\alpha_{m,k}r)dr}{\displaystyle\pi\int_0^1 r[J_m(\alpha_{m,k}r)]^2 dr}, \quad m = 0, 1, \ldots, k = 1, 2, \ldots \tag{28}$$

By (15)

$$A_{m,k} = \frac{\displaystyle\int_0^{2\pi}\int_0^1 v(r,\theta) J_m(\alpha_{m,k}r)\cos(m\theta) r dr d\theta}{\displaystyle\pi\int_0^1 r[J_m(\alpha_{m,k}r)]^2 dr} = 0, \quad m = 0, 1, \ldots, k = 1, 2, \ldots \tag{29}$$

The equation (23) implies that $a_m(r) = 0$. By using the same argument, it is possible to prove that $b_m(r) = 0$. Then it can be concluded that $v \equiv 0$ en Ω.
Proof (of Theorem 2). The functions

$$v_1(r,\theta) = J_m(\alpha_{m,k}r)\cos(m\theta) \text{ and } v_2(r,\theta) = J_m(\alpha_{m,k}r)\sin(m\theta),$$

are eigenfunctions associated to the eigenvalue $\lambda = \alpha_{m,k}^2$, this is to say

$$\Delta v_1 = -\alpha_{m,k}^2 v_1,\ v_1/\partial\Omega \equiv 0, \quad \Delta v_2 = -\alpha_{m,k}^2 v_2,\ v_2/\partial\Omega \equiv 0.$$

By multiplying v_1 and v_2 and then integrating the product, we obtain

$$\langle v_1, v_2\rangle = \iint_\Omega v_1(x,y)v_2(x,y)dxdy \tag{30}$$
$$= \int_0^1 \int_0^{2\pi} v_1(r,\theta)v_2(r,\theta)rdrd\theta \tag{31}$$
$$= \int_0^1 rJ_m^2(\alpha_{m,k}r)dr \int_0^{2\pi} \frac{1}{2}\sin 2m\theta d\theta \tag{32}$$
$$= 0. \tag{33}$$

Let

$$A = \frac{\iint_\Omega vv_1dxdy}{\iint_\Omega v_1^2dxdy} = \frac{\langle v, v_1\rangle}{||v_1||^2} \quad \text{y} \quad B = \frac{\iint_\Omega vv_2dxdy}{\iint_\Omega v_2^2dxdy} = \frac{\langle v, v_2\rangle}{||v_2||^2}$$

by (30), then $\langle v_1, v_2\rangle = 0$, and so

$$\left\langle v - \frac{\langle v, v_1\rangle}{||v_1||^2}v_1 - \frac{\langle v, v_2\rangle}{||v_2||^2}v_2, v_2\right\rangle = \left\langle v - \frac{\langle v, v_1\rangle}{||v_1||^2}v_1 - \frac{\langle v, v_2\rangle}{||v_2||^2}v_2, v_1\right\rangle = 0. \tag{34}$$

Let $w = v - Av_1 - Bv_2$, then $\langle w, v_1\rangle = \langle w, v_2\rangle = 0$, $-w = -v + Av_1 + Bv_2$.

$$\begin{aligned} -\Delta w &= -\Delta v - A(-\Delta v_1) - B(-\Delta v_2) \\ &= \lambda v - A\lambda v_1 - B\lambda v_2, \\ &= \lambda(v - Av_1 - Bv_2), \\ &= \lambda w, \end{aligned}$$

$w/\Delta\Omega \equiv 0, \lambda = \alpha_{m,k}^2$. Therefore,

$$\int_0^1 \int_0^{2\pi} w(r,\theta)J_q(\alpha_{q,l}r)\cos(q\theta)rdrd\theta = \int_0^1 \int_0^{2\pi} w(r,\theta)J_q(\alpha_{q,l}r)\sin(q\theta)rdrd\theta = 0$$

for any integer $q \geq 0$ and $l \geq 1$ Let

$$a_q(r) = \int_0^{2\pi} w(r,\theta)\cos q\theta d\theta = c_qJ_q(\alpha_{q,l}r),$$
$$b_q(r) = \int_0^{2\pi} w(r,\theta)\sin q\theta d\theta = c_qJ_q(\alpha_{q,l}r),$$

$w(r,\theta) = v(r,\theta) - Av_1(r,\theta) - Bv_2(r,\theta)$, $v(1,\theta) = 0$, then $a_q(1) = 0$ y $a_q(r)$ satisfies (10), and by following the same argument applied in Theorem 1, it is possible to prove that $a_q(r) = b_q(r) = 0$, for any integer $q \geq 0$ and $r \in (0,1]$. Therefore, $w \equiv 0$, $v - Av_1 - Bv_2 = 0$. This is to say that $v = Av_1 + Bv_2 = AJ_m(\alpha_{m,k}r)\cos m\theta + BJ_m(\alpha_{m,k}r)\sin m\theta$

4 Conclusions

The problem (3) has non-zero solutions $v(r,\theta) = J_m(\alpha.r)(A\cos(m\theta)+B\sin(m\theta))$, only for a countable set of values of α, given by

$$0 < \alpha_{m,1} < \alpha_{m,2} < \cdots < \alpha_{m,k} < \cdots$$

The sequence of positive roots denoted by $\alpha_{m,k}$ goes to infinity as $k \to \infty$.
If $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^2$, then the eigenvalues of (3) are $\lambda = \alpha^2_{m,k}$, where m and k are positive integers, and the corresponding eigenfunctions are $v(r,\theta) = J_m(\alpha.r)(A\cos(m\theta) + B\sin(m\theta))$, where A and B are constants. Additionally, $\alpha = \alpha_{m,k}$ corresponds to the k-th positive root of the Bessel's function $J_m(\alpha.r)$.
Eigenvalues and eigenfunctions of the Laplacian operator problems are concerned with fundamental modes of vibration of a membrana in 2D. This is important for science of real drums and persussion.

References

[1] M. Protter and H. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Inc., 1967.

[2] P. T. Georgi, *Fourier Series* Dover Publications, Inc., 1976.

[3] B. Epstein, *Partial Differential Equations*, Robert E Krieger Publishing Company, 1975.

[4] K. E. Gustafson, *Introduction to Partial Differential Equations and Hilbert Space methods*, Dover Publications, Inc., 1999.

[5] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, 2th ed., Cambridge Mathematical Library, Cambridge Univ. Press, 1995.

[6] R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Wiley-Interscience, 1953.

[7] David Gilbarg and Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.

[8] E. L. Ince, *Ordinary Differential Equations*, Dover Publicatoins, 1956.

Received: April 10, 2018; Published: May 30, 2018