Design of controller on synchronization of memristor-based neural networks with time-varying delays

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ABSTRACT

In this paper, synchronization of memristor-based neural networks (MNNs) with time-varying delays is investigated. By employing the Newton–Leibniz formulation and inequality technique, the controller with state or output coupling is designed to obtain global exponential synchronization of MNNs. The obtained delay-dependent conditions can be checked easily and they also enrich and improve the results in earlier publications. Finally, one numerical example is given to demonstrate the effectiveness of the obtained results.

1. Introduction

The practical memristor device was realized by scientists at Hewlett–Packard Laboratories and the finding was published in 2008 [1] since memristor was originally theorized by Chua in 1971 [2]. Memristor was predicted as the fourth circuit element (the other three are resistor, capacitor and inductor) and it could play the role as resistor in circuit system. In the past few years, memristor has received increasing research attention for its potential applications in the next generation computer and powerful brainlike neural computers [3]. In addition, it has been shown that memristors are proposed to work as synaptic weights in artificial neural networks [4,18]. Due to this feature, the model of memristor-based neural networks (MNNs) can be built to emulate the human brain where synapses are implemented with memristors.

As is well known, synchronization of neural networks is significant and it has received great attention due to their potential applications in many different areas such as secure communication [19–23], information science [24–27, 29–31], and biological system [32–35]. In addition, synchronization control has been used to investigate the dynamic properties of neural networks. Moreover, the results in [17] show that memristor-based nonlinear hybrid system plays an important role in the security of secure communication due to the special feature of memristor. Therefore, it is significant to study synchronization of MNNs.

Motivated by the above discussion, in this paper, we focus our attention on the design of the controller for the synchronization of MNNs. The contributions of this paper are as follows. Firstly, by using the nonsmooth analysis of control theory, the synchronization of MNNs with discontinuous right-hand side is investigated. Different from continuous neural networks, the system of MNNs is discontinuous since the parameters concerning memristors change according to its state. The classical solution is not applicable, so the existence of solutions for MNNs is a delicate problem. Also, this problem brings challenges to investigate the synchronization of MNNs. Secondly, a general controller with state or output coupling is proposed. In [9,12], the synchronization of MNNs was obtained with memoryless controller. But the controller considered in our paper contains the information of the size of \( \tau(t) \). Thirdly, the coupling matrix in [12,13] is required to be symmetrical while the coupling matrices in our paper are random and can be easily solved by using the MATLAB tool boxes. Finally, by using a new lemma and the transform scaling, our results are true for MNNs which in terms of differential inclusion.

The organization of this paper is as follows. The system and some preliminaries are introduced in Section 2. In Section 3, a delay-dependent controller is designed to obtain the sufficient conditions for the synchronization of MNNs. Then, numerical simulations are given to demonstrate the effectiveness of the obtained results in Section 4. Finally, conclusions are drawn in Section 5.
2. System description and preliminaries

In this paper, we consider the memristor-based neural networks as follows:

\[
x_i(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij}x_j(t)g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}x_j(t-t_j(t)) + g_i(x_i(t-t_1(t))), \quad t \geq 0, \quad i = 1, 2, \ldots, n,
\]

where

\[
a_{ij}(x_j(t)) = \begin{cases} \sigma_{ij}^+ & x_j(t) < 0, \\ \sigma_{ij}^- & x_j(t) \geq 0, \end{cases}
\]

\[
b_{ij}(x_j(t-t_j(t))) = \begin{cases} b_{ij}^+ & x_j(t-t_j(t)) < 0, \\ b_{ij}^- & x_j(t-t_j(t)) \geq 0, \end{cases}
\]

\[x_i(t)\] is the state variable of the \(i\)-th neuron. \((a_{ij}(x_j(t)))\) and \((b_{ij}(x_j(t-t_j(t))))\) denote the feedback connection weight and delayed feedback connection weight, respectively. \(g_j : \mathbb{R} \rightarrow \mathbb{R}\) is bounded continuous function, \(r_j(t)\) corresponds to the transmission delay, \(i, j = 1, 2, \ldots, n\) and all constant numbers. The initial condition of system (1) is \(x(s) = \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)\).

Obviously, system (1) is a continuous-time system, then its solution is different from the classical solution and cannot be defined in the conventional sense. In order to obtain the solution of system (1), some definitions and lemmas are given.

**Definition 1.** For a system with discontinuous right-hand side:

\[
\frac{dx}{dt} = F(x), \quad x(0) = x_0, \quad x \in \mathbb{R}^n, \quad t \geq 0,
\]

where \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is discontinuous. A set-valued map is defined as

\[
\Phi(x) = \bigcap_{\delta} \bigcap_{\sigma(x) = 0} \mathcal{C}[B(x, \delta), N] \cup \text{closure of the convex hull of the set } E, E \subset \mathbb{R}^n,
\]

where \(\mathcal{C}[B(x, \delta)]\) is the closure of the convex hull of the set \(E, E \subset \mathbb{R}^n, B(x, \delta) = \{y \in \mathbb{R}^n : \|y-x\| < \delta, x, y \in \mathbb{R}^n, \delta \in \mathbb{R}^+\}, \) and \(N \subset \mathbb{R}^n, \mu(N)\) is Lebesgue measure of set \(N\).

A solution in Filippov’s sense [5] of system (2) with the initial condition \(x(0) = x_0 \in \mathbb{R}^n\) is an absolutely continuous function \(x(t), t \in [0, T], T > 0,\) which satisfies \(x(0) = x_0\) and differential inclusion:

\[
\frac{dx}{dt} \in \Phi(x) \quad \text{for a.a. } t \in [0, T].
\]

If \(F(x)\) is bounded, then the set-valued function \(\Phi(x)\) is nonempty, bounded and closed, convex, and it is upper semicontinuous [5], then the solution \(x(t)\) of system (2) with the initial condition exists and it can be extended to the interval \([0, +\infty)\) in the sense of Filippov.

By applying the theories of set-valued maps and differential inclusions [5–7], then system (1) can be rewritten as the following differential inclusion:

\[
x_i(t) \in x_i(t) + \sum_{j=1}^{n} a_{ij}(x_j(t))g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(x_j(t-t_j(t)))g_j(x_j(t-t_j(t)) + g_i(x_i(t-t_1(t))), \quad t \geq 0, \quad i = 1, 2, \ldots, n,
\]

where

\[
co[a_{ij}(x_j(t))] = \begin{cases} a_{ij}^+ & x_j(t) < 0, \\ a_{ij}^- & x_j(t) > 0, \end{cases}
\]

\[
co[b_{ij}(x_j(t-t_j(t)))] = \begin{cases} b_{ij}^+ & x_j(t-t_j(t)) < 0, \\ b_{ij}^- & x_j(t-t_j(t)) > 0, \end{cases}
\]

Throughout this paper, we consider system (3) as the drive system. Then the corresponding response system is as follows:

\[
y_i(t) \in -y_i(t) + \sum_{j=1}^{n} co[a_{ij}(y_j(t))]g_j(y_j(t)) + \sum_{j=1}^{n} co[b_{ij}(y_j(t-t_j(t)))g_j(y_j(t-t_j(t))) + u_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n,
\]

where \(y_i(t)\) is the state variable of the \(i\)-th neuron, \(u_i(t)\) is the appropriate control input to obtain a certain control objective, other parameters are the same as in systems (3). The initial condition of (4) is \(y(s) = \psi(s) = (\psi_1(s), \psi_2(s), \ldots, \psi_n(s))^T \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)\).

**Definition 2.** A function \(x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T\) is a solution of (1), with the initial condition \(x(s) = \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)\), if \(x(t)\) is an absolutely continuous function and satisfies the differential inclusion (3).

Throughout this paper, the following assumptions are given for system (1).

\(H1\) For \(j \in 1, 2, \ldots, n\), \(g_j\) is bounded and there exists constant \(l_j > 0\) such that

\[
0 \leq g_j(s_1) - g_j(s_2) \leq l_j, \quad g_j(0) = 0,
\]

for all \(s_1, s_2 \in \mathbb{R}, s_1 \neq s_2\).

\(H2\) The transmission delay \(r_j(t)\) is a differential function and there exist \(r, \mu > 0\) such that

\[
0 \leq r_j(t) \leq r, \quad r_j(t) \leq \mu,
\]

for all \(t \geq 0, i = 1, 2, \ldots, n\).

**Lemma 1.** Suppose that the assumption \((H1)\) holds, then solution \(x(t)\) with the initial condition \(x(s) = \phi(s) = (\phi_1(s), \phi_2(s), \ldots, \phi_n(s))^T \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)\) of (1) exists and it can be extended to the interval \([0, +\infty)\) in the sense of Filippov.

Define the synchronization error as \(e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T\) where \(e_i(t) = y_i(t) - x_i(t)\) for all \(i = 1, 2, \ldots, n\). Then based on theories of set-valued maps and differential inclusions, we can obtain the following error system:

\[
\dot{e}_i(t) = -e_i(t) + \sum_{j=1}^{n} co[a_{ij}(y_j(t))]g_j(y_j(t)) - co[a_{ij}(x_j(t))]g_j(x_j(t)) + \sum_{j=1}^{n} co[b_{ij}(y_j(t-t_j(t))]g_j(y_j(t-t_j(t))) - co[b_{ij}(x_j(t-t_j(t))]g_j(x_j(t-t_j(t)))) + u_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n,
\]

The aim of this paper is to design a controller \(u(t)\) to let the response system (4) synchronize with the drive system (3). Since the information on the size of \(e(t)\) is available, the controller in the following form is considered:

\[
u(t) = K_1e(t) + K_2e(t-r(t))
\]
or
\[ u(t) = K_1(g(y(t)) - g(x(t))) + K_2(g(y(t) - r(t)) - g(x(t) - r(t))), \] where \( K_1 \) and \( K_2 \) are constant gain matrices.

Before giving our main results, we give the following definition and lemma.

**Definition 3.** System (3) and system (4) are said to be globally exponentially synchronized, if there exist constants \( \varepsilon > 0 \) and \( M > 1 \), such that
\[ \|y(t) - x(t)\| \leq M \sup_{-\tau \leq s \leq 0} \|\phi(s) - \phi(s)\| e^{-\varepsilon t}, \quad t \geq 0, \] where \( \varepsilon \) is called the estimated rate index of exponential synchronization, \( \phi(s) \) is the initial condition of system (3) and \( \phi(s) \) is the initial condition of system (4).

**Lemma 2.** Under assumption (H 1), we have
\[ \|\phi(t) - \phi(t)\| \leq A_0(t)\|\phi(t) - \phi(t)\|, \]
where \( A_0 = \max\{a_{y,j}, |\pi_{y,j}|\} \), \( B_0 = \max\{a_{y,j}, |\pi_{y,j}|\} \), \( a_{y,j} = \min\{a_{y,j} - a_{y,j}\} \), \( \pi_{y,j} = \max\{a_{y,j} - a_{y,j}\} \), \( B_0 = \max\{b_{y,j}, b_{y,j}\} \), \( \pi_{y,j} = \max\{b_{y,j} - b_{y,j}\} \), \( B_0 = \max\{b_{y,j}, b_{y,j}\} \), \( g(\varepsilon(t)) = g(y(t)) \).}

**Theorem 1.** Suppose that the assumptions (H 1) and (H 2) hold. If there exist symmetric positive definite matrices \( P_i, Q_i, \) \( i = 1, 2, 3, 4 \), three positive diagonal matrices \( E, F, G \), and matrices \( M, R, N, W_1, W_2 \), such that the following LMI holds:
\[ A = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} & PB & A_{16} \\ * & A_{22} & 0 & 0 & LG & A_{26} \\ * & * & -Q_2 & 0 & 0 & \epsilon \\ * & * & -Q_4 & 0 & 0 & \epsilon \\ * & * & * & A_{55} & 0 & 0 \\ * & * & * & * & A_{66} & 0 \end{bmatrix} < 0, \]
where \( A_{11} = -2P + W_1 + W_1^T + Q_1 + Q_2 + M + M^T, \quad A_{12} = W_2 + N - M, \quad A_{14} = PA + LE + LF, \quad A_{22} = -(1 - \mu)Q_1 - N - N^T, \quad A_{26} = -R - R^T, \quad A_{44} = Q_3 + \epsilon Q_4 - 2\epsilon - 2F, \quad A_{55} = -(1 - \mu)Q_3 - 2G, \quad A_{66} = -R - R^T, \) and the gain matrices of control law (6) are
\[ K_i = P_i \omega, \quad i = 1, 2. \]

Then the drive system (3) can globally exponentially synchronize with the response system (4) via the control input (6), the exponential synchronization rate index is \( \rho / 2 \).

**Proof.** Consider the following Lyapunov–Krasovskii functional candidate for system (5) as
\[ V(t) = e^T(t)\Phi(t) e^T(t) + \int_{t-\tau}^t e^T(s)Q_1 e^T(s) ds + \int_t^{\infty} e^T(s)Q_2 e^T(s) ds \]
\[ + \int_{t-\tau}^t g^T(s)Q_3 g(s) ds + \int_0^{\infty} g^T(s)Q_4 g(s) ds \]
\[ + \int_0^{\infty} g^T(s)Q_5 g(s) ds, \]
\[ \tau \leq t \leq \infty. \]

Calculating the upper right derivation of \( V(t) \) along the trajectories of the error system (5), and combining with Lemma 2, we have
\[ D^+ V(t) = 2e^T(t)\Phi(t) e^T(t) + e^T(t)Q_1 e^T(t) - (1 - \rho)(e^T(t)Q_2 e^T(t) - e^T(t-r)Q_2 e^T(t-r)) \]
\[ + g^T(t)Q_3 g(t) - (1 - \rho)^2 g^T(t)Q_4 g(t) \]
\[ + g^T(t)Q_5 g(t) - (1 - \rho)^3 g^T(t)Q_5 g(t) \]
\[ + \int_0^{\infty} g^T(s)Q_3 Q_4 g(s) ds \]
\[ \leq e^T(t)\Phi(t) e^T(t) \]
\[ + \int_{t-\tau}^t g^T(s)Q_3 Q_4 g(s) ds \]
\[ \leq e^T(t)\Phi(t) e^T(t) + \int_{t-\tau}^t g^T(s)Q_3 Q_4 g(s) ds. \]

From (H 1), it can be deduced that for any positive diagonal matrices \( E, F, G \), we have
\[ 0 \leq 2g^T(e(t))Ee(t) - 2g^T(e(t))Eg(e(t)), \]
\[ 0 \leq 2g^T(e(t))Ee(t) - 2g^T(e(t))Eg(e(t)), \]
\[ 0 \leq 2g^T(e(t))Ee(t) - 2g^T(e(t))Eg(e(t)). \]
0 \leq 2gT(t)F(Lc(t) - g(c(t))). \tag{13}

0 \leq 2gT(t)(c(t) - c(t - \tau))(L^T \dot{c}(t) - \dot{c}(t - \tau))). \tag{14}

From the Newton–Leibniz formulation \(c(t) - c(t - \tau) - \int_{t-\tau}^{t} \dot{c}(s) \, ds = 0\), the following equation is true for any matrices \(M, N, R\) with appropriate dimensions

\[
2 \left( c(t) - c(t - \tau) - \int_{t-\tau}^{t} \dot{c}(s) \, ds \right)^T \\
\times \left( M c(t) + N \left( c(t) - c(t - \tau) \right) + R \int_{t-\tau}^{t} \dot{c}(s) \, ds \right) = 0. \tag{15}
\]

It follows from (11)–(15) that

\[
D^+ V(t) \leq c^T(t) \left( -2P + 2P \dot{K}_1 + Q_1 + Q_2 + 2M \sigma(t) + 2c^T(t) P K_2 \right) \\
+ N - M^T \sigma(t) - \int_{t-\tau}^{t} c^T(s) \, ds - c^T(t - \tau) \right) \\
- 2c^T(t - \tau) \left( M^T g(t) \right) \int_{t-\tau}^{t} \dot{c}(s) \, ds - c^T(t - \tau) \right) \\
- 2c^T(t - \tau) \dot{c}(t) \left( -2M^T g(t) \right) \int_{t-\tau}^{t} \dot{c}(s) \, ds - c^T(t - \tau) \right) \\
- 2c^T(t - \tau) \dot{c}(t) \left( -2M^T g(t) \right) \int_{t-\tau}^{t} \dot{c}(s) \, ds - c^T(t - \tau) \right)
\]

where \(\delta(t) = \left( c^T(t) \left( c^T(t - \tau) \right)^T \right) \left( \int_{t-\tau}^{t} \dot{c}(s) \, ds \right)^T \).

\[
\begin{bmatrix}
Y_{11} & Y_{12} & 0 & Y_{14} & PB & Y_{16} \\
Y_{22} & 0 & 0 & 0 & Y_{26} \\
* & * & -Q_2 & 0 & 0 \\
* & * & * & Y_{44} & 0 \\
* & * & * & * & Y_{55} \\
* & * & * & * & Y_{66}
\end{bmatrix}
\]

and \(Y_{11} = -2P + 2P \dot{K}_1 + Q_1 + Q_2 + M + M^T, Y_{12} = PK_2 + N - M^T, Y_{14} = PA + LE + LF, Y_{16} = R - M^T, Y_{22} = -(1 - \mu)Q_1, N - N^T, Y_{26} = -R - N^T, Y_{44} = Q_3 + rQ_4 - 2E - 2F, Y_{55} = -(1 - \mu)Q_3 - 2G, Y_{66} = -(1 - \mu)Q_3 - 2G^T\).

By setting \(K_i = P^{-1} W_i, i = 1, 2\), we know that \(Y = \Lambda\). It follows from (10) and (16) that

\[
D^+ V(t) \leq \lambda_{\max}(\Lambda) \left\| e(t) \right\|^2 - \lambda_{\max}(L^T Q_L) \int_{t-\tau}^{t} \left\| \dot{e}(s) \right\|^2 \, ds. \tag{17}
\]

On the other hand, it is easy to check that for all \(t \geq 0\)

\[
\rho_{\sigma \lambda}(P) \left\| e(t) \right\|^2 \leq V(t) \leq \lambda_{\max}(P) \left\| e(t) \right\|^2 + \omega \int_{t-\tau}^{t} \left\| \dot{e}(s) \right\|^2 \, ds, \tag{18}
\]

where \(\omega = \lambda_{\max}(Q_1 + Q_2 + L^T Q_L) + r_{\sigma \theta \max}(L^T Q_L)\).

Choose a sufficiently small positive constant \(\rho > 0\) such that

\[
\rho_{\omega \lambda}(P) + \lambda_{\max}(\Lambda) < 0, \tag{19}
\]

\[
\rho_{\omega \lambda}(L^T Q_L) < 0,
\]

hence, from (17) and (18), we have

\[
D^+ V(t) + \rho V(t) \leq 0, \quad t \geq 0.
\]

Then

\[
V(t) \leq V(0) e^{-\rho t}. \tag{20}
\]

Then we have

\[
\| y(t) - x(t) \| \leq M \sup_{r \leq \tau \leq 0} \| \varphi(s) - \varphi(\delta(s)) \| e^{-\rho/2} \leq 0, \quad t \geq 0.
\]

where \(M = (\lambda_{\max}(P) + \omega) \lambda_{\max}(P)^{1/2}\).

Therefore, the drive system (3) can globally exponentially synchronize with the response system (4) via the control input (6), and the exponential synchronization rate index is \(\rho/2\). The proof is completed. \(\square\)

In the case of the time-varying delay \(\epsilon(t)\) being constant, we have the following result.

**Corollary 1.** Suppose that the assumptions (H 1) and (H 2) hold. If there exist symmetric positive definite matrices \(P, Q_i, i = 1, 2, 3, 4\), three positive diagonal matrices \(E, F, G\), and matrices \(M, N, R, W_1, W_2\), such that the following LMI holds:

\[
\Lambda = \begin{bmatrix}
\Delta_{11} & \Delta_{12} & 0 & \Delta_{14} & PB & \Delta_{18} \\
\Delta_{22} & 0 & 0 & LG & \Delta_{26} \\
* & * & -Q_2 & 0 & 0 \\
* & * & * & Y_{44} & 0 \\
* & * & * & * & Y_{55} \\
* & * & * & * & Y_{66}
\end{bmatrix} < 0, \tag{21}
\]

where \(\Delta_{11} = -2P + 1W_1 + W_2 + Q_1 + Q_3 + M + M^T, \Delta_{12} = 2W_2 - N - M^T, \Delta_{14} = PA + LE + LF, \Delta_{16} = R - M^T, \Delta_{22} = -Q_3 - N^T, \Delta_{26} = -R - N^T, \Delta_{44} = Q_3 + rQ_4 - 2E - 2F, \Delta_{55} = -Q_3 - 2G, \Delta_{66} = -(1 - \mu)Q_3 - 2G^T, \Delta_{15} = Q_2 + rQ_4 - 2E - 2F, \Delta_{17} = \rho_{\omega \lambda}(L^T Q_L) < 0, \quad \text{and the gain matrices of control law (6) are}

\[
K_i = P^{-1} W_i, \quad i = 1, 2.
\]

Then the drive system (3) can globally exponentially synchronize with the response system (4) via the control input (6), the exponential synchronization rate index is \(\rho/2\).

**Remark 1.** Synchronization problem of MNNs has been studied in [8-10,12-14,16]. The delays used are differentiable functions or even constants. In our paper, the derivatives of the time-varying delays do not need to be small than one compared with the assumptions in [9-15]. Moreover, by using Lemma 2 and the transform scaling, our results are true for all the subsystems of (5).

**Theorem 2.** Suppose that the assumptions (H 1) and (H 2) hold. If there exist symmetric positive definite matrices \(P, Q_i, i = 1, 2, 3, 4\), three positive diagonal matrices \(E, F, G\), an inverse matrix \(Q\), and matrices \(M, N, R, W_1, W_2\), such that the following LMI holds:

\[
\Delta = \begin{bmatrix}
\Delta_{11} & \Delta_{12} & \Delta_{13} & 0 & \Delta_{15} & QB & \Delta_{17} \\
\Delta_{22} & W_2 & W_2 & 0 & 0 & LG & \Delta_{27} \\
* & * & * & 0 & QA & QB & 0 \\
* & * & * & * & \Delta_{33} & 0 & 0 \\
* & * & * & * & * & \Delta_{55} & 0 \\
* & * & * & * & * & \Delta_{66} & 0 \\
* & * & * & * & \Delta_{44} & \Delta_{77}
\end{bmatrix} < 0, \tag{22}
\]

where \(\Delta_{11} = -Q - Q^T + W_1 + W_2 + Q_1 + Q_3 + M + M^T, \Delta_{12} = W_2 - N - M^T, \Delta_{13} = P - Q - Q^T + W_2, \Delta_{15} = Q^T + LE + LF, \Delta_{17} = -Q - M^T, \Delta_{19} = -R - N^T, \Delta_{22} = -(1 - \mu)Q_3 - N^T, \Delta_{27} = -(1 - \mu)Q_3 - 2G, \Delta_{33} = -R - N^T, \Delta_{44} = Q_3 + rQ_4 - 2E - 2F, \Delta_{55} = -Q_3 - 2G, \Delta_{66} = -(1 - \mu)Q_3 - 2G^T, \Delta_{77} = -R - R^T, \quad \text{and the gain matrices of control law (6) are}

\[
K_i = Q^{-1} W_i, \quad i = 1, 2.
\]
Then the drive system (3) can globally exponentially synchronize with the response system (4) via the control input (5), the exponential synchronization rate index is $\rho/2$.

**Proof.** Consider the following Lyapunov–Krasovskii functional candidate for system (5) as

$$\nabla V(t) = e^T(t)P(t)e(t) + \int_{t-\tau(t)}^{t} e^T(s)Q_1e(s)ds + \int_{t-\tau(t)}^{t} e^T(s)Q_2e(s)ds \\
+ \int_{t-\tau(t)}^{t} g^T(\xi(s))Q_3g(\xi(s))ds \\
+ \int_{t-\tau(t)}^{t} g^T(\xi(s))Q_4g(\xi(s))ds \, dt.$$

Calculating the upper right derivation of $V(t)$ along the trajectories of the error system (5), and combining with Lemma 2, we have

$$D^+ \nabla V(t) \leq 2e^T(t)P(t)e(t) + 2e^T(t)Q_1e(t) + 2e^T(t)Q_2e(t) + 2e^T(t)Q_3e(t) + 2e^T(t)Q_4e(t)$$

$$+ \int_{t-\tau(t)}^{t} e^T(s)Q_1e(s)ds + \int_{t-\tau(t)}^{t} e^T(s)Q_2e(s)ds + \int_{t-\tau(t)}^{t} g^T(\xi(s))Q_3g(\xi(s))ds + \int_{t-\tau(t)}^{t} g^T(\xi(s))Q_4g(\xi(s))ds \leq e^T(t)(-2Q_1(t-K_1)e(t) + Q_2e(t) + 2e^T(t)Q_3e(t) + 2e^T(t)Q_4e(t))$$

$$+ \int_{t-\tau(t)}^{t} e^T(s)Q_1e(s)ds + \int_{t-\tau(t)}^{t} g^T(\xi(s))Q_3g(\xi(s))ds + \int_{t-\tau(t)}^{t} g^T(\xi(s))Q_4g(\xi(s))ds.$$

From (H1), it can be deduced that for any positive diagonal matrices $E, F, G$, we have

$$0 \leq 2g^T(\xi(t))E(\xi(t)) + 2g^T(\xi(t))E(\xi(t)).$$

$$0 \leq 2g^T(\xi(t))F(\xi(t)) - g(\xi(t)),$$

and $\text{det}(E) > 0, \text{det}(F) > 0$. From the Newton–Leibniz formulation $e(t) - e(t - \tau) - \int_{t-\tau(t)}^{t} e(s)ds = 0$, the following equation is true for any matrices $M, N, R$ with appropriate dimensions

$$2 \left( e(t) - e(t - \tau) - \int_{t-\tau(t)}^{t} e(s)ds \right)^T$$

$$\times \left( M(t) + N(t) - \int_{t-\tau(t)}^{t} e(s)ds \right) = 0.$$ (27)

It follows from (23)–(27) that

$$D^+ \nabla V(t) \leq e^T(t)(-2Q_1(t-K_1)e(t) + Q_2e(t) + 2e^T(t)Q_3e(t) + 2e^T(t)Q_4e(t))$$

$$+ \int_{t-\tau(t)}^{t} e^T(s)Q_1e(s)ds + \int_{t-\tau(t)}^{t} g^T(\xi(s))Q_3g(\xi(s))ds + \int_{t-\tau(t)}^{t} g^T(\xi(s))Q_4g(\xi(s))ds \leq e^T(t)Qe(t) + g^T(\xi(t))Qg(\xi(t))$$

On the other hand, it is easy to check that for all $t \geq 0$

$$\lambda_{\text{min}}(P)e(t)^2 \leq V(t) \leq \lambda_{\text{max}}(P)e(t)^2 + \omega \int_{t-\tau(t)}^{t} \|\xi(s)\|^2 ds,$$ (30)

where $\omega = \lambda_{\text{max}}(P_{11} + P_{22} + 2P_{33}) + \rho_{\text{max}}(L^TQ_4L)$. Choose a sufficiently small positive constant $\rho$ so that $\rho \lambda_{\text{max}}(P) + \lambda_{\text{max}}(Q) < 0$, $\rho_{\text{max}}(L^TQ_4L) < 0$.

**Remark 2.** In Theorems 1 and 2, some approaches are given to choose the estimation gain matrices $K_1$ and $K_2$, which can be helpful for the design of controllers to let the drive system synchronize with the corresponding response one. In addition, the criteria for the exponential synchronization are presented in

$$\text{Then the drive system (3) can globally exponentially synchronize with the response system (4) via the control input (5), the exponential synchronization rate index is } \rho/2.$$
the forms of LMI s, which can be more conveniently verified than those in the literature [12,28,33].

**Corollary 2.** Suppose that the assumptions (H 1) and (H 2) hold. If there exist symmetric positive definite matrices \( P, Q_i \), \( i = 1, 2, 3, 4 \), three positive diagonal matrices \( E, F, G \), an inverse matrix \( Q \), and matrices \( M, N, R, W_1, W_2 \) such that the following LMI holds:

\[
\Delta = \begin{bmatrix}
\Delta_{11} & \Delta_{12} & 0 & \Delta_{15} & \Delta_{16} & \Delta_{17} \\
* & \Delta_{22} & \Delta_{23} & 0 & \Delta_{25} & \Delta_{26} \\
* & * & -Q_1 & 0 & 0 & 0 \\
* & * & * & -Q_2 & 0 & 0 \\
* & * & * & * & -Q_3 & 0 \\
* & * & * & * & * & -Q_4 \\
\end{bmatrix} < 0, \tag{32}
\]

where \( \Delta_{11} = \begin{bmatrix} -Q - Q^T + W_1 + W_1^T + Q_1 + Q_2 + M + M^T \\ P - Q - Q^T + W_1^T \\ Q_5 = QA + LE + LF \\ R - M^T \\ Q_3 + \rho Q - 2E - 2F \\ -Q_3 - 2G \\ -R - R^T \\
\end{bmatrix} \), and the gain matrices of control law (6) are

\[
K_i = Q^{-1}W_i, \quad i = 1, 2.
\]

Then the drive system (3) can globally exponentially synchronize with the response system (4) via the control input (6), the exponential synchronization rate index is \( \rho / 2 \).

**Remark 3.** In [9,12,13], the sufficient algebraic conditions on the synchronization of MNNS with time-varying delays were obtained with memoryless controller. In addition, the coupling matrix in [12,13] is required to be symmetrical. But in our paper, we propose a general controller (6) where the information of the size of \( \tau (t) \) is considered. The delay-dependent feedback controller (6) is necessary since delay is ubiquitous in practice. Moreover, the coupling matrices \( K_1 \) and \( K_2 \) do not need to be symmetrical, and the values of them can be easily solved by using the MATLAB tool boxes.

**Remark 4.** In Theorems 1 and 2, the designing laws in the synchronization of MNNS are proposed via state coupling (6). Also, synchronization via output coupling is not considered in most papers. However, this is important because in many real networks only output signals can be measured. For different coupling strategies, state and output coupling, different theoretical synchronization criteria will be derived. To take this into account, in the following we use the output coupling (7) to investigate the synchronization of MNNS.

**Theorem 3.** Suppose that the assumptions (H 1) and (H 2) hold. If there exist symmetric positive definite matrices \( P, Q_i \), \( i = 1, 2, 3, 4 \), three positive diagonal matrices \( E, F, G \), and matrices \( M, N, R, W_1, W_2 \), such that the following LMI holds:

\[
\Delta = \begin{bmatrix}
\Delta_{11} & \Delta_{12} & 0 & \Delta_{15} & \Delta_{16} \\
* & \Delta_{22} & \Delta_{23} & 0 & \Delta_{25} \\
* & * & -Q_1 & 0 & 0 \\
* & * & * & -Q_2 & 0 \\
* & * & * & * & -Q_3 \\
* & * & * & * & * \\
\end{bmatrix} < 0, \tag{33}
\]

where \( \Delta_{11} = \begin{bmatrix} -2P + Q_1 + Q_2 + M + M^T \\ N - M^T \\ PA + W_1 + LE + LF \\ P + Q_2 + M + M^T \\ -Q_1 - N^T \\ Q_4 + \rho Q_4 - 2E - 2F \\
\end{bmatrix} \), and the gain matrices of control law (7) are

\[
K_i = Q^{-1}W_i, \quad i = 1, 2.
\]

Then the drive system (3) can globally exponentially synchronize with the response system (4) via the control input (7), the exponential synchronization rate index is \( \rho / 2 \).

**Theorem 4.** Suppose that the assumptions (H 1) and (H 2) hold. If there exist symmetric positive definite matrices \( P, Q_i \), \( i = 1, 2, 3, 4 \), three positive diagonal matrices \( E, F, G \), an inverse matrix \( Q \), and matrices \( M, N, R, W_1, W_2 \), such that the following LMI holds:

\[
\Delta = \begin{bmatrix}
\Delta_{11} & \Delta_{12} & 0 & \Delta_{15} & \Delta_{16} & \Delta_{17} \\
* & \Delta_{22} & \Delta_{23} & 0 & \Delta_{25} & \Delta_{26} \\
* & * & -Q_1 & 0 & 0 & 0 \\
* & * & * & -Q_2 & 0 & 0 \\
* & * & * & * & -Q_3 & 0 \\
* & * & * & * & * & -Q_4 \\
\end{bmatrix} < 0, \tag{34}
\]

where \( \Delta_{11} = \begin{bmatrix} -Q - Q^T + Q_1 + Q_2 + M + M^T \\ N - M^T \\ P - Q - Q^T + W_1^T \\ Q_5 = QA + W_1 + LE + LF \\ Q_7 = Q_8 + W_2 \\ Q_17 = R - M^T \\
\end{bmatrix} \), \( \Delta_{22} = \begin{bmatrix} -Q_1 - N^T \\ Q_3 + \rho Q_4 - 2E - 2F \\
\end{bmatrix} \), and the gain matrices of control law (7) are

\[
K_i = Q^{-1}W_i, \quad i = 1, 2.
\]

Then the drive system (3) can globally exponentially synchronize with the response system (4) via the control input (7), the exponential synchronization rate index is \( \rho / 2 \).

4. **Numerical example**

In this section, one example is provided to verify the effectiveness of results obtained in the previous section.

**Example 1.** Consider the two-dimensional memristor-based neural network with time-varying delays

\[
x_i(t) = -x_i(t) + \sum_{j=1}^{2} a_{ij}(x_j(t))g_j(x_j(t)) + \sum_{j=1}^{2} b_{ij}(x_j(t - \tau_j(t)))g_j(x_i(t - \tau_i(t))), \quad t \geq 0, \quad i = 1, 2, \tag{35}
\]

where

\[
\begin{align*}
\Delta_{11}(x_1(t)) &= \begin{cases} -1.5, & x_1(t) < 0, \\
-2.5, & x_1(t) \geq 0, \end{cases} \\
\Delta_{12}(x_2(t)) &= \begin{cases} 3, & x_2(t) < 0, \\
2.5, & x_2(t) \geq 0, \end{cases} \\
\Delta_{21}(x_1(t)) &= \begin{cases} 1, & x_1(t) < 0, \\
0.8, & x_1(t) \geq 0, \end{cases} \\
\Delta_{22}(x_2(t)) &= \begin{cases} -2, & x_2(t) < 0, \\
-1.5, & x_2(t) \geq 0, \end{cases} \\
\end{align*}
\]

\[
\begin{align*}
b_{11}(x_1(t - \tau_1(t))) &= \begin{cases} -3, & x_1(t - \tau_1(t)) < 0, \\
-2.5, & x_1(t - \tau_1(t)) \geq 0, \end{cases} \\
b_{12}(x_2(t - \tau_2(t))) &= \begin{cases} 0.1, & x_2(t - \tau_2(t)) < 0, \\
0.08, & x_2(t - \tau_2(t)) \geq 0, \end{cases} \\
b_{21}(x_1(t - \tau_1(t))) &= \begin{cases} 0.1, & x_1(t - \tau_1(t)) < 0, \\
0.2, & x_1(t - \tau_1(t)) \geq 0, \end{cases} \\
b_{22}(x_2(t - \tau_2(t))) &= \begin{cases} -2.2, & x_2(t - \tau_2(t)) < 0, \\
-2.5, & x_2(t - \tau_2(t)) \geq 0, \end{cases} \\
\end{align*}
\]

with \( \tau_1(t) = t - 0.5 \cos(t), 0 \leq \tau_1(t) \leq 1.5, \tau_2(t) \leq 0.5 \) and take the activation function as \( g(x_i) = \tanh(x_i), i,j = 1, 2 \). Fig. 1 depicts the phase trajectories of system (35).

Consider (35) as the drive system. Then the corresponding response system is as follows:

\[
y_i(t) = -y_i(t) + \sum_{j=1}^{2} a_{ij}(y_j(t))g_j(y_j(t)) + \sum_{j=1}^{2} b_{ij}(y_j(t - \tau_j(t)))g_j(y_i(t - \tau_i(t)))
\]
Fig. 1. Phase plot of system (35) with the initial condition $x_1(0) = 0.2, x_2(0) = 0.6, \forall s \in [-1, 0)$.

Fig. 2. Choose 20 random initial conditions, the synchronization errors $e_1(t)$ and $e_2(t)$ with the controller $u(t) = K_1 x_1(t) + K_2 x_2(t) - r(t)$.

\[
\begin{align*}
x_t(t) &= \sum_{i} g_i(y_i(t - t_i(t))) + u(t), & t \geq 0, & i = 1, 2. \\
\end{align*}
\]

The other parameters are the same as in system (35). Define errors $e_i(t) = y_i(t) - x_i(t), \ i = 1, 2$. The controller is

\[
u(t) = K_{1x} x_1(t) + K_{2x} x_2(t) - r(t).
\]

Therefore, it follows from Theorem 1 that the drive system (35) can globally exponentially synchronize the response system (36), and the gain matrices of control law (37) are

\[
F = \begin{bmatrix} 54.7622 & 0 \\ 0 & 54.7776 \end{bmatrix}, \quad G = \begin{bmatrix} 38.9667 & 0 \\ 0 & 38.2303 \end{bmatrix}, \quad W_1 = \begin{bmatrix} -247.7910 & 604.3070 \\ -617.3596 & -237.3839 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 14.9358 & 7.6085 \\ 7.5011 & 16.4336 \end{bmatrix}, \quad M = 10^3 \begin{bmatrix} 0.0287 & 1.6611 \ -1.6608 & 0.0290 \end{bmatrix}, \quad N = 10^3 \begin{bmatrix} -0.0017 & -1.6609 \\ 1.6610 & -0.0018 \end{bmatrix}, \quad R = 10^3 \begin{bmatrix} 0.0287 & -1.6610 \\ 1.6610 & 0.0287 \end{bmatrix}.
\]

Choose 20 random initial conditions, Fig. 2 shows the trajectories of the synchronization errors between the drive system (35) and the response system (36) under the controller (37). We can see systems (35) and (36) achieve exponential synchronization with 20 random initial conditions. The state trajectories of

Fig. 3. State trajectories of variable $x_1(t)$ and $y_1(t)$ with the initial condition $x_1(0) = 0.2, y_1(0) = -1.2, \forall s \in [-1, 0)$.

Fig. 4. State trajectories of variable $x_2(t)$ and $y_2(t)$ with the initial condition $x_2(0) = 0.6, y_2(0) = 2.4, \forall s \in [-1, 0)$.
variables $x_1(t), y_1(t)$ with the initial condition $x_1(s) = 0.2, y_1(s) = -1.2, y_2(s) = 1$ are shown in Fig. 3. State trajectories of variable $x_2(t)$ and $y_2(t)$ with the initial condition $x_1(s) = 0.6, y_1(s) = 2.4, y_2(s) = -1$ are shown in Fig. 4.

5. Conclusions

In this paper, the synchronization control has been investigated for MNNS with time-varying delays. By constructing proper Lyapunov functional and employing inequality technique, a general controller is designed to achieve the exponential synchronization of the systems. The designing laws are proposed with state or output coupling which result in different theoretical synchronization criteria. The results of our paper are general and less conservative compared with some previous works. In addition, the coupling matrices $K_1$ and $K_2$ do not need be symmetrical, and the values of them can be easily solved by using the MATLAB tool boxes.

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