POLYNOMIAL NORMAL FORMS FOR VECTOR FIELDS ON $\mathbb{R}^3$

JIAZHONG YANG

Abstract

The present paper is devoted to studying a class of smoothly (C^∞) finitely determined vector fields on $\mathbb{R}^3$. Given any such generic local system of the form $\dot{x} = Ax + \cdots$, where A is a 3×3 matrix, we find the minimal possible number $i(A)$ such that the vector field is $i(A)$-jet determined, and we find the number $\mu(A)$ of moduli in the C^∞ classification. We also give a list of the simplest normal forms, that is, polynomials of degree $i(A)$ containing exactly $\mu(A)$ parameters.

1. Introduction and main results

Let A be a fixed $n \times n$ matrix. Denote by V_A the set of germs at the origin of smooth (C^∞) vector fields on $\mathbb{R}^n$ of the form

$$\dot{x} = Ax + \cdots \tag{1}$$

(the dots denote nonlinear terms). This paper is devoted to the following questions.

(1) Find $i(A)$—the *index of finite determinacy* of a *generic* vector field $X \in V_A$. By the index of finite determinacy, we mean the minimal number k such that X is k-determined, that is, C^∞-equivalent to any local vector field Y such that $j^k Y = j^k X$, where j^k is the k-jet at the origin.

(2) Find $\mu(A)$—the number of moduli distinguishing closed *generic* vector fields of V_A that are not C^∞-equivalent. By *closed vector fields of* V_A, we mean vector fields with $i(A)$-jet at the origin.

(3) Find the simplest normal form to which a *generic* vector field of V_A can be reduced by a smooth change of coordinates. By the *simplest normal form*, we mean a normal form that is a degree $i(A)$-polynomial and contains exactly $\mu(A)$ parameters.

As far as we know, complete answers to these questions are known only if $n = 1$ and $n = 2$ (see [7]). More exactly, Takens, in [7], studies vector fields on $\mathbb{R}$ (resp., $\mathbb{R}^2$) having a vanishing (resp., a pair of purely imaginary) eigenvalue(s). In this paper, we

DUKE MATHEMATICAL JOURNAL
Vol. 106, No. 1, © 2000
Received 13 July 1999. Revision received 13 January 2000.
2000 *Mathematics Subject Classification*. Primary 34K17; Secondary 37C15.

give a complete answer for the 3-dimensional case. Most of the results of this paper are corollaries of the results of [8], where vector fields with degenerated nonlinear parts are also considered.

The numbers $i(A)$ and $\mu(A)$ depend on the structure of resonant relations

$$\lambda_i = \alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n, \quad \alpha_1 + \cdots + \alpha_n \geq 2 \tag{2}$$

(α's are nonnegative integers) between the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix A. If there are no resonant relations, then by the Poincaré-Dulac-Chen theorem any vector field of the form (1) is C^∞-equivalent to its linear approximation $\dot{x} = Ax$ (see [1, Chap. 3]), and therefore $i(A) = 1$ and $\mu(A) = 0$. Such smoothly linearizable systems are nonresonant vector fields.

The simplest vector fields beyond the nonresonant ones are those that have exactly one resonant relation (2) (which is possible only if $n \geq 2$). In this case it is easy to see that vector field (1) is C^∞-equivalent to the resonant normal form $\dot{x}_i = \lambda_i x_i + \delta_i x_{i-1}$, for $i = 1, \ldots, n-1$, and $\dot{x}_n = \lambda_n x_n + a x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}$, where $\delta_i = 0, 1$ depending on the existence of Jordan blocks or not, $\lambda_n = \alpha_1\lambda_1 + \cdots + \alpha_{n-1}\lambda_{n-1}$. (Notice that in this case, necessarily, $\alpha_i = 0$ if λ_i has multiplicities.) Scaling the coordinate x_n generically, we can reduce a to 1; therefore, in this case, $\mu(A) = 0$, and $i(A)$ equals the order of the resonant relation. (The order of a resonant relation (2) is the number $\alpha_1 + \cdots + \alpha_n$.)

If there are more than one, yet still a finite number of resonant relations (2), then the resonant normal form for vector fields of the form (1) is a polynomial of degree $l(A)$, where $l(A)$ is the maximal order of resonant relations. In this case, the vector fields are necessarily hyperbolic (see [5]), and, by the Dulac-Chen theorem, any vector field (1) can be reduced to a resonant normal form by a smooth change of coordinates (see [1, Chap. 6]). Therefore, $i(A) \leq l(A)$, and $\mu(A)$ does not exceed the number of resonant relations. We note that in this case, which is possible only if $n \geq 3$, answers to the questions posed above are much less trivial, and already in the 3-dimensional case l can be arbitrarily big. (Take, e.g., a matrix with eigenvalues 1, 2, and N, where N is a big positive integer.) It turns out, however, that there exist cases where $\mu(A) = 0$ whereas l is arbitrarily big, as well as cases where $i(A)$ is much smaller than the maximal order of resonant relations.

When there are infinitely many resonant relations, the resonant normal form for vector fields (1) is no longer polynomial and the maximal order of resonant relations is equal to infinity. Nevertheless, in this case, because of the Belitskii theorem and the Ichikawa theorem (more details are given shortly), there are germs of smoothly finitely determined vector fields. Consequently, to know the corresponding numbers $i(A)$ and $\mu(A)$ of those germs becomes particularly interesting.

We study the above cases separately, formulating the respective results. First, we consider the case where vector fields admit finitely many resonant relations between their eigenvalues. We use the following definition.

Definition 1
Let p_1, p_2, and p_3, $p_1 \le p_2 \le p_3$, be nonnegative integers without common factor, $(p_1, p_2, p_3) \ne (1, 1, 1)$. We say that a 3×3 matrix A has a resonant type $p_1 : p_2 : p_3$ if there exists $\lambda \ne 0$ such that $(\lambda_1, \lambda_2, \lambda_3) = \lambda(p_1, p_2, p_3)$ up to enumeration of the eigenvalues.

THEOREM 1
Let A be a 3×3 matrix admitting finite and more than one resonant relations. Then it has resonant type $p_1 : p_2 : p_3$, where p_1 and p_2 are coprime.

(1) *If $p_1 = p_2 = 1$, then $i(A) = p_3$, $\mu(A) = 0$ for nondiagonalizable A and $\mu(A) = \max(0, p_3 - 3)$ for diagonalizable A.*
(2) *If $p_1 = 1$ and $p_2 > 1$, then $\mu(A) = 0$ and $i(A) = \max(p_2, p_3 - (p_2 - 1)[p_3/p_2])$, where $[a]$ is the integer part of a.*
(3) *If $p_1 > 1$, then $\mu(A) = \max(0, r - 2)$ and $i(A) = l$, where r is the number of resonant relations and l is the maximal order of the resonant relations.*

Example 1

(1) If the matrix A has resonant type $1 : 2 : 2m$, $m \ge 2$, then $\mu(A) = 0$ (though there are $m+2$ resonant relations) and $i(A) = m$ (though the maximal order of resonant relations is $2m$).
(2) If the matrix A has resonant type $1 : 1 : p$ with $p \le 3$, then $\mu(A) = 0$. (If $p \ge 4$, then the relation is also true provided that A is not diagonalizable.)
(3) If the matrix A has resonant type $3 : 5 : p$, then $\mu(A) = 0$ if and only if either $p \le 29$ or $p \in \{31, 32, 34, 37\}$.

The classification of generic vector fields on $\mathbb{R}^3$ in the case of finite number of resonant relations is as follows.

THEOREM 2
Let A be a 3×3 matrix admitting finite and more than one resonant relations. Then a generic vector field of the form $\dot{x} = Ax + \cdots$ is C^∞-equivalent to one of the following normal forms:

$$\dot{x}_1 = \lambda x_1, \qquad \dot{x}_2 = \lambda x_2 + x_1, \qquad \dot{x}_3 = p\lambda x_3 + x_2^p; \tag{3}$$

$$\dot{x}_1 = \lambda x_1, \qquad \dot{x}_2 = \lambda x_2, \qquad \dot{x}_3 = p\lambda x_3 + x_1^p \pm x_2^p + \sum_{i=2}^{p-2} \mu_i x_1^i x_2^{p-i}; \tag{4}$$

$$\dot{x}_1 = \lambda x_1, \qquad \dot{x}_2 = p_2\lambda x_2 + x_1^{p_2}, \qquad \dot{x}_3 = p_3\lambda x_3 + x_1^{p_3-p_2s} x_2^s, \quad s = \left[\frac{p_3}{p_2}\right]; \tag{5}$$

$$\dot{x}_1 = \lambda x_1, \qquad \dot{x}_2 = p\lambda x_2 + \delta x_3, \qquad \dot{x}_3 = p\lambda x_3 + x_1^p, \quad \delta \in \{0, 1\}; \tag{6}$$

$$\begin{gathered} \dot{x}_1 = p_1\lambda x_1, \qquad \dot{x}_2 = p_2\lambda x_2, \\ \dot{x}_3 = p_3\lambda x_3 + x_1^\alpha x_2^\beta + x_1^{\alpha+p_2} x_2^{\beta-p_1} + \sum_{i=2}^{[\beta/p_1]} \mu_i x_1^{\alpha+ip_2} x_2^{\beta-ip_1}, \end{gathered} \tag{7}$$

where μ's are moduli. The normal forms (3) and (4) hold when the linear approximation has the resonant type $1 : 1 : p$. (They correspond, resp., to the nondiagonalizable and diagonalizable linear approximations.) The normal forms (5) and (6) correspond to the resonant type $1 : p_2 : p_3$ ((5) and (6) correspond, resp., to $1 < p_2 < p_3$ and $1 < p_2 = p_3 = p$). Finally, the normal form (7) holds if the linear approximation has the resonant type $p_1 : p_2 : p_3$ with $p_1 \geq 2$ and $p_3 = \alpha p_1 + \beta p_2$, where α is the minimal number such that $p_3 - \alpha p_1$ is divisible by p_2.

Theorem 2 has the following corollary. For any fixed $\lambda \neq 0$, all vector fields having a linear part of (3) (resp., (5) and (6)), in the generic case, are smoothly equivalent to each other. A similar statement holds for vector fields of the form (7). More precisely, we show that vector fields having normal form (7) are smoothly equivalent to each other if and only if they have the same collections of moduli. (Two collections of moduli $\{\mu_1, \ldots, \mu_k\}$ and $\{\tilde{\mu}_1, \ldots, \tilde{\mu}_k\}$ are said to be the same if $\mu_j = \tilde{\mu}_j$ for $j = 1, \ldots, k$.) The corresponding statement for vector fields of the form (4), however, does not hold (e.g., two vector fields with distinct collections of moduli $\{\mu_2, \ldots, \mu_{p-2}\}$ and $\{\mu_{p-2}, \ldots, \mu_2\}$ are smoothly equivalent under the change of coordinates $(x_1, x_2, x_3) \to (x_2, x_1, x_3)$).

We now turn to the vector fields having infinite number of resonant relations (2). We use the following definition.

Definition 2
A tuple $(\lambda_1, \ldots, \lambda_n)$ is called 1-resonant if there is a relation

$$(k, \lambda) =: k_1\lambda_1 + \cdots + k_n\lambda_n = 0, \tag{8}$$

where $k_1, \ldots, k_n$ are nonnegative integers with $\sum k_i \neq 0$, and any other such relations $(\tilde{k}, \lambda) = 0$ can be derived from (8) by multiplying an integer l; that is, $\tilde{k}_j = lk_j$, $j = 1, \ldots, n$.

A tuple $(\lambda_1, \dots, \lambda_n)$ is called strongly 1-resonant if it is 1-resonant and any resonant relation (2) is a trivial derivation $\lambda_i = l(k, \lambda) + \lambda_i$ with some integer $l \geq 0$.

A tuple $(\lambda_1, \dots, \lambda_n)$ is called quasi-strongly 1-resonant if all the mutually distinct λ's form a strongly 1-resonant tuple.

The definition of 1-resonant (strongly, quasi-strongly, resp.) for a matrix and for a vector field can be given correspondingly.

Example 2
The tuple $(1, -1, \alpha)$ is strongly 1-resonant for any irrational α and is not 1-resonant for any rational α.

The tuple $(0, 1, p)$ is 1-resonant, yet not strongly 1-resonant, if p is a positive integer, since in this case there is an extra resonant relation $\lambda_3 = p\lambda_2$.

The tuple $(0, 1, 1, \alpha, \alpha)$ is quasi-strongly 1-resonant for any irrational α.

It is known by the Ichikawa theorem (see [5], [6]) that any vector field (1) with a fixed linear part admitting an infinite number of resonant relations is finitely determined in the formal context (on the level of formal series) if and only if it is 1-resonant and the nonlinear part does not belong to a certain set of infinite codimension. (This set depends on the linear approximation only.) By the Belitskii theorem (see [3]), such a vector field is finitely determined in the C^∞ context if and only if it is finitely determined in the formal context. Moreover, according to [3], two 1-resonant vector fields are smoothly equivalent if and only if they are formally equivalent. Therefore, the Ichikawa result holds in the C^∞ category, too. In other words, in the case of an infinite number of resonant relations, $i(A)$ is well defined if and only if the tuple of eigenvalues of A is 1-resonant. In what follows, we assume so.

Note that if $n = 1$ or $n = 2$, then any 1-resonant vector field is strongly 1-resonant. For classification of strongly 1-resonant vector fields, we refer the reader to [10] and [3, Chap. 4]. Starting from dimension 3, there are quasi-strongly 1-resonant vector fields as well as those 1-resonant systems that are neither strongly nor quasi-strongly 1-resonant. The classification of such vector fields is more complicated (see [8] and [9]).

We notice that the simplest example of a nonstrongly 1-resonant tuple is $(0, \lambda, p\lambda)$, where $\lambda \neq 0$ and p is a positive integer. In fact, in a 3-dimensional case, it is easy to show that the form $(0, \lambda, p\lambda)$ is unique.

LEMMA 1
Any 1*-resonant vector field on* $\mathbb{R}^3$ *is either strongly* 1*-resonant or has the resonant type* $0 : 1 : p$.

We leave to the reader the proof of this statement. Now we state the main result concerning *all* 1-resonant vector fields on $\mathbb{R}^3$.

THEOREM 3

Let X be any 1-resonant vector field having the resonant type $0 : 1 : p$. Then $i(A) = \max(3, p)$ and $\mu(A) = 3$.

THEOREM 4

A generic vector field on R^3 with a fixed linear approximation of the resonant type $0 : 1 : p$ is C^∞-equivalent to one of the following normal forms:

$$\dot{x}_1 = x_1^2 + \mu_1 x_1^3, \qquad \dot{x}_2 = \lambda x_2 + \mu_2 x_1 x_2, \qquad \dot{x}_3 = p\lambda x_3 + \mu_3 x_1 x_3 + x_2^p; \tag{9}$$

$$\dot{x}_1 = x_1^2 + \mu_1 x_1^3, \qquad \dot{x}_2 = \lambda x_2 + x_3, \qquad \dot{x}_3 = \lambda x_3 + \mu_2 x_1 x_2 + \mu_3 x_1 x_3; \tag{10}$$

$$\dot{x}_1 = x_1^2 + \mu_1 x_1^3, \qquad \dot{x}_2 = \lambda x_2 + \mu_2 x_1 x_2, \qquad \dot{x}_3 = \lambda x_3 + \mu_3 x_1 x_3; \tag{11}$$

$$\begin{gathered} \dot{x}_1 = x_1^2 + \mu_1 x_1^3, \qquad \dot{x}_2 = \lambda x_2 + \mu_2 x_1 x_2 + \mu_3 x_1 x_3, \\ \dot{x}_3 = \lambda x_3 - \mu_3 x_1 x_2 + \mu_2 x_1 x_3. \end{gathered} \tag{12}$$

The normal form (9) holds if $p > 1$, (10) holds if $p = 1$ and A is not diagonalizable, and (11) and (12) hold if $p = 1$ and A is diagonalizable.

Like the previous case, Theorem 4 has a corollary. For any fixed $\lambda \neq 0$, all vector fields having a linear part of (9) (resp., (10)), in the generic case, are smoothly equivalent to each other if and only if they have the same collections of moduli. The corresponding conclusion for vector fields of the form (11) (resp., (12)) is as follows. Two such vector fields are smoothly equivalent if and only if either the collections of moduli are identical or they have the forms (μ_1, μ_2, μ_3) and (μ_1, μ_3, μ_2) (resp., (μ_1, μ_2, μ_3) and $(\mu_1, \mu_2, -\mu_3)$). This is because of the enumeration of x_2 and x_3 in the diagonalizable case.

2. Preliminaries

In this section, we recall the basic techniques of normalization of vector fields. For a detailed explanation, please refer to [7] (see also [4, Chap. 3], [8]).

Given a vector field, the starting point of normalization is its resonant normal form, which means that the linear part has been put into the Jordan normal form and the nonlinear part contains the resonant terms only. By the results of [4, Chap. 3], to further simplify a vector field in resonant normal form, it suffices to perform the so-called resonant transformations, changes of coordinates with an identical linear part and resonant nonlinear parts. More exactly, the following statement holds.

PROPOSITION 1

Given a vector field X, any resonant transformation brings one resonant normal form of X to another resonant normal form; any transformation having an identical linear part and bringing one resonant normal form of X to another resonant normal form must be a resonant transformation.

This proposition implies that to normalize a vector field with a finite number of resonant relations one can perform polynomial changes of coordinates. We do so in the proof of Theorems 1 and 2. When a vector field admits an infinite number of resonant relations, we find that it is more convenient to normalize it *jet-by-jet*. Denote by $j^l X$ the l-jet of an X at zero, and call two vector fields X and $\tilde{X}$ l-jet equivalent if there is a diffeomorphism Φ such that $j^l \Phi_* X = j^l \tilde{X}$.

LEMMA 2

(i) *(See [7].) Let X and $\tilde{X}$ be vector fields such that $j^k X = j^k \tilde{X}$. If the equation $j^{k+1}[X, \varphi] = j^{k+1}(\tilde{X} - X)$, where $[X, \varphi]$ denotes the Lie bracket of X and φ, has a solution φ with $j^1 \varphi = 0$, then X and $\tilde{X}$ are $(k+1)$-jet equivalent.*

(ii) *(See [2].) If two vector fields X and $\tilde{X}$ are l-jet equivalent for any $l < \infty$, then they are formally equivalent.*

The following lemmas are useful to prove our results.

LEMMA 3

If for any vector field Y, $j^k Y = 0$, the equation $[X, \varphi] = Y$ has a solution φ with $j^1 \varphi = 0$, then X and $j^k X$ are formally equivalent.

Proof

Because of the second statement of Lemma 2, to prove Lemma 3 it suffices to prove that X and $j^k X$ are l-jet equivalent for any $l < \infty$. Moreover, it is obvious that we need only consider the case $l > k$. Let $\tilde{X} = j^k X$, $Y = \tilde{X} - X$. Then $j^k Y = 0$. If the equation $[X, \varphi] = Y$ has a solution φ, $j^1 \varphi = 0$, then, according to Lemma 2, X and $\tilde{X}$ are $(k+1)$-jet equivalent. Assume that we have proved that X and $\tilde{X}$ are l-jet equivalent for $l \geq k+1$; that is, there exists a diffeomorphism Φ such that $j^l \Phi_* X = j^k \tilde{X}$. In what follows, we show that X and $\tilde{X}$ are $(l+1)$-jet equivalent. To this end, denote $\bar{X} = j^l X$. Then $j^l(\bar{X} - X) = 0$ and, as assumed, the equation $[X, \varphi] = \bar{X} - X$ has a solution φ with a vanishing 1-jet. Therefore, the equation $j^{l+1}[X, \varphi] = j^{l+1}(\bar{X} - X)$ is solvable. Again, by Lemma 2, there exists a diffeomorphism $\bar{\Phi}$ such that $j^{l+1} \bar{\Phi}_* X = j^{l+1} \bar{X}$. It follows from $j^{l+1} \bar{X} = j^l X$ that X and $j^l X$ are $(l+1)$-jet equivalent, noticing that $j^{l+1} \Phi_* \bar{\Phi}_* X = j^k X$. □

From Lemma 3, one sees that to prove that a vector field X is formally k-jet determined it suffices to prove the solvability of φ, $j^1\varphi = 0$, in the equation $[X, \varphi] = Y$, where Y is any vector field such that $j^kY = 0$. In fact, it suffices to prove the solvability of this equation where Y is any formal *resonant* (with respect to the tuple of eigenvalues of X) vector field with a vanishing k-jet. In other words, we have the following lemma.

LEMMA 4

Let X be a smooth vector field $X(0) = 0$. If for any formal resonant (with respect to the tuple of eigenvalues of X) vector field Y, $j^kY = 0$, the equation

$$\left[j^kX, \varphi\right] = Y \tag{13}$$

has a formal solution φ such that $j^1\varphi = 0$ and X is smoothly k-jet determined.

Proof

We need only to show that X is formally k-jet determined, recalling the Belitskii theorem about the coincidence of smooth finite determinacy and formal finite determinacy. Because of Lemma 3, it suffices to show that under the assumption of the lemma, the equation $[j^kX, \varphi] = Z$, where $j^kZ = 0$, always has a vanishing 1-jet solution.

Since for any homogeneous vector field H there is a φ, $j^1\varphi = 0$, such that $[j^1X, \varphi] - H$ contains resonant monomials only, by jet-by-jet, one can see that for any vector field Z, $j^kZ = 0$, there is a ϕ, $j^1\phi = 0$, such that $Y =: [j^kX, \phi] - Z$ is a resonant vector field. Now the lemma follows from the assumption that for any resonant vector field Y with $j^kY = 0$ the equation $[j^kX, \varphi] = Y$ has a vanishing 1-jet solution. □

Equation (13) is called a homological equation.

3. Proof of the results

3.1. Proof of Theorems 1 and 2

Since Theorems 1 and 2 deal with vector fields having a finite number of resonant relations, by Proposition 1, the normalization of such vector fields can be fulfilled under polynomial resonant transformations. Using this observation, we prove them in an explicit way, specifying the changes of coordinates.

Let X be any vector field that has a resonant type $p_1 : p_2 : p_3$, where $0 < p_1 \leq p_2 \leq p_3$. Then we have three cases: (1) resonant type $1 : 1 : p$, $p > 1$; (2) resonant type $1 : p_2 : p_3$, $p_2 > 1$; and (3) resonant type $p_1 : p_2 : p_3$, $p_1 > 1$.

Case 1: Resonant type $1:1:p,\ p>1$

Vector fields with resonant type $1:1:p$ admit $(p+1)$-resonant relations, and the resonant normal form is

$$\begin{gathered}\dot{x}_1=\lambda x_1,\qquad \dot{x}_2=\lambda x_2+\delta x_1,\\ \dot{x}_3=p\lambda x_3+a_1x_1^p+a_2x_1^{p-1}x_2+\cdots+a_{p+1}x_2^p,\end{gathered}\tag{14}$$

where $\delta=0$ or 1.

(i) $\delta=1$. If $\delta=1$ in (14), then the change of coordinates

$$\bigl(y_1,y_2,y_3\bigr)=\bigl(x_1,x_2,x_3+\alpha_1x_1^p+\alpha_2x_1^{p-1}x_2+\cdots+\alpha_{p+1}x_2^p\bigr)\tag{15}$$

brings (14) to a vector field of the form $\dot{y}_1=\lambda y_1$, $\dot{y}_2=\lambda y_2+y_1$, and

$$\dot{y}_3=p\lambda y_3+(a_1+\alpha_2)y_1^p+(a_2+2\alpha_3)y_1^{p-1}y_2+\cdots+\bigl(a_p+p\alpha_{p+1}\bigr)y_1y_2^{p-1}+a_{p+1}y_2^p.$$

It follows that one can always choose suitable α_i such that all the resonant terms except $a_{p+1}y_2^p\partial/\partial y_3$ can be eliminated. In the generic case, a_{p+1} is different from zero and thus can be scaled to 1. Therefore, we obtain normal form (3), which contains no moduli; that is, $\mu(A)=0$.

Change of coordinates (15) exhausts all the possible resonant transformations. Therefore, because of Proposition 1, normal form (3) is the simplest; that is, the term $y_2^p\partial/\partial y_3$ is not removable.

One can also prove the minimality of the normal form in a geometric way. Vector field (3) is not smoothly equivalent to its linearized system because it has exactly one C^∞-invariant surface (in coordinates of (3) it is given by $x_1=0$), whereas the linearized system has an infinite number of C^∞-invariant surfaces. (They have the form $x_1=0$, $x_3=\alpha x_1^p$, where $\alpha\in\mathbb{R}$.) One can find all the possible invariant surfaces of (3) in the following way. Consider the possible solutions of equation $x_1=f(x_2,x_3)$, where f is an unknown formal series. By expanding f in terms of its variables, one can prove formally that the equation has only one trivial solution $x_1=0$ along the vector field, which means that the vector field has only one invariant surface of the form $x_1=0$. Similarly, one can show that the equations $x_2=f(x_1,x_3)$ and $x_3=f(x_1,x_2)$ have no solution. Therefore, there are no invariant surfaces of such form. Using the same arguments, one can discuss the existence of invariant surfaces of the linearized system.

(ii) $\delta=0$. If $\delta=0$ in (14), then any resonant transformation keeps (14) unchanged. To further simplify (14), however, one can turn to linear changes of coordinates, observing the relation $\lambda_1=\lambda_2$. In fact, a linear change of coordinates $x_1=\alpha_1y_1+\alpha_2y_2$, $x_2=\alpha_3y_1+\alpha_4y_2$, $x_3=\pm y_3$, $\alpha_1\alpha_4-\alpha_2\alpha_3\neq 0$, keeps $\dot{y}_1=\lambda y_1$ and $\dot{y}_2=\lambda y_2$ and transfers $\dot{x}_3$ to $\dot{y}_3=p\lambda y_3+b_1y_1^p+b_2y_1^{p-1}y_2+\cdots+b_{p+1}y_2^p$, where b_i are functions of $\alpha_1,\ldots,\alpha_4$.

The problem of simplifying the vector field now becomes the normalization of the function $f := b_1 y_1^p + b_2 y_1^{p-1} y_2 + \cdots + b_{p+1} y_2^p$, that is, the maximal elimination and scalings of b_i within all possible $\alpha_1, \ldots, \alpha_4$. It is easy to see that, in the generic case, if $p = 2$, then f can be normalized to $y_1^2 \pm y_2^2$. If $p = 3$, then one can reduce f to $y_1^3 + y_2^3$. For $p > 3$, one can scale b_1 to 1, b_{p+1} to ± 1 (or to 1 if p is odd), and eliminate two more terms, b_2 and b_p. Therefore, one obtains the normal form (4), which contains $p - 3$ parameters.

Case 2: Resonant type $1 : p_2 : p_3,\ p_2 > 1$
In this case, there are also two subcases: (i) $p_3 > p_2$ and (ii) $p_3 = p_2$.

(i) $p_3 > p_2$. The resonant normal form in this case is given by

$$\dot{x}_1 = \lambda x_1, \qquad \dot{x}_2 = p_2 \lambda x_2 + a_0 x_1^{p_2}, \qquad \dot{x}_3 = p_3 \lambda x_3 + P(x_1, x_2), \tag{16}$$

where $P(x_1, x_2)$ is a polynomial containing resonant terms only; that is,

$$P(x_1, x_2) = a_1 x_1^{p_3 - p_2 s} x_2^s + a_2 x_1^{p_3 - p_2(s-1)} x_2^{s-1} + \cdots + a_{s+1} x_1^{p_3},$$

where $s = [p_3/p_2]$. The lowest order of P is $p_3 - (p_2 - 1)s$.

If $a_0 \neq 0$ in (16), then the transformation $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3 - a_2/(s a_0) x_1^{p_3 - p_2 s} x_2^s - a_3/((s-1)a_0) x_1^{p_3 - p_2(s-1)} x_2^{s-1} - \cdots - a_{s+1}/a_0 x_1^{p_3 - p_2} x_2$ brings (16) to

$$\dot{y}_1 = \lambda y_1, \qquad \dot{y}_2 = p_2 \lambda y_2 + a_0 y_1^{p_1}, \qquad \dot{y}_3 = p_3 \lambda y_3 + a_1 y_1^{p_3 - p_2 s} y_2^s. \tag{17}$$

Generically $a_0 \neq 0$ and $a_1 \neq 0$; therefore, one can scale them to 1 and obtain (5), a moduli-free normal form.

As in the previous case, one can also explain the invariance of $a_0 \neq 0$ and $a_1 \neq 0$ in a geometric way. In fact, in the generic case, the system has only one smooth invariant surface. (In terms of (16) it is given by $x_1 = 0$; see the discussion in case 1.) If $a_0 = 0$, then the system has an infinite number of smooth invariant surfaces given by $x_2 = \alpha x_1^{p_2}$. If $a_1 = 0$ ($a_0 \neq 0$), then in terms of (16) there is an extra invariant surface given by $x_3 = a_2/(s a_0) x_1^{p_3 - p_2 s} x_2^s + \cdots + a_{s+1}/a_0 x_1^{p_3 - p_2} x_2$.

(ii) $p_2 = p_3 = p$. The resonant normal form in this case is

$$\dot{x}_1 = \lambda x_1, \qquad \dot{x}_2 = p \lambda x_2 + \delta x_3 + a x_1^p, \qquad \dot{x}_3 = p \lambda x_3 + b x_1^p, \tag{18}$$

where $\delta \in \{0, 1\}$.

If $\delta = 1$ in (18), then the change of coordinates $(y_1, y_2, y_3) = (x_1, x_2, x_3 + a x_1^p)$ reduces (18) to $\dot{y}_1 = \lambda y_1$, $\dot{y}_2 = p \lambda y_2 + y_3$, $\dot{y}_3 = p \lambda y_3 + b x_1^p$, where b generically can be scaled to 1. Thus, the final normal form is also moduli-free.

If $\delta = 0$ in the linear part of (18), then the genericity assumption means that the nonlinear part does not vanish. In terms of (18), if $a = 0$ ($b \neq 0$), then b can be scaled to 1. If $b = 0$ and $a \neq 0$, then scale a and enumerate x_2 and x_3. If $ab \neq 0$, then let $y_1 = x_1$, $y_2 = \alpha_1 x_2 + \beta_1 x_3$, and $y_3 = \alpha_2 x_2 + \beta_2 x_3$, where $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, and change (18) to

$$\dot{y}_1 = \lambda y_1, \qquad \dot{y}_2 = p\lambda y_2 + \big(a\alpha_1 + b\beta_1\big)y_1^p, \qquad \dot{y}_3 = p\lambda y_3 + \big(a\alpha_2 + b\beta_2\big)y_1^p.$$

Up to enumeration, it is always possible to put $a\alpha_1 + b\beta_1$ to zero and $a\alpha_2 + b\beta_2$ to 1. We obtain normal form (6).

Case 3: Resonant type $p_1 : p_2 : p_3,\ p_1 > 1$
Observe that if $p_1 > 1$ and there is more than one resonant relation between the eigenvalues, then p_1 and p_2 must be coprime. Consequently, any resonant relation has a form $p_3 = \alpha p_1 + \beta p_2$. Denote by α the minimal possible number such that $p_3 - \alpha p_1$ is divisible by p_2. Then p_3 has the combinations

$$p_3 = \alpha p_1 + \beta p_2 = (\alpha + p_2)p_1 + (\beta - p_1)p_2 = \cdots = (\alpha + lp_2)p_1 + (\beta - lp_1)p_2,$$

where $l = [\beta/p_1]$. The resonant normal form consists of $(l+1)$-resonant terms, and the maximal order of these resonant terms is $\alpha + \beta + l(p_2 - p_1)$. Notice that in this case all the possible resonant changes of coordinates have the form $(x_1, x_2, x_3) \to (x_1, x_2, x_3 + \theta_1 x_1^{\alpha} x_2^{\beta} + \cdots + \theta_l x_1^{\alpha + lp_2} x_2^{\beta - lp_1})$, which keeps the original normal form unchanged. Therefore, taking two possible scalings into consideration, we obtain normal form (7).

Following the normalization process, we see that in terms of (7), vector fields with distinct collections of parameters are not smoothly equivalent.

Theorems 1 and 2 are proved. □

3.2. Proof of Theorems 3 and 4
Given a vector field X with resonant type $0 : 1 : p$, because of Lemmas 2–4, it suffices to prove the following statements.

(1) The $i(A)$-jet of X can be normalized to one of normal forms (9)–(12).
(2) Assume that $j^{i(A)}X$ has one of the forms listed in (9)–(12). Then homological equation (13) has a solution φ with a vanishing 1-jet for any resonant vector field Y with a vanishing k-jet.
(3) The vector fields $j^{i(A)-1}X$ and $j^{i(A)}X$ are not equivalent; in other words, X is not $(i(A)-1)$-jet determined.

Below we study these problems step by step.

(1) Normalization of the $i(A)$-jet of X
Given a vector field X, we first normalize its center variable, since for all the resonant

types $0:1:p$, the resonant normal form restricted to the center manifold is the same: $\dot{x}_1 = x_1 f_1(x_1)$, where $f_1(x_1)$ is a formal series of x_1. Assume the restricted vector field is generic, that is, $f_1'(0) \neq 0$. Then, by the results of [7], it is smoothly reducible to $\dot{x}_1 = x_1^2 + \mu_1 x_1^3$, where μ_1 is a parameter.

It remains to show the following. If X has the resonant type $0:1:p$, $p>1$, then its $(\max(3,p))$-jet can be normalized to (9), and if X has the resonant type $0:1:1$, then its 3-jet can be reduced to one of (10)–(12).

Case 1: Resonant type $0:1:p,\ p>1$
A generic vector field X in this case is smoothly equivalent to the form

$$\dot{x}_1 = x_1^2 + \mu_1 x_1^3, \qquad \dot{x}_2 = \lambda x_2 + x_2 f_2(x_1), \qquad \dot{x}_3 = p\lambda x_3 + x_3 f_3(x_1) + x_2^p g(x_1), \tag{19}$$

where f_i and g are formal series of x_1.

In terms of (19), we assume the following genericity conditions hold. (Their geometric invariance will be explained later.) We have

$$f_3'(0) - p f_2'(0) \notin \{0, 1, 2, \dots\}, \quad g(0) \neq 0. \tag{20}$$

The normalization of $(\max(3,p))$-jet of X means the removability of the *extra* resonant terms under condition (20). We distinguish between two subcases: (i) $p=2$ and (ii) $p \geq 3$.

(i) $p=2$. We need to show that for any numbers A, B, and C, the resonant terms $Ax_1^2 x_2 \partial/\partial x_2$ and $(Bx_1^2 x_3 + Cx_1 x_2^2)\partial/\partial x_3$ can be eliminated from $j^3 X$. To this end, take the change of coordinates

$$\Phi : (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (x_1, x_2 + \alpha_1 x_1 x_2, x_3 + \alpha_2 x_1 x_3 + \alpha_3 x_2^2),$$

where α_i are numerical parameters. Then Φ transfers $j^3 X$ to the form (we drop all the tildes) $j^3 \Phi_* X = j^3 X + (0, \alpha_1 x_1^2 x_2, \alpha_2 x_1^2 x_3 + \xi x_1 x_2^2)$, where $\xi = (2 f_2'(0) - f_3'(0))\alpha_3 + g(0)(\alpha_2 - 2\alpha_1)$.

The elimination of the mentioned terms is equivalent to the solvability of $\alpha_{1,2,3}$ from the equations $\alpha_1 + A = 0$, $\alpha_2 + B = 0$, $\xi + C = 0$, for any numbers A, B, C, which is clear because of the genericity condition $f_3'(0) - 2f_2'(0) \neq 0$.

Since $g(0) \neq 0$, one can scale it to 1 and obtain normal form (9).

(ii) $p \geq 3$. One sees that the change of coordinates $\Phi_s : (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (x_1, x_2 + \alpha_1 x_1^s x_2, x_3 + \alpha_2 x_1^s x_3)$, where $\alpha_{1,2}$ are numerical parameters, preserves the $(s+1)$-jet of a vector field and brings the effect $(0, s\alpha_1 x_1^{s+1} x_2, s\alpha_2 x_1^{s+1} x_3)$ to the $(s+2)$-jet of the system. Therefore, by taking $\alpha_1 = -A_s/s$, $\alpha_2 = -B_s/s$ in Φ_s one can remove the terms $A_s x_1^{s+1} x_2 \partial/\partial x_2$ and $B_s x_1^{s+1} x_3 \partial/\partial x_3$. The composition of all such Φ_s, $s = 1, \dots, p-2$, normalizes the p-jet of X to (9). (Scale $g(0)$ to 1, if necessary.)

Case 2: Resonant type 0 : 1 : 1

A generic system in this case is smoothly equivalent to the form

$$\dot{x}_1 = x_1^2 + \mu_1 x_1^3, \qquad \dot{x}_2 = x_2 g_1(x_1) + x_3 g_2(x_1), \qquad \dot{x}_3 = x_2 g_3(x_1) + x_3 g_4(x_1), \tag{21}$$

where $g_i(x_1)$ are formal series, $g_1(0) = g_4(0) = \lambda$, $g_2(0) = 0$ or 1, $g_3(0) = 0$.

We consider two cases: (i) $g_2(0) = 1$ and (ii) $g_2(0) = 0$.

(i) $g_2(0) = 1$. The case $g_2(0) = 1$ corresponds to vector fields having two identical eigenvalues and a nondiagonalizable linear part. In terms of (21), we assume the following genericity condition holds:

$$g_3'(0) \neq 0. \tag{22}$$

As in the previous normalization, we perform the change of coordinates

$$\Phi : (x_1, x_2, x_3) \longrightarrow \mathrm{id} + x_1\big(\alpha_1 x_1, \alpha_2 x_2 + \alpha_3 x_3, \alpha_4 x_2 + \alpha_5 x_3\big) + x_1^2(0, \alpha_6 x_2, 0),$$

where α_j are parameters. Then Φ brings X to a system whose 3-jet is given by

$$j^3\Phi_* X = j^3 X + x_1\big(0, \xi_1 x_2 + \xi_2 x_3, -\xi_1 x_3\big) + x_1^2\big(0, \xi_3 x_2 + \xi_4 x_3, \xi_5 x_2 + \xi_6 x_3\big),$$

where

$$\begin{cases} \xi_1 = \alpha_4, \\ \xi_2 = \alpha_5 - \alpha_2, \\ \xi_3 = g_1'(0)\alpha_1 + g_2'(0)\alpha_4 - \alpha_2 - g_3'(0)\alpha_3, \\ \xi_4 = g_2'(0)\alpha_1 - g_2'(0)\alpha_2 + \big(g_1'(0) - g_4'(0) - 1\big)\alpha_3 + g_2'(0)\alpha_5 - \alpha_6, \\ \xi_5 = g_3'(0)\alpha_1 + g_3'(0)\alpha_2 + \big(g_4'(0) - g_1'(0) - 1\big)\alpha_4 - g_3'(0)\alpha_5, \\ \xi_6 = g_4'(0)\alpha_1 + g_3'(0)\alpha_3 - g_2'(0)\alpha_4 - \alpha_5. \end{cases}$$

It is easy to see that the system of equations $\xi_i = A_i$, $i = 1, \dots, 6$, is linear in terms of $\alpha_1, \dots, \alpha_6$ and is solvable with respect to them for any A_i if and only if (22) holds. In other words, under the genericity condition (22), the terms $(A_1 x_1 x_2 + A_2 x_1 x_3 + A_3 x_1^2 x_2 + A_4 x_1^2 x_3)\partial/\partial x_2$ and $(A_5 x_1^2 x_2 + A_6 x_1^2 x_3)\partial/\partial x_3$ are removable, which results in the normalization of $j^3 X$.

(ii) $g_2(0) = 0$. If $g_2(0) = 0$ in (21), then X has two identical eigenvalues and the linear part is diagonalizable. Introduce the matrix

$$M = \begin{pmatrix} g_1'(0) & g_2'(0) \\ g_3'(0) & g_4'(0) \end{pmatrix}, \tag{23}$$

and denote by ν_1, ν_2 its eigenvalues.

In terms of (21) and (23), we assume the following genericity condition holds:

$$\nu_1 - \nu_2 \notin \{0, \pm 1, \pm 2, \dots\}. \tag{24}$$

One sees that if ν_1 and ν_2 are distinct real numbers (resp., a pair of conjugated complex numbers), then there exists a nondegenerate matrix A such that $A^{-1}MA = \operatorname{diag}(\nu_1, \nu_2)$ (resp., $A^{-1}MA = \left(\begin{smallmatrix} \alpha & \beta \\ -\beta & \alpha \end{smallmatrix}\right)$, where α and β are real numbers in $\nu_{1,2} = \alpha \pm i\beta$). We consider only the case where $\nu_{1,2}$ are real. We prove that in this case j^3X can be normalized to (11). If $\nu_{1,2}$ are complex, then similar discussion can be given, and, correspondingly, j^3X can be normalized to (12).

Notice that the change of coordinates $\Phi : (x_1, x_2, x_3) \to (x_1, (x_2, x_3)A^t)$ preserves j^1X and normalizes j^2X to $j^2\Phi_*X = (x_1^2, \lambda x_2 + \nu_1 x_1 x_2, \lambda x_3 + \nu_2 x_1 x_3)$. To normalize the 3-jet, it remains to show that all the resonant terms of homogeneous degree 3, except $\mu_1 x_1^3 \partial/\partial x_1$, are removable.

Consider the change of coordinates $\Phi : (x_1, x_2, x_3) \to \mathrm{id} + x_1(0, \alpha_1 x_2 + \alpha_2 x_3, \alpha_3 x_2 + \alpha_4 x_3)$, where α_i are numerical parameters. It preserves j^2X and brings the effect

$$x_1^2\big(0, \alpha_1 x_2 + (\nu_2 - \nu_1 + 1)\alpha_2 x_3, (\nu_1 - \nu_2 + 1)\alpha_3 x_2 + \alpha_4 x_3\big)$$

to the terms of homogeneous degree 3. It follows that by choosing suitable α_i one can remove the mentioned terms of degree 3, recalling the genericity conditions $\nu_1 - \nu_2 \neq \pm 1$.

The normalization of the $i(A)$-jet of X for all the resonant types $0 : 1 : p$ is done.

(2) The solvability of the homological equation

Below we show that for any resonant vector field Y with a vanishing $i(A)$-jet homologic equation (13) has a solution φ with a vanishing 1-jet. Consequently, by Lemma 4, X is smoothly $i(A)$-determined. According to the value of p and the diagonalizability of j^1X, we have three cases. We treat one case in more detail, since the other cases can be studied in a similar way.

Case 1: Resonant type $0 : 1 : p,\ p > 1$

Let $k = \max(3, p)$, and let Y be any resonant vector field with a vanishing k-jet; that is, let

$$Y = \big(F_1(x_1)x_1, F_2(x_1)x_2, F_3(x_1)x_3 + F_4(x_1)x_2^p\big),$$

where $j^{k-1}F_{1,2,3}(x_1) = 0$ and $j^{k-p}F_4(x_1) = 0$.

We look for a solution φ, $j^1\varphi = 0$, of the equation (13) in the form

$$\varphi = \big(\varphi_1(x_1)x_1, \varphi_2(x_1)x_2, \varphi_3(x_1)x_3 + \varphi_4(x_1)x_2^p\big).$$

In other words, we shall prove the solvability of φ_i from the following system, which is derived from (13). We have

$$\begin{cases} \big(x_1 + 2\mu_1 x_1^2\big)\varphi_1(x_1) - \big(x_1^2 + \mu_1 x_1^3\big)\varphi_1'(x_1) = F_1(x_1), \\ \big(x_1^2 + \mu_1 x_1^3\big)\varphi_2'(x_1) = \mu_2 x_1 \varphi_1(x_1) - F_2(x_1), \\ \big(x_1^2 + \mu_1 x_1^3\big)\varphi_3'(x_1) = \mu_3 x_1 \varphi_1(x_1) - F_3(x_1), \\ \big(\mu_3 - p\mu_2\big)x_1\varphi_4(x_1) - \big(x_1^2 + \mu_1 x_1^3\big)\varphi_4'(x_1) = F_4(x_1) + \varphi_3(x_1) - p\varphi_2(x_1). \end{cases} \tag{25}$$

It is clear that formally the first three equations are always solvable with respect to $\varphi_{1,2,3}(x_1)$ because of the conditions $j^{k-1}F_{1,2,3}(x_1) = 0$. Moreover, $\varphi_{1,2,3}(x_1)$ have a vanishing $(k-2)$-jet at zero. To see the solvability of φ_4, rewrite the last equation

$$(\mu_3 - p\mu_2)\varphi_4(x_1) - \big(x_1 + \mu_1 x_1^2\big)\varphi_4'(x_1) = H(x_1), \tag{26}$$

where $H(x_1)$ is a function of x_1, and let

$$\varphi_4(x_1) = \beta_0 + \beta_1 x_1 + \cdots, \qquad H(x_1) = h_0 + h_1 x_1 + \cdots.$$

Then equation (26) leads to an infinite system of β_i:

$$(\mu_3 - p\mu_2)\beta_0 = h_0, \qquad (\mu_3 - p\mu_2 - i)\beta_i = (i-1)\mu_1\beta_{i-1} + h_i, \quad i = 1, 2, \ldots.$$

The solvability of β_i now follows immediately from the genericity condition (20), $\mu_3 - p\mu_2 \notin \{0, 1, 2, \ldots\}$.

It is clear that φ obtained in this way has a vanishing 1-jet at zero.

Case 2: Resonant type $0 : 1 : 1$*, nondiagonalizable linear part*

Let Y be any resonant vector field with a vanishing 3-jet, that is,

$$Y = \big(F_1 x_1, F_2 x_2 + F_3 x_3, F_4 x_2 + F_5 x_3\big),$$

where F_i are functions of x_1 with a vanishing 2-jet. We look for a solution φ of (13) in the form

$$\varphi = \big(\varphi_1 x_1, \varphi_2 x_2 + \varphi_3 x_3, \varphi_4 x_2 + \varphi_5 x_3\big), \tag{27}$$

where φ_i are function x_1. With this notation, the solvability of (13) is equivalent to that of the system

$$\begin{cases} \big(x_1 + 2\mu_1 x_1^2\big)\varphi_1 - \big(x_1^2 + \mu_1 x_1^3\big)\varphi_1' = F_1, \\ -\big(x_1^2 + \mu_1 x_1^3\big)\varphi_2' - \mu_2 x_1 \varphi_3 + \varphi_4 = F_2, \\ -\big(x_1^2 + \mu_1 x_1^3\big)\varphi_3' - \mu_3 x_1 \varphi_3 + \varphi_5 - \varphi_2 = F_3, \\ \mu_2 x_1 \varphi_2 + \mu_3 x_1 \varphi_4 - \mu_2 x_1 \varphi_5 - \big(x_1^2 + \mu_1 x_1^3\big)\varphi_4' = F_4 - \mu_2 x_1 \varphi_1, \\ \mu_2 x_1 \varphi_3 - \varphi_4 - \big(x_1^2 + \mu_1 x_1^3\big)\varphi_5' = F_5 - \mu_3 x_1 \varphi_1. \end{cases} \tag{28}$$

From the first equation of (28), one can obtain φ_1. Moreover, it has a vanishing 1-jet because of the assumption $j^2 F_1 = 0$. To solve other φ_j, rearrange the remaining equations of (28) as follows: change the second equation by the difference of itself and the last equation, change the last equation by the sum of itself and the second equation, and change the fourth equation by the sum of itself and the third equation multiplied by $\mu_2 x_1$. One has the following system of equations:

$$\begin{cases} \left(x_1^2 + \mu_1 x_1^3\right)\left(\varphi_5' - \varphi_2'\right) - 2\mu_2 x_1 \varphi_3 + 2\varphi_4 = \tilde{F}_2, \\ \varphi_5 - \varphi_2 - \left(x_1^2 + \mu_1 x_1^3\right)\varphi_3' - \mu_3 x_1 \varphi_3 = F_3, \\ \mu_3 x_1 \varphi_4 - \left(x_1^2 + \mu_1 x_1^3\right)\varphi_4' - \mu_2\mu_3 x_1^2 \varphi_3 - \mu_2 x_1\left(x_1^2 + \mu_1 x_1^3\right)\varphi_3' = \tilde{F}_4, \\ -\left(x_1^2 + \mu_1 x_1^3\right)\left(\varphi_2' + \varphi_5'\right) = \tilde{F}_5, \end{cases} \tag{29}$$

where $\tilde{F}_j$ are certain functions of the known functions F_j and φ_1. From the second equation of system (29), one can express $\varphi_5 - \varphi_2$ in terms of φ_3. Substituting $\varphi_5 - \varphi_2$ to the first equation, one obtains φ_4 in terms of φ_3; then the third equation becomes an equation of φ_3:

$$\left(\mu_2 x_1^2 + \cdots\right)\varphi_3 + \left(2\mu_2 x_1^3 + \cdots\right)\varphi_3' + o\left(x_1^4\right)\varphi_3'' + o\left(x_1^5\right)\varphi_3''' = F(x_1),$$

where F is a known function of x_1, $j^2 F(x_1) = 0$.

As in the proof of the previous case, one can show that the above equation is solvable with respect to φ_3 under the condition $\mu_2 \neq 0$. Consequently, one solves φ_4. The solvability of φ_2 and φ_5 now becomes clear.

Following the derivation of $\tilde{F}_j$, it is easy to check that the solution obtained in this way has a vanishing 1-jet. We leave the details to the reader.

Case 3: Resonant type $0 : 1 : 1$*, diagonalizable linear part*

Take Y and φ as in the last case. Then one can rewrite equation (13) as

$$\begin{cases} \left(x_1 + 2\mu_1 x_1^2\right)\varphi_1 - \left(x_1^2 + \mu_1 x_1^3\right)\varphi_1' = F_1, \\ \nu_1 x_1 \varphi_1 - \left(x_1^2 + \mu_1 x_1^3\right)\varphi_2' = F_2, \\ \nu_2 x_1 \varphi_1 - \left(x_1^2 + \mu_1 x_1^3\right)\varphi_5' = F_5, \\ (\nu_1 - \nu_2) x_1 \varphi_3 - \left(x_1^2 + \mu_1 x_1^3\right)\varphi_3' = F_3, \\ (\nu_2 - \nu_1) x_1 \varphi_4 - \left(x_1^2 + \mu_1 x_1^3\right)\varphi_4' = F_4. \end{cases} \tag{30}$$

As in previous cases, one can solve $\varphi_{1,2,5}(x_1)$ from the first three equations. Moreover, they have a vanishing 1-jet because $j^2 F_{1,2,5} = 0$ at zero.

To solve φ_3, consider the fourth equation. In fact, let

$$\varphi_3(x_1) = \beta_2 x_1^2 + \beta_3 x_1^3 + \cdots, \qquad F_3(x_1) = h_2 x_1^3 + h_3 x_1^4 + \cdots.$$

Then the fourth equation can be written in the form

$$(\nu_1 - \nu_2 - 2)\beta_2 = h_2, \qquad (\nu_1 - \nu_2 - i)\beta_i = (i-1)\mu_1\beta_{i-1} + h_i, \quad i = 3, 4, \ldots.$$

Now the solvability of φ_3 follows from the genericity condition $\nu_1 - \nu_2 \neq 2, 3, \ldots$.

Arguing in the same manner and using the genericity condition $\nu_1 - \nu_2 \neq -2, -3, \ldots$, we can solve φ_4 from the last equation of (30).

Recall that we have used the genericity conditions $\nu_1 - \nu_2 \neq 0, \pm 1$ in normalizing the matrix M (see (23)) and the terms of homogeneous degree 3.

(3) The index of finite determinacy and the genericity conditions

As shown above, the index of finite determinacy $i(A)$ is not bigger than $\max(3, p)$. On the other hand, it is clear that $i(A)$ is not less than 3, since the vector field restricted to the center manifold is 3-jet determined. Therefore, for generic vector fields having resonant types $0 : 1 : p$, $p = 1, 2, 3$, the index of finite determinacy is exactly 3. If a vector field X has a resonant type $0 : 1 : p$, $p > 3$, then the index of finite determinacy cannot be less than p. In other words, X is not $(p-1)$-jet determined. This is so because of the following observation: System (9) has only two invariant surfaces (given by $x_1 = 0$ and $x_2 = 0$ in the local coordinates), whereas the system

$$\dot{x}_1 = x_1^2 + \mu_1 x_1^3, \qquad \dot{x}_2 = \lambda x_2 + \mu_2 x_1 x_2, \qquad \dot{x}_3 = p\lambda x_3 + \mu_3 x_1 x_3$$

has at least three invariant surfaces (given by the coordinate planes).

Now we give an explanation about the genericity conditions listed in (20), (22), and (24). We show that these conditions, though given in terms of local coordinates, in fact, do not depend on them.

Take genericity condition (20). If $p \geq 3$, then the condition that $f_3'(0) - pf_2'(0) \neq 0, 1, \ldots$ is related to the existence of smooth invariant surfaces of the $(p-1)$-jet of X. If these inequalities hold, then $j^{p-1}X$ has exactly three invariant surfaces. (In coordinates (19) they are the coordinate planes.) If the condition is violated, that is, if $f_3'(0) - pf_2'(0) = k$ for some nonnegative integer k, then, except for these three invariant surfaces, the 2-jet of X has an infinite number of other invariant surfaces. (In coordinates (19), they are given by the equation $x_3 = \alpha x_1^k x_2^p$, where α is a real number.)

Similar explanations about genericity conditions (22) and (24) can be given correspondingly.

Finally, one sees from the proof of the theorems that two vector fields in normal form (9) or (10) with distinct collections of parameters $\{\mu_1, \mu_2, \mu_3\}$ are not formally equivalent; therefore the corollary following Theorem 4 holds.

Theorems 3 and 4 are proved. □

Acknowledgments. The author is very grateful to M. Zhitomirskii for introducing normal form theory to him and for useful discussions. This paper was finally finished during a post-doctoral visit to the Mathematics Department at the State University of Campinas; the author would like to thank the department for its hospitality.

References

[1] V. ARNOLD and YU. ILYASHENKO, "Ordinary differential equations" in *Dynamical Systems, Vol. 1*, Encyclopaedia Math. Sci. **1**, Springer, Berlin, 1988, 1–148.

[2] G. BELITSKII, *Equivalence and normal forms of germs of smooth mappings*, Russian Math. Surveys **33** (1978), 107–177.

[3] ———, *Smooth equivalence of germs of C^∞ of vector fields with one zero or a pair of pure imaginary eigenvalues*, Funct. Anal. Appl. **20** (1986), 253–259.

[4] A. BRUNO, *Local Methods in Nonlinear Differential Equations*, Springer Ser. Soviet Math., Springer, Berlin, 1989.

[5] F. ICHIKAWA, *Finitely determined singularities of formal vector fields*, Invent. Math. **66** (1982), 199–214.

[6] ———, *On finite determinacy of formal vector fields*, Invent. Math. **70** (1982), 45–52.

[7] F. TAKENS, *Normal forms for certain singularities of vector fields*, Ann. Inst. Fourier (Grenoble) **23** (1973), 163–195.

[8] J. YANG, *Polynomial Normal Forms of Vector Fields*, thesis, Technion-Israel Institute of Technology, 1997.

[9] ———, *Normal forms of quasi-strongly* 1*-resonant vector fields*, in preparation.

[10] M. ZHITOMIRSKII, *Local classification of differential* 1*-forms and vector fields*, thesis, Kharkov University, 1983.

Departamento de Matemática, Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica, Caixa Postal 6065, Campinas 13081-970, São Paulo, Brazil; yang@ime.unicamp.br; current: Department of Mathematics, Peking University, Beijing 100871, China; yang@sxx0.math.pku.edu.cn