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THE FINITE ELEMENT IMMERSED BOUNDARY METHOD: MODEL, STABILITY, AND NUMERICAL RESULTS

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Abstract. *The Immersed Boundary Method (IBM) has been designed by Peskin for the modeling and the numerical approximation of fluid-structure interaction problems. Recently, a finite element version of the IBM as been developed, which offers interesting features for both the analysis of the problem under consideration and the robustness and flexibility of the numerical scheme. In this paper we shall review our model and present numerical results in two and three dimensions fully confirming the good performance of our scheme. A preliminary stability analysis shows a condition linking the time step size and the discretization parameter along the immersed boundary. This condition is confirmed by our numerical experiments.*

1 INTRODUCTION

Fluid-structure interaction systems often involve the resolution of the fluid dynamic equations on a moving (that is, time dependent) domain. Several approaches have been considered in order to deal with such problem. A classical way to overcome the difficulties due to the reconstruction of the mesh at each time step, is the introduction of the arbitrary Lagrangian-Eulerian (ALE) formulation [1, 2, 3, 4, 5, 6], transporting the equations to a fixed arbitrary reference configuration. Although this approach has been used successfully, its accurate implementation is expensive in real applications. In order to be able to solve at low cost fluid structure interaction problems undergoing moderate deformations, aeronautical engineers have developed transpiration techniques, introducing suitable modifications of the interface boundary conditions (see, e.g., [7, 8, 9, 10]).

The fictitious domain method can also be used in order to simulate an incompressible viscous flow around moving rigid bodies; the idea consists in extending the equations to a simple domain where a structured grid can be used and then considering suitable Lagrange multipliers to enforce the boundary conditions along the moving bodies, see [11, 12]. Unfortunately, this method is not able to deal with the interaction between fluid and flexible solids with large deformations.

A completely different approach is due to Peskin who developed the immersed boundary method (IBM) (see [13, 14]) to study flow patterns around heart valves. The immersed boundary method is designed to handle a flexible boundary immersed in a fluid, hence it is particularly suited for biological fluid dynamic problems (see, e.g., [15, 16, 17, 18, 19, 20, 21, 22, 23]. As we have already mentioned, the computation requirement to evolve or adapt the mesh becomes considerably expensive in many fluid-structure interaction systems. In the IBM, the structure is thought as a part of the fluid where additional forces are applied, and where additional mass may be localized. Therefore, instead of separating the system in its two components coupled by interface conditions, as it is conventionally done (see, e.g. [24, 25]), the incompressible Navier-Stokes equations, with a nonuniform mass density and an applied elastic force density, are used in order to describe the coupled motion of the hydroelastic system in a unified way. The advantage of this method is that the fluid domain can have a simple shape, so that structured grids can be used. On the other hand, the immersed boundary is typically not aligned with the grid and it is represented using Lagrangian variables, defined on a curvilinear mesh moving through the domain. Another fundamental assumption of the IBM is that the immersed structure has a fiber-like one dimensional structure, which may have a mass but occupies no volume in the fluid domain (see [14, 15]).

The original numerical approach to the IBM is based on finite differences for the spatial discretization. This employs two independent grids, one for the Eulerian variables in the fluid and the other for the Lagrangian variables associated with the immersed boundary. The main difficulty in the spatial discretization consists in the construction of suitable approximation of the Dirac delta function which is used to take into account the interaction equations, see [14]. The temporal discretization that is currently used by Peskin and his coauthors is a second-order accurate Runge-Kutta method, based on the midpoint rule (see, e.g., [26, 27]).

More recently, the finite element method has been applied to the spatial discretization of the IBM in [28, 29, 30, 31]. In particular, in [30, 31] it has been proposed the EIBM, *extended immersed boundary method* which is based on the idea of considering the submerged elastic solid occupying a finite volume in the fluid domain. This was done by replacing the kinematic and dynamic matching of the fluid-solid interface and the effect of the immersed solid with nodal forces calculated in the context of finite element formulations. The equations in both fluid and solid domains are approximated using finite elements, while the continuity between the fluid and the solid domains are enforced interpolating the velocities and the distribution of forces delta function with the reproducing kernel particle method (RKPM). This enables the construction of discretized delta function which belongs to $\mathbf{C}^n$ with n chosen according to the required smoothness.

Our approach to the discretization of the IBM is completely based upon the finite element method. Our aim is to deal with the delta function, which is related to the forces exerted by the immersed structure on the fluid and viceversa, in a variational way. So that there is no need to construct a regularization of the delta function, but its effect is taken into account by its action on the test functions.

The outline of the paper is the following. In Section 2 we recall the formulation of the IBM and present the variational formulation used for the finite element discretization. Finally, in Section 3 we report some numerical results.

2 FORMULATION OF THE IMMERSED BOUNDARY METHOD AND ITS FINITE ELEMENT APPROXIMATION

Let Ω be the two or a three dimensional domain containing the fluid and the flexible or elastic structure. As usual the Navier-Stokes equations describe the dynamics of a viscous incompressible fluid with respect to the Eulerian variables denoted by $\mathbf{x}$:

$$\begin{aligned} &\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\mathbf{grad}\,\mathbf{u}\right) - \mu\Delta\mathbf{u} + \mathbf{grad}\,p = \mathbf{F} && \text{in } \Omega\times]0,T[\\ &\nabla\cdot\mathbf{u} = 0 && \text{in } \Omega\times]0,T[. \end{aligned} \tag{1}$$

Here ρ and μ denote the density and the viscosity of the fluid. The unknowns $\mathbf{u}(\mathbf{x},t)$ and $p(\mathbf{x},t)$ represent the velocity and the pressure, respectively. On the right hand side of the first equation in (1), $\mathbf{F}$ denotes the density of the body force acting on the fluid. It usually contains a singular vector field, which is zero everywhere, except possibly on the surface representing the immersed structure. We assume that μ is constant, while ρ can be a function of $(\mathbf{x},t)$, since we consider the structure as a part of the fluid carrying an additional mass.

The immersed boundary is considered as an elastic incompressible material filling a two or three dimensional space or laying along an immersed boundary in the form of a simple closed curve or surface. Let $\mathbf{q}$ denote the Lagrangian coordinates in the initial solid domain Ω_0, labeling a material point of it. The position of a point in the current solid domain at a time t is denoted by $\mathbf{X}(\mathbf{q},t)$, hence this represents the position in Ω of the material point which was labeled by $\mathbf{q}$ at the initial time. Usually the fluid is assumed to have a uniform mass density ρ_0, while the mass density of the immersed structure can be described introducing the *excess* Lagrangian mass density $M(\mathbf{q})$, that is the difference between the mass of the elastic material and the mass of the fluid displaced by it. Then we can express the density of the fluid and the force exerted by the structure on the fluid in terms of $M(\mathbf{q})$, the excess of mass density, and of $\mathbf{f}(\mathbf{q},t)$, the force density that the immersed material applies on the fluid, as follows:

$$\rho(\mathbf{x},t) = \rho_0 + \int_{\Omega_0} M(\mathbf{q})\delta(\mathbf{x}-\mathbf{X}(\mathbf{q},t))\,d\mathbf{q}, \qquad \text{in } \Omega\times]0,T[, \tag{2}$$

$$\mathbf{F}(\mathbf{x},t) = \int_{\Omega_0} \mathbf{f}(\mathbf{q},t)\delta(\mathbf{x}-\mathbf{X}(\mathbf{q},t))\,d\mathbf{q}, \qquad \text{in } \Omega\times]0,T[, \tag{3}$$

where δ is the Dirac delta function in $\mathbb{R}^d$ $(d = 2,3)$. The force $\mathbf{F}$ given in (3) is the right hand side of (1) and takes into account the interaction between the fluid and the immersed structure. This is a crucial point in the modeling of different applications, since the expression of $\mathbf{f}$ takes into account the elasticity properties of the structure.

In order to compute the position of the immersed structure another relation which enforces the no slip condition for a viscous fluid has to be considered:

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{u}(\mathbf{X}(\mathbf{q},t),t) \qquad \text{in } \Omega_0\times]0,T[. \tag{4}$$

The above equation means that the structure moves at the same velocity as the fluid.

To summarize, the resolution of the immersed boundary method requires to find $\mathbf{u}$, p and $\mathbf{X}$ which satisfy:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\mathbf{grad}\,\mathbf{u}\right) - \mu\Delta\mathbf{u} + \mathbf{grad}\,p = \mathbf{F} \quad \text{in } \Omega\times]0,T[\tag{5}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times]0,T[\tag{6}$$

$$\rho(\mathbf{x},t) = \rho_0 + \int_{\Omega_0} M(\mathbf{q})\delta(\mathbf{x}-\mathbf{X}(\mathbf{q},t))\,d\mathbf{q} \quad \text{in } \Omega\times]0,T[\tag{7}$$

$$\mathbf{F}(\mathbf{x},t) = \int_{\Omega_0} \mathbf{f}(\mathbf{q},t)\delta(\mathbf{x}-\mathbf{X}(\mathbf{q},t))\,d\mathbf{q} \quad \text{in } \Omega\times]0,T[\tag{8}$$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{u}(\mathbf{X}(\mathbf{q},t),t) \quad \text{in } \Omega_0\times]0,T[\tag{9}$$

$$\mathbf{u}(\mathbf{x},t) = 0 \quad \text{on } \partial\Omega\times]0,T[\tag{10}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega \tag{11}$$

$$\mathbf{X}(\mathbf{q},0) = \mathbf{X}_0(\mathbf{q}) \quad \text{in } \Omega_0. \tag{12}$$

Conditions (10) and (11) represent boundary and initial conditions relative to the Navier-Stokes equation (5)-(6); other boundary conditions could also be used. The last equation (12) is the initial condition for (9) which drives the motion of the immersed structure. Since $\mathbf{q}$ is the Lagrangian variable associated to a material point in the initial configuration, then $\mathbf{X}_0(\mathbf{q}) = \mathbf{q}$, if we assume that the reference configuration is unstressed.

We shall consider two basic examples. In two space dimensions we model a massless closed curve immersed in a fluid and in three dimensions we consider the analogous situation of a massless elastic membrane. In these two cases, we have

$$\mathbf{f} = \kappa \frac{\partial^2 \mathbf{X}}{\partial s^2}, \tag{13}$$

and

$$\mathbf{f} = \kappa \Delta_{r,s} \mathbf{X}, \tag{14}$$

respectively, where κ is the elasticity constant assuming that the material is homogeneous and isotropic.

In order to work with these two cases in a unified way, let B_t denote the immersed boundary, that is the one dimensional curve in the two dimensional case or the membrane in the three dimensional example. The main mathematical result that allows us to deal with the approximation of the Dirac mass appearing in (8), is the following lemma.

Lemma 1 *Assume that, for all $t \in [0, T]$, the immersed boundary B_t is Lipschitz continuous. Assume, moreover, that $\mathbf{X}$ is regular enough so that the right hand side of* (15) *makes sense. Then for all $t \in]0, T[$, the force density $\mathbf{F}(t)$, defined formally in* (8)*, is a distribution function belonging to $H^{-1}(\Omega)^d$ ($d = 2, 3$) defined as follows: for all $\mathbf{v} \in H^1_0(\Omega)$*

$${}_{H^{-1}}< \mathbf{F}(t), \mathbf{v} >_{H^1_0} = \int_D \mathbf{f}(\mathbf{s}, t) \cdot \mathbf{v}(\mathbf{X}(\mathbf{s}, t)) \, d\mathbf{s} \quad \forall t \in [0, T]. \tag{15}$$

We consider discrete space sequences $\mathbf{V}_h \subset H^1_0(\Omega)^d$, $\mathbf{S}_h \subset L^2_0(\Omega)$ and an approximation of the immersed curve (membrane, resp.). For the sake of simplicity we discard the nonlinear term in the Navier–Stokes equations; numerical results for the fully nonlinear system are in progress. Then, the spatial finite element semidiscretizations of the problem under consideration in two and three space dimensions, respectively, read

Problem 1 *Given $\mathbf{f} \in L^2(]0, L[\times]0, T[)$, $\mathbf{u}_{0h} \in \mathbf{V}_h$ and $\mathbf{X}_{h\,0} \in \mathbf{S}_h$, for all $t \in]0, T[$, find $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$ and $\mathbf{X}_h(t) \in \mathbf{S}_h$, such that*

$$\rho \left(\frac{d}{dt}(\mathbf{u}_h(t), \mathbf{v}) + (\mathbf{u}_h \cdot \mathbf{grad}\, \mathbf{u}_h, \mathbf{v}) \right) + \mu(\mathbf{grad}\, \mathbf{u}_h(t), \mathbf{grad}\, \mathbf{v})$$

$$- (\nabla \cdot \mathbf{v}, p_h(t)) = < \mathbf{F}_h(t), \mathbf{v} > \qquad \forall \mathbf{v} \in \mathbf{V}_h \tag{16}$$

$$(\nabla \cdot \mathbf{u}_h(t), q) = 0 \qquad \forall q \in Q_h \tag{17}$$

$$< \mathbf{F}_h(t), \mathbf{v} > = \sum_{i=0}^{m-1} \kappa \left(\frac{\partial \mathbf{X}_{h\,i+1}}{\partial s}(t) - \frac{\partial \mathbf{X}_{h\,i}}{\partial s}(t) \right) \mathbf{v}(\mathbf{X}_{h\,i}(t)) \qquad \forall \mathbf{v} \in \mathbf{V}_h \tag{18}$$

$$\frac{\partial \mathbf{X}_{h\,i}}{\partial t}(t) = \mathbf{u}_h(\mathbf{X}_{h\,i}(t), t) \qquad \forall i = 0, 1, \cdots, m \tag{19}$$

$$\mathbf{u}_h(\mathbf{x}, 0) = \mathbf{u}_{0h}(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega \tag{20}$$

$$\mathbf{X}_{h\,i}(0) = \mathbf{X}_0(s_i) \qquad \forall i = 1, \cdots, m. \tag{21}$$

and

Problem 2 *Given* $\mathbf{f} \in L^2(D\times]0,T[)$, $\mathbf{u}_{0h} \in \mathbf{V}_h$ *and* $\mathbf{X}_{0h} \in \mathbf{S}_h$, *for all* $t \in]0,T[$, *find* $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$ *and* $\mathbf{X}_h(t) \in \mathbf{S}_h$, *such that*

$$
\begin{aligned}
&\rho\left(\frac{d}{dt}(\mathbf{u}_h(t),\mathbf{v}) + (\mathbf{u}_h\cdot\mathbf{grad}\,\mathbf{u}_h,\mathbf{v})\right) + \mu(\mathbf{grad}\,\mathbf{u}_h(t),\mathbf{grad}\,\mathbf{v}) \\
&\qquad - (\nabla\cdot\mathbf{v}, p_h(t)) = <\mathbf{F}_h(t),\mathbf{v}> && \forall \mathbf{v}\in\mathbf{V}_h \quad (22)\\
&(\nabla\cdot\mathbf{u}_h(t), q) = 0 && \forall q\in Q_h \quad (23)\\
&<\mathbf{F}_h(t),\mathbf{v}> = -\kappa\sum_{e\in\mathcal{E}_h}\int_e \mathbf{v}(\mathbf{X}_h(t))\cdot\left[\frac{\partial\mathbf{X}_h(t)}{\partial\mathbf{n}}\right] d\sigma && \forall \mathbf{v}\in\mathbf{V}_h \quad (24)\\
&\frac{\partial\mathbf{X}_{h\,i}}{\partial t}(t) = \mathbf{u}_h(\mathbf{X}_{h\,i}(t),t) && \forall i = 1,\cdots,\mathcal{N}_{hs} \quad (25)\\
&\mathbf{u}_h(\mathbf{x},0) = \mathbf{u}_{0h}(\mathbf{x}) && \forall \mathbf{x}\in\Omega \quad (26)\\
&\mathbf{X}_{h\,i}(0) = \mathbf{X}_{0h}(s_i) && \forall i = 1,\cdots,\mathcal{N}_{hs}. \quad (27)
\end{aligned}
$$

We refer the interested reader to [28, 29, 32] for the derivation of our numerical schemes.

3 NUMERICAL RESULTS

In this section we report some numerical results in two and three dimensions which confirm the promising features of our approach.

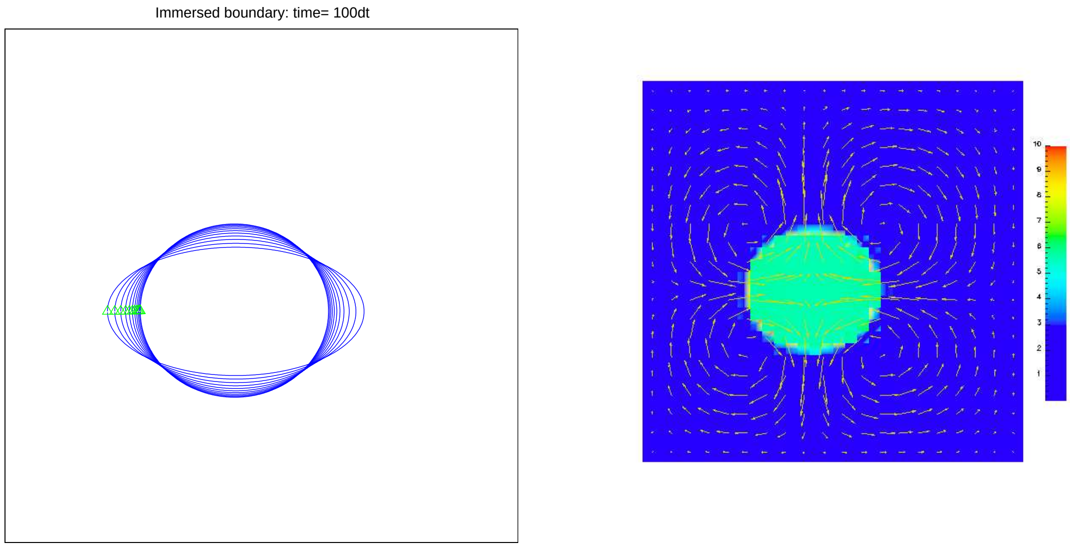
In two dimensions the computational domain is a square, we consider a mesh of subsquares and the stable $Q2-P1$ Stokes element (i.e., biquadratic velocities and discontinuous linear pressures). The immersed boundary (which is a curve) is approximated by piecewise linear elements. In Figure 3 we show the evolution of an elastic ellipse subjected only to its elastic force. It can be clearly observed that the ellipse tends to a circular configuration.

Our second example, reported in Figure 2, shows an elastic ellipse in a moving fluid.

Figure 3 shows the interaction between two immersed boundaries which are pushed one against the other by the fluid motion.

We conclude this section with some preliminary three dimensional example. In Figure 4 we present the evolution of an elastic membrane immersed in a periodically moving fluid. The membrane is clamped at the boundary of the box containing the fluid. We explicitly observe that we do not need enforce numerically the membrane to be clamped to the box, since the boundary conditions imposed to the fluid mean no motion at the box boundary.

The last example is the three dimensional generalization of the results presented in Figure 3. The initial configuration of the immersed boundary is an ellipsoid which is subjected only to its elastic force. The membrane tends to a spherical shape.



(a) Evolution of the boundary (b) Fluid pressure and velocity after 100 steps

Figure 1: Evolution of an immersed boundary subjected to its elastic force

4 CONCLUSIONS

In this paper we have presented a finite element formulation of the popular immersed boundary method. The main advantage in using a finite element formulation with respect to the traditional finite difference approach is that the Dirac mass present in the formulation can be dealt with in a natural variational way. The reported numerical results show the excellent behavior of our scheme. The reader interested in some preliminary numerical analysis, including a stability analisys, is referred to [28, 29, 32, 33].

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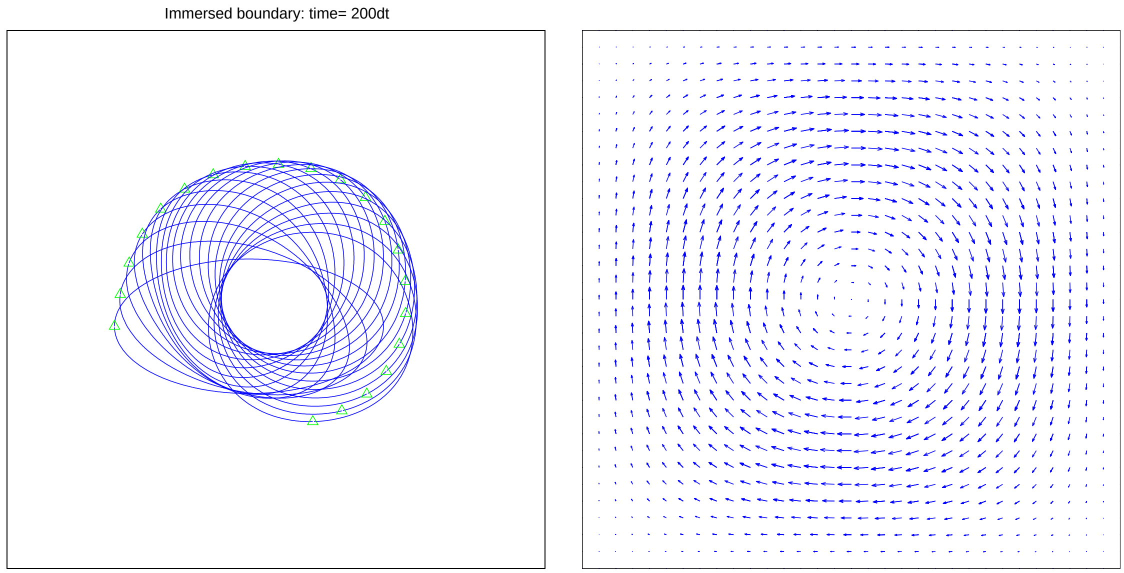

(a) The rotating boundary (b) The rotating fluid

Figure 2: Ellipse driven by a rotating fluid

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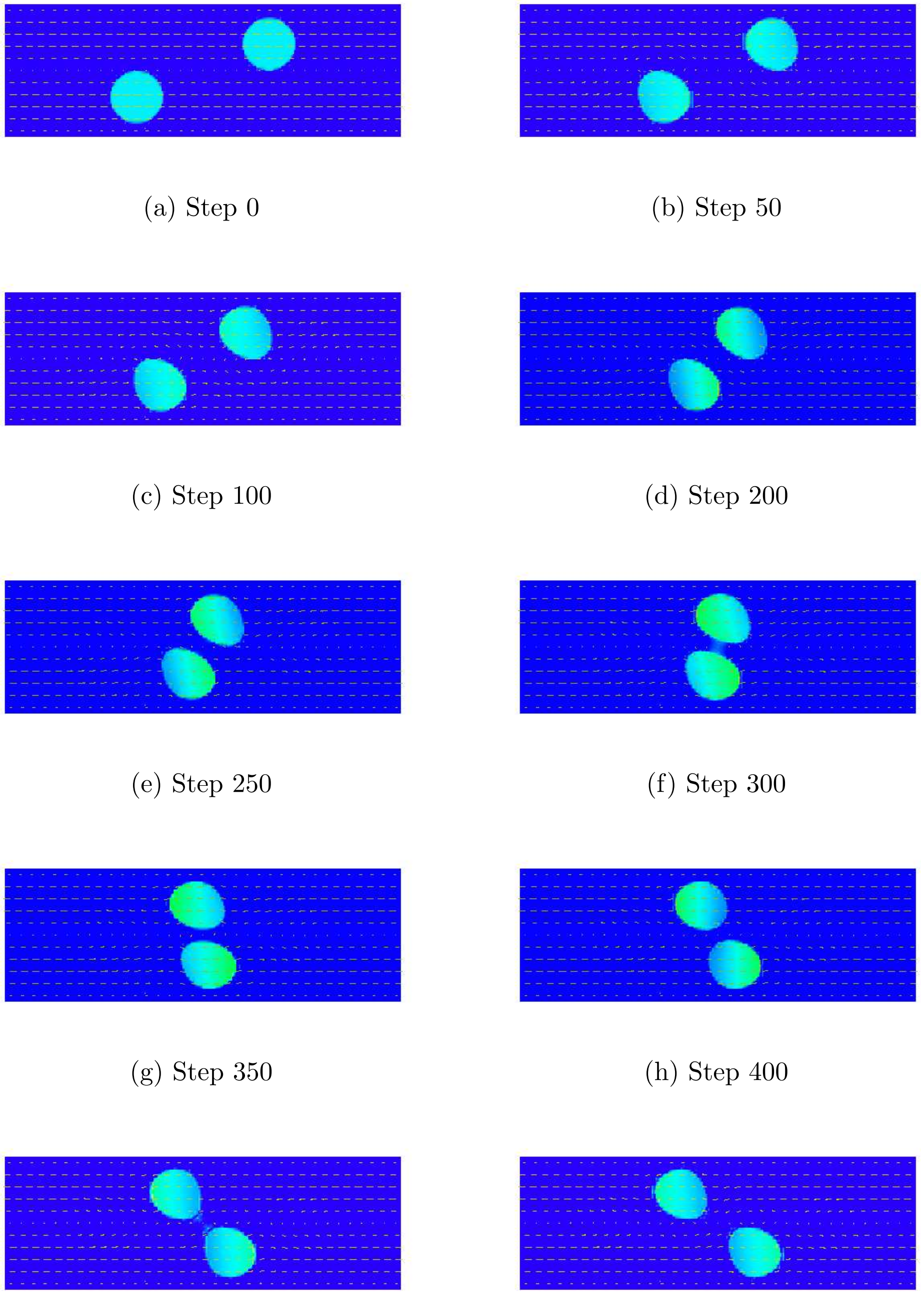

(a) Step 0

(b) Step 50

(c) Step 100

(d) Step 200

(e) Step 250

(f) Step 300

(g) Step 350

(h) Step 400

(i) Step 450

(j) Step 500

Figure 3: Two immersed boundaries driven by laminar flows

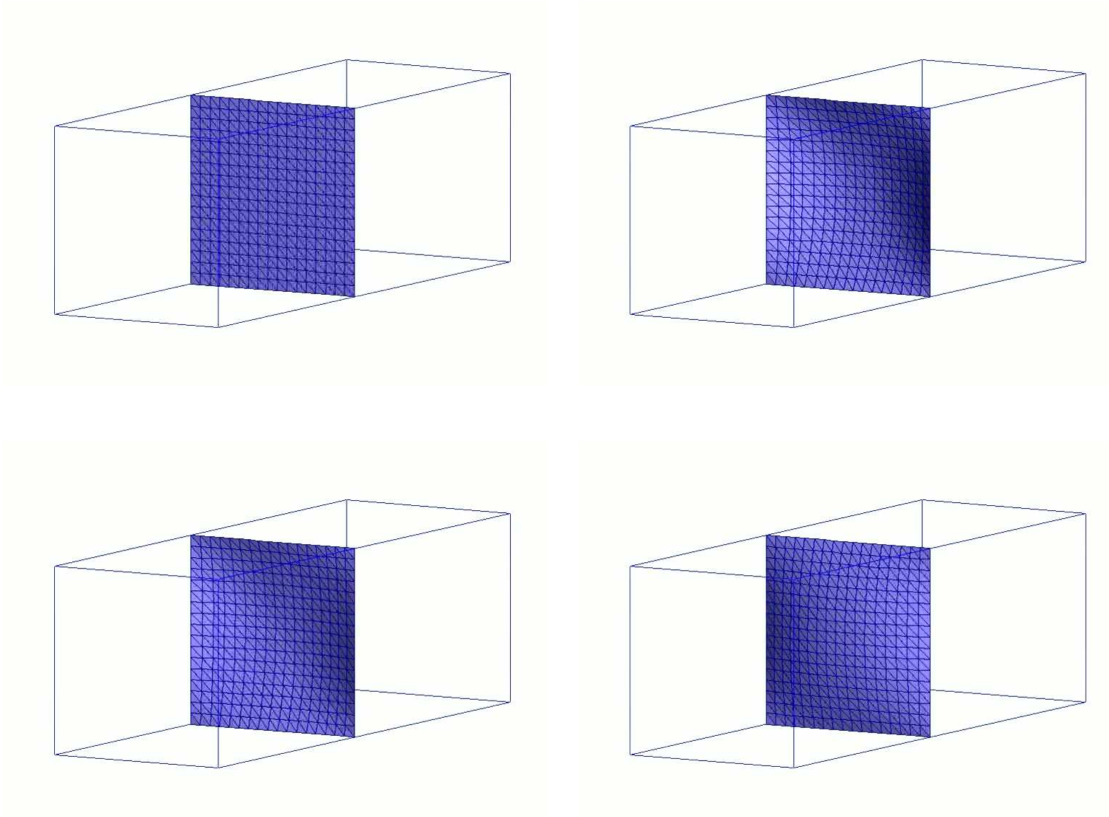

Figure 4: Membrane driven by a periodically moving fluid

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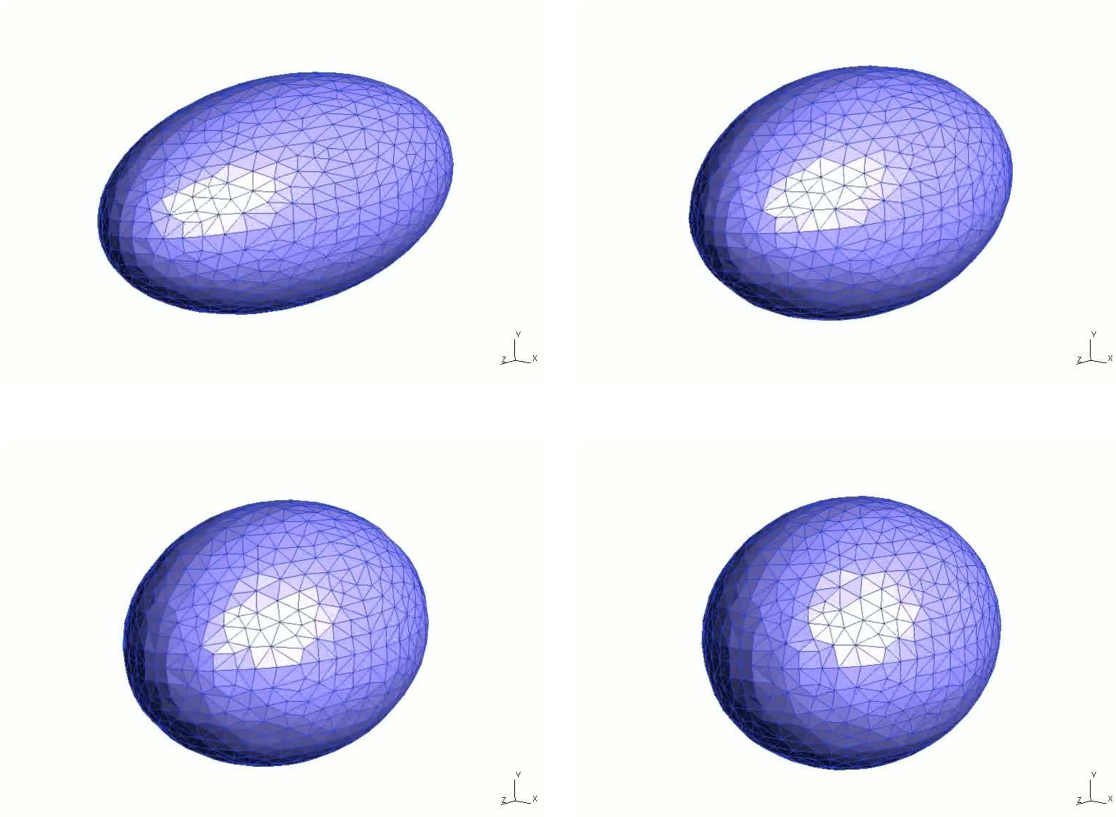

Figure 5: Ellipsoid immersed in a static fluid

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