ICOSAHEDRAL SETS IN PG(5, 2)

RON SHAW
School of Mathematics, University of Hull, Hull, HU6 7RX, England
Emeritus Professor Ron Shaw,
School of Mathematics,
University of Hull,
Hull HU6 7RX,
England
Abstract

Starting out from the 15 pairs of opposite edges and the 20 faces of a coloured icosahedron, a simple new construction is given of a “double-five” of planes in $PG(5, 2)$. This last is a recently discovered configuration consisting of a set $\psi$ of $(15 + 20 =) 35$ points in $PG(5, 2)$ which admits two distinct decompositions $\psi = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_5 = \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4 \cup \beta_5$ into a set of five mutually skew planes. Moreover $\lambda_r = \alpha_r \cap \beta_r$ is a line, for each $r$, while $n_{rs} = \alpha_r \cap \beta_s$ is a point, for $r \neq s$. The new construction illuminates why the symmetry group of $\psi$ is isomorphic to $A_5 \times Z_2$. The set $\psi$ is a set of hyperbolic type, and it has a cubic equation.

Use instead of the 15 pairs of opposite edges and the 12 vertices of the icosahedron yields a set $\phi \subset PG(5, 2)$ of $(15 + 12 =) 27$ points. The set $\phi$ is of elliptic type, but, like $\psi$, has cubic equation and icosahedral symmetry. The sets $\psi$ and $\phi$ are of Tonchev type 3b, see [15, Table I].
1. Introduction

1.1. A new configuration in \( PG(5, 2) \)

Consider a configuration \( \psi \) of 35 points in \( PG(5, 2) \) which admits two distinct decompositions

\[
\psi = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_5 = \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4 \cup \beta_5 \tag{1.1}
\]

into a set of five planes such that the following rather stringent incidence requirements are satisfied:

\[
\begin{align*}
\alpha_r \cap \alpha_s &= \emptyset & \beta_r \cap \beta_s &= \emptyset \\
\alpha_r \cap \beta_r &= \text{a line } \lambda_r & \alpha_r \cap \beta_s &= \text{a point } n_{rs}
\end{align*}
\tag{1.2}
\]

(with \( r \neq s \)). The 35 points of such a configuration \( \psi \), to be called a double-five of planes, can conveniently be displayed as an array

\[
\psi = \begin{pmatrix}
\lambda_1 & n_{12} & n_{13} & n_{14} & n_{15} \\
n_{21} & \lambda_2 & n_{23} & n_{24} & n_{25} \\
n_{31} & n_{32} & \lambda_3 & n_{34} & n_{35} \\
n_{41} & n_{42} & n_{43} & \lambda_4 & n_{45} \\
n_{51} & n_{52} & n_{53} & n_{54} & \lambda_5
\end{pmatrix}.
\tag{1.3}
\]

Here the 7-point set of the \( r \)th row is the plane \( \alpha_r \), and the 7-point set of the \( s \)th column is the plane \( \beta_s \), the five \( \alpha \)-planes being mutually skew, as are the five \( \beta \)-planes. Thus the 20 “off-diagonal” points \( n_{rs}, r \neq s \), of \( \psi \) are distinct, and the remaining 15 “diagonal” points of \( \psi \) form 5 mutually skew lines \( \lambda_r, r = 1, \ldots, 5 \).

Clearly the incidence requirements are extremely stringent, but nevertheless the author’s conjecture that such a configuration should exist was, in [12], shown to be true.

**Theorem 1.1.** A double-five of planes exists in \( PG(5, 2) \) and is a projectively unique figure.

**Proof.** The main method of construction of a double-five in [12] started out from a fixed choice of the six elements \( \{ n_{i4}, n_{i5}; i = 1, 2, 3 \} \), which, by the incidence requirements, have to be a basis for the vector space \( V(6, 2) = PG(5, 2) \cup \{0\} \). Then a straightforward check showed that the array (1.3) can always be completed to form a double-five. In fact one finds that it can be completed in only two ways, to form double-fives, say \( \psi, \psi' \). Moreover the change of basis involving the interchange \( n_{i4} \leftrightarrow n_{i5} \) leads to the interchange \( \psi \leftrightarrow \psi' \). Consequently the incidence requirements 1.2 are so restrictive as
to entail that a double-five is a projectively unique figure in \( PG(5, 2) \); that is the set of double-fives forms a single \( GL(6, 2) \)-orbit.

For more details of the proof, and also other (non-icosahedral) constructions of a double-five, see [12].

**Theorem 1.2.** Given a double-five \( \psi \), displayed as in the array (1.3), then \( \psi \) determines uniquely:

1) the five planes \( \alpha_r \) and the five planes \( \beta_r, r = 1, \ldots, 5 \), there being no other planes lying on \( \psi \);
2) the five solids (that is \( PG(3, 2) \)'s) \( \sigma_r = \text{join}(\alpha_r, \beta_r) \);
3) the set \( \Sigma = \Sigma(\psi) = \{ \lambda_1, \ldots, \lambda_5 \} \) of five skew lines, where \( \lambda_r = \alpha_r \cap \beta_r \);
4) the set of fifteen “diagonal points” which lie on these five lines;
5) the set \( \{ n_{rs} \equiv \alpha_r \cap \beta_s; r \neq s \} \) of twenty “off-diagonal” points;
6) a privileged hyperplane \( \varpi = \varpi(\psi) \), to be referred to as the even hyperplane, namely that spanned by the five lines \( \lambda_r \);
7) a privileged point \( u = u(\psi) \in \varpi \), to be referred to as the nucleus of \( \psi \), namely that point which satisfies

\[
    n_{rs} + n_{sr} = u, \quad \text{for all } r \neq s. \tag{1.4}
\]

**Proof.** See [12]. Of course the uniqueness claimed in respect of items 2) - 5) follows from that in item 1).

**Remark 1.** We refer to \( \varpi \) as the even hyperplane of \( \psi \) because it consists of those vectors having even weight relative to bases of the kind \( \{ n_{i4}, n_{i5} \} \) mentioned above. The same is true relative to the bases considered later in Sec. 3. On the other hand the twenty off-diagonal points \( n_{rs} \) all have odd weights, and so lie off \( \varpi \). We refer to \( u \) as the nucleus of the double-five \( \psi \) for the following reason: the fifteen diagonal points constitute a parabolic quadric \( \mathfrak{P}_4 \subset \varpi = PG(4, 2) \) (in fact, see Sec. 4.2, the only \( \mathfrak{P}_4 \) lying on \( \psi \)), whose nucleus is \( u \).

**1.2. The symmetry group of a double-five**

Let \( G(\psi) \subset GL(6, 2) \) denote the symmetry group of the double-five \( \psi \). If the hyperplane \( \varpi \) has equation \( \chi(x) = 0 \), then consider the transvection \( J \in GL(6, 2) \) defined by

\[
    Jx = x + \chi(x)u, \quad x \in V(6, 2). \tag{1.5}
\]

Noting that \( Jx = x \) for \( x \in \varpi \), and, by Eq. (1.4), \( Jn_{rs} = n_{sr} \), we see that \( J \) belongs to \( G(\psi) \) and achieves the interchange

\[
    J\alpha_r = \beta_r, \quad J\beta_r = \alpha_r. \tag{1.6}
\]
Theorem 1.3. The symmetry group $G(\psi)$ of a double-five \(\psi\) has the structure

$$G(\psi) = G_0(\psi) \times Z_2, \quad \text{with} \quad G_0(\psi) \cong A_5, \quad \text{and} \quad Z_2 = \langle J \rangle,$$

where $G_0(\psi)$ denotes the subgroup consisting of those symmetries which do not interchange the $\alpha$-planes and $\beta$-planes.

Proof. Clearly any $A \in G(\psi)$ either permutes the $\alpha$-planes, and also the $\beta$-planes, amongst themselves in the manner

$$A\alpha_r = \alpha_{\rho(r)}, \quad A\beta_r = \beta_{\rho(r)},$$

for some permutation $\rho$ of 12345, or else $A$ maps the $\alpha$-planes onto the $\beta$-planes, and vice versa, in which case $A \circ J$ satisfies Eq. (1.8). It was shown in [12] that for any even permutation $\rho$ of 12345, but for no odd permutation, there exists $A = A(\rho) \in GL(6,2)$ satisfying Eq. (1.8), with such an $A$ easily seen to be unique. \(\blacksquare\)

Remark 2. By theorems 1.1, 1.3, the double-fives in $PG(5,2)$ form a single $GL(6,2)$-orbit with stabilizer group of order 120. So the length of this orbit is $|GL(6,2)|/120 = 2^{12}.3^3.7^2.31 = 167989248$.

1.3. Sets of hyperbolic and elliptic types

Given a hyperbolic quadric $\mathcal{H}_5$ in $PG(5,2)$, then any hyperplane $\pi \subset PG(5,2)$ intersects the 35-point set $\mathcal{H}_5$ in either the 15 points of a parabolic quadric $\mathcal{P}_4$ or in the 19 points of a hyperbolic cone $\Pi_0\mathcal{H}_3$. Given instead an elliptic quadric $\mathcal{E}_5$ in $PG(5,2)$, then any hyperplane $\pi \subset PG(5,2)$ intersects the 27-point set $\mathcal{E}_5$ in either the 15 points of a parabolic quadric $\mathcal{P}_4$ or in the 11 points of an elliptic cone $\Pi_0\mathcal{E}_3$. See, for example, [8]. Sets of points whose intersection properties with hyperplanes mimic those of hyperbolic and elliptic quadrics are of interest to design theorists and code theorists, see [15], and one therefore (in the present $PG(5,2)$ case) makes the following definitions.

Definition 1.4. A 35-point set $\psi \subset PG(5,2)$ is said to be a set of hyperbolic type if it has the intersection property

$$|\psi \cap \pi| \in \{15, 19\}, \quad \text{for any hyperplane} \ \pi \subset PG(5,2).$$

(1.9)

A 27-point set $\phi \subset PG(5,2)$ is said to be a set of elliptic type if it has the intersection property

$$|\phi \cap \pi| \in \{11, 15\}, \quad \text{for any hyperplane} \ \pi \subset PG(5,2).$$

(1.10)

Lemma 1.5. A double-five $\psi \subset PG(5,2)$ is a set of hyperbolic type.
Proof. The mere fact that a double-five \( \psi \subseteq PG(5,2) \) is the disjoint union \( \psi = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \cup \alpha_5 \) of one set of five planes already implies that \( \psi \) has the requisite intersection property with hyperplanes. For since \( |\alpha_r \cap \pi| \in \{3,7\}, r = 1, \ldots, 5 \), it follows that \( |\psi \cap \pi| \) is either \( 5 \times 3 = 15 \) or else \( 7 + 4 \times 3 = 19 \). (Of course \( \pi \) can not contain two of the planes \( \alpha_r \), for then \( \pi \) would contain their join, namely \( PG(5,2) \), and so not be a hyperplane.)

1.4. Plan

Note from equation 1.7 that a double-five has the same symmetry group \( A_5 \times Z_2 \) as that of a regular icosahedron. The chief aim of the present article is to show that the fact that a double-five has icosahedral symmetry is not a coincidence.

We commence, in Sec. 2, by spelling out in some detail various facts concerning the regular icosahedron, and especially concerning its colouring; in that section we also agree upon certain labelling conventions. These preliminary considerations bear fruit in Sec. 3.2, where we describe an elegant construction of a double-five out of the 15 pairs of opposite edges and the 20 faces of a coloured icosahedron. The resulting clear view of its symmetries is described in Sec. 3.3, where, in terms of a basis arising from our icosahedron, we explain how all 120 elements of \( G(\psi) \) can be written down with ease. In Secs. 3.4-3.6 we consider (i) the parabolic quadric \( \mathfrak{P}_4 \), and its six spreads of lines, formed by the 15 diagonal points of \( \psi \) (ii) the 105 lines lying on \( \psi \), and (iii) the \( G(\psi) \)-orbits of hyperplanes.

Arising from the same coloured icosahedron, we show in Sec. 4.1 that there are five other double-fives which share the same diagonal \( \mathfrak{P}_4 \) as \( \psi \), but which make use of different spreads of lines on \( \mathfrak{P}_4 \). In Sec. 4.2, starting out again from the coloured icosahedron, but using instead the 15 pairs of opposite edges and the 12 vertices of the icosahedron, we construct a set \( \phi \) of 27 points in \( PG(5,2) \), called an icosahedral twenty-seven, which has the same icosahedral symmetry \( G(\psi) \) as \( \psi \), and which is a set of elliptic type. We also show that our sets \( \psi \) and \( \phi \) are of type 3b in the classification of Tonchev [15, Table I]. In Sec. 4.2 we further show that the double-five \( \psi \) contains five subsets \( \phi_r = \psi \setminus (\alpha_r \Delta \beta_r) \), each of elliptic type 3b. Moreover, by adjoining to \( \phi \) eight suitable points (namely the eight points corresponding, see Sec. 3.2, to the eight icosahedral faces of colour \( r \)) the elliptic set \( \phi \) is seen to lie upon five double-fives \( \psi_r \).

In Sec. 5.1 we provide explicit coordinate forms for the planes \( \alpha_r, \beta_r \). In Sec. 5.2 we point out that a result in [10], or in [7], implies that both \( \psi \) and \( \phi \) have cubic equations, and, in a suitable basis, we provide coordinate expressions for these cubics. (In fact, see [12], all of the hyperbolic and elliptic sets in \( PG(5,2) \) listed in [15, Table I] — other of course than the quadrics themselves, listed as type 1 — have cubic equations.) Finally, in Sec. 6, we touch upon certain duality considerations, and also enquire whether our methods are capable of generalization.
2. Icosahedral Preliminaries

We need to remind ourselves of certain facts concerning the regular icosahedron $I$. Let $V, E, F$ denote the sets consisting respectively of the 12 vertices, 30 edges, 20 faces of $I$. The 6 antipodal pairs of vertices yield the 6 diagonals of $I$, the set $D = \{D_1, D_2, D_3, D_4, D_5, D_6\}$ of these diagonals forming a set of equiangular lines in Euclidean 3-dimensional space. The symmetry group of $I$ is $G(I) = G_0(I) \times Z_2$, of order 120, where $Z_2$ is generated by the symmetry which sends each vertex $X$ into its antipode $^\ast X$, and where the group $G_0 = G_0(I)$ of direct symmetries is isomorphic to the alternating group $A_5$ of order 60.

**Definition 2.1.** A pair of opposite edges of $I$ will be termed an Edge.

Let $P$ denote the set of 15 Edges. Each of the $\binom{6}{2} = 15$ unordered pairs $\{D_i, D_j\}$ of diagonals gives rise, in an obvious manner, to a corresponding Edge, say $E_{ij} \in P$, and so we may also identify $P$ with the set consisting of the 15 pairs of diagonals. Now $G_0$ acts transitively upon $P$; however $G_0$ is not transitive upon the set $T$ consisting of the $\binom{6}{3} = 20$ unordered triples of diagonals. Indeed $T$ splits into two $G_0$-orbits of length 10, say

$$T = \Omega \cup \Omega^* , \quad (2.1)$$

with $\Omega$ comprising ten triples of acute type and $\Omega^*$ 10 triples of obtuse type, cf. [9]. We may identify the orbit $\Omega$ with the ten pairs of opposite faces of $I$. (Caution: we do not refer to a pair of opposite faces as a Face! — since this latter term is later used for something different, see Sec. 2.5. In our later finite geometry deliberations, it turns out that opposite edges of $I$ map on to the same point of $PG(5, 2)$, while opposite faces map on to different points.)

2.1. The coloured icosahedron

The group $G_0$ acts imprimitively upon the sets $P, F$. The 15 Edges form 5 blocks of size 3, with the three Edges of a block being mutually orthogonal. The set $F$ of the 20 faces can be partitioned in two distinct ways, one “left-handed” and one “right-handed”, into 5 blocks of size 4, with the four faces of a block constituting, when appropriately extended, the four faces of a regular tetrahedron. (Cf. Coxeter [4, §3.6] for the dual situation with regard to the regular dodecahedron.)

In Fig. ?? the five blocks of edges of the icosahedron are assigned the five “colours” 1, 2, 3, 4, 5. (For the moment the vertex labels in this diagram should be ignored.) The same colours are assigned, in two enantiomorphic ways, to the two systems of blocks of faces: if a face has colour $r$ in the left-handed system, then its opposite face is assigned colour $r$ in the right-handed system. A face coloured $r$ in the left-handed system and $s$ in the right-handed system is labelled $rs$ in the figure. Now these face-colourings have
the property that the labels $pq, rs$ of any two adjacent faces use four distinct colours $pqrs$. Consequently, as in the figure, the edge-colouring can be tied in nicely with the two face-colourings by assigning the fifth colour $t$ to the shared edge of two adjacent faces. A simple way to construct such a diagram is to start out from any colouring of the five blocks of edges, and then, when travelling along an edge of colour $r$, assign left-hand colour $r$ to the face in “forward-left” position and right-hand colour $r$ to the face in “forward-right” position. See Fig. ?? for a local view of the colouring, valid at any location (for appropriate vertices $A, B, C, D, E, F$).

Each direct symmetry $g \in G_0(3)$ gives rise to its own even permutation $g_c$, say, of the five colours. For example consider the cases where $g$ is a rotation through an angle (a) $\frac{2\pi}{5}$, (b) $\frac{2\pi}{3}$, (c) $\frac{2\pi}{2}$ about (a) the diagonal $D_6$, (b) the line joining the centres of the faces 54 and 45, (c) the line joining the mid-points of the edges of the pair $E_{56}$. Then from Fig. ?? we see immediately that $g_c$ (for a suitable sense of the rotation, in cases (a), (b)) is given by

\begin{align}
(a) \quad & g_c = (12345), \\
(b) \quad & g_c = (123), \\
(c) \quad & g_c = (12)(34) .
\end{align}

So under $g \mapsto g_c$ the group $G_0$ is mapped onto the group $A_6^{\text{colour}} \cong A_5$ of all even permutations of the five colours 1, 2, 3, 4, 5.

### 2.2. Diagonal-labelling

We will adopt a numbering of the 6 diagonals such that the two $\frac{2\pi}{3}$-rotations about the diagonal $D_6$ give rise to the cyclic permutations $(D_1D_2D_3D_4D_5)$ and $(D_5D_4D_3D_2D_1)$ of the remaining five diagonals, so that the environment of one end of the diagonal 6(= $D_6$) is $2 \ 6 \ 4 \ 1 \ 5$. The environments of the neighbouring ends of the other five diagonals are then determined:

\begin{align}
2 & \ 6 & \ 5 \\
3 & \ 1 & \ 5 \\
6 & \ 2 & \ 1 \\
4 & \ 3 & \ 2 \\
5 & \ 4 & \ 3 \\
1 & \ 6 & \ 4
\end{align}

Under this choice of numbering we will often identify the set $D$ of the six diagonals with the set $\{1, 2, 3, 4, 5, 6\}$, and so interpret the alternating and symmetric groups $A_6$ and $S_6$ in terms of permutations of the diagonals. Each direct symmetry $g \in G_0(3)$ thus gives rise to a permutation $g_d \in S_6$. For $g$ as in Eq. (2.2) we read off from Fig. ?? the corresponding $g_d$:

\begin{align}
(a) \quad & g_d = (12345), \\
(b) \quad & g_d = (425)(361), \\
(c) \quad & g_d = (14)(56) .
\end{align}

Observe that these $g_d$ are all even permutations, and so the image, $A_6^{\text{diag}}$ say, of $G_0$ in $S_6$ under the mapping $g \mapsto g_d$ lies in the subgroup $A_6$ of $S_6$. 

9
With our choice of numbering of the diagonals, the two $A_5^{\text{diag}}$-orbits of triples in Eq. (2.1) are

$$\Omega = \{126, 236, 346, 456, 516, 124, 235, 341, 452, 513\} \quad (2.5)$$

and

$$\Omega^* = \{136, 246, 356, 416, 526, 123, 234, 345, 451, 512\}. \quad (2.6)$$

Note, for any permutation $ijklmn$ of 123456, that the complementary triples $ijk, lmn$ lie on different orbits.

Let $\Lambda_r$ denote the block $\{E_{ij}, E_{kl}, E_{mn}\}$ of three perpendicular Edges of colour $r$. Then $\Lambda_r$ determines a syntheme $ijklmn$ consisting of the three disjoint duads $ij, kl, mn$. In the adopted colouring and numbering, the five blocks of Edges $\Lambda_1, \ldots, \Lambda_5$ correspond to the following five synthemes:

$$\Lambda_1 \leftrightarrow 345216 \quad \Lambda_2 \leftrightarrow 451326 \quad \Lambda_3 \leftrightarrow 125436 \quad \Lambda_4 \leftrightarrow 123546 \quad \Lambda_5 \leftrightarrow 234156 \quad (2.7)$$

where, of course, each of the fifteen duads $ij$ formed from the numbers 123456 occurs just once. In other words the five synthemes constitute a total. Our terminology is that of Sylvester, see [13], [14]; other terminology is used, see e.g. [2, Ch. 6]. Note that we can summarize the preceding result in the form

$$\Lambda_r = \{E_{r+2r-2}, E_{r-1r+1}, E_{r6}\}, \quad (2.8)$$

provided that we allow $r$ to take the values 1, 2, 3, 4, 5 modulo 5.

**Remark 3.** The pair of opposite edges of an Edge are the short sides of a golden rectangle, and so each of the five blocks of Edges yields a set of three mutually perpendicular golden rectangles, the twelve vertices of which are the twelve vertices of the icosahedron. See [3, Fig. 11.2b], where it is noted that this property of the regular icosahedron was described by Fra Luca Pacioli as “the twelfth almost incomprehensible effect” in his De Divina Proportione (1509). It should be noticed that each set $\{R, R', R''\}$ of three golden rectangles has a natural cyclic order, with $R$ piercing $R'$, $R'$ piercing $R''$, and $R''$ piercing $R$. This cyclic order has been followed in Eqs. (2.7), (2.8).

**Remark 4.** A purely combinatorial recipe can be given to harmonize the cyclic order for the five synthemes within any total. For, given any two synthemes $\Lambda = d_1d_2d_3$ and $\Lambda' = d'_1d'_2d'_3$ belonging to the same total, we may initially order $\Lambda'$ relative to $\Lambda$ by demanding that the duad $d'_i$ should be disjoint from the duad $d_i$, for each $i = 1, 2, 3$. Then our recipe is that the cyclic order for $\Lambda'$, relative to the agreed cyclic order for $\Lambda$, should be the opposite to the order $d'_1d'_2d'_3$. 

10
Remark 5. It should be stressed that the six diagonal symbols $i = 1, 2, 3, 4, 5, 6$ are not simply related to the five colours $r = 1, 2, 3, 4, 5$, and in many ways it would have been preferable to use a different set of symbols, e.g. $r, o, y, b, g$, for the colours. (So no great significance — other than that of convenience, cf. Eq. (2.8) — should be read into the fact that in Fig. ?? we chose to colour the edges $E_{6r}$ with the colour $r$.) However one of the many fascinations of the icosahedron is, see Sec. 2.4 below, how $G_0$ manages to be faithfully represented both as the group $A_{\text{colour}}^5$ consisting of the even permutations of five colours, and as a subgroup $A_{\text{diag}}^5$ of the group $A_6$ consisting of all the even permutations of the six diagonals.

2.3. Vertex-labelling

Each diagonal $D_i, i = 1, 2, \ldots, 6$, joins an antipodal pair $\{V_i, V_i(= \hat{V}_i)\}$ of vertices of $\mathcal{I}$. Which vertex is labelled $i$; and which $\hat{i}$, is an extra fussy detail which needs to be settled, for each $i = 1, 2, \ldots, 6$, when we extend our numbering of the 6 diagonals to a labelling of the 12 vertices. It will prove convenient to use the labels $1, 2, 3, 4, 5, 6$ for the vertices forming one of the pentagonal “caps” on $\mathcal{I}$, referred to as the arctic cap, and hence $1, 2, 3, 4, 5, 6$ for the vertices of the antipodal cap, the antarctic cap. A vertex-labelling of this type will be termed a polar labelling. For the icosahedron in Fig. ??, we have adopted a polar labelling with the vertices $6 = V_6$ and $\hat{6} = \hat{V}_6$ the north and south poles. The environments of the vertices $1, 2, 3, 4, 5, 6$ are then, cf. Eq. (2.3) and Fig. ??, as follows:

\[
\begin{array}{cccccccc}
2 & 6 & 5 & 3 & 6 & 1 & 4 & 6 \\
3 & 4 & \hat{5} & 4 & 1 & 5 & 2 & 1 \\
1 & 5 & 6 & 3 & 2 & \hat{4} & 3 & 2
\end{array}
\]

(2.9)

In the notation of Eq. (2.8) the fifteen Edges can be classified into:

(a) five tropical Edges $E_{r+2r-2} = \{V_{r+2}V_{r-2}, \hat{V}_{r+2}\hat{V}_{r-2}\}$;
(b) five equatorial Edges $E_{r-1r+1} = \{V_{r-1}V_{r+1}, \hat{V}_{r-1}\hat{V}_{r+1}\}$;
(c) five polar Edges $E_{r6} = \{V_rV_6, \hat{V}_r\hat{V}_6\}$.

(2.10)

Let $f_{rs}$ denote the face in Fig. ?? with colour label $rs$. Then, in our polar labelling of the vertices, the twenty faces $f_{rs}$, $r \neq s$, can be classified into:

(a) five arctic faces $f_{r-1r+2} = V_rV_{r+1}V_6$;
(b) five antarctic faces $f_{r+2r-1} = \hat{V}_r\hat{V}_{r+1}\hat{V}_6$;
(c) five north equatorial faces $f_{r+1r} = V_rV_{r+1}\hat{V}_{r+3}$;
(d) five south equatorial faces $f_{rr+1} = \hat{V}_r\hat{V}_{r+1}V_{r+3}$.

(2.11)
Each symmetry \( g \in G(3) \) gives rise to a permutation \( g_v \) of the twelve vertex labels. For \( g \) as in Eq. (2.4) we read off from Fig. ?? the corresponding \( g_v \):

\[
\begin{align*}
(a) g_v &= (12345)(12345), \\
(b) g_v &= (425)(425)(361)(361), \\
(c) g_v &= (14)(56)(14)(56)(22)(33).
\end{align*}
\]

(2.12)

Of course the permutation representation \( g \to g_v \) is imprimitive, since \( G(3) \) maps antipodal pairs of vertices into antipodal pairs.

2.4. The group \( G_0 \) as an \( A_5 \) subgroup of \( A_6 \)

As in Sec. 2.2 the mapping \( g \to g_d \) injects \( G_o \cong A_5 \) into the subgroup \( A_6 \) of the group \( S_6 \) of permutations of the six diagonals \( D \). Now of course \( A_6 \) possesses one obvious class of six \( A_5 \) subgroups, namely the six stabilizers \( \text{stab}_0(i) = \text{stab}(i) \cap A_6 \) of the six symbols \( i = 1, 2, \ldots, 6 \). The image \( A_5^{\text{diag}} \) of \( G_0 \) in \( A_6 \) is not however one of this class, since \( A_5^{\text{diag}} \) is transitive (indeed doubly-transitive) on \( D \). So \( A_5^{\text{diag}} \) belongs to the “other” class of six \( A_5 \) subgroups of \( A_6 \), namely the six stabilizers \( \text{stab}_0(T_i) = \text{stab}(T_i) \cap A_6 \) of the six totals \( T_i \). See the Appendix.

Now the total in Eq. (2.7) is the total \( T_6 \) in the array (A.2); indeed, due to our colouring and numbering conventions, we have

\[
\Lambda_{r6} = \Lambda_r, \quad r = 1, \ldots, 5,
\]

(2.13)

where \( \Lambda_{ij} \) is as defined in Eq. (A.1). Consequently we have

\[
A_5^{\text{diag}} = \text{stab}_0(T_6).
\]

(2.14)

Since the block/syntheme \( \Lambda_r \) was assigned colour \( r \), and since for, \( \sigma \in \text{stab}(T_6) \), we have, from Eqs. (A.4), (2.13), \( \sigma(\Lambda_r) = \Lambda_{\rho(r)} \), we can identify \( A_5^{\text{colour}} \) with \( \text{stab}_0(6) \).

Remark 6. The involutory outer automorphism \( \theta \) of \( S_6 \) considered in the appendix thus swaps \( g_c \) with \( g_d \), see after Eq. (A.4), and so the effect of \( \theta \) (= \( \theta^{-1} \)) upon the elements of \( A_5^{\text{colour}} = \text{stab}_0(6) \) can be effortlessly read off from Fig. ?? . For example, in the case of the previously considered rotations through angles \( (a) \frac{2\pi}{5}, \quad (b) \frac{2\pi}{3}, \quad (c) \frac{2\pi}{7}, \)

, see Eqs. (2.2), (2.4), we see immediately that

\[
\begin{align*}
(a) \theta((12345)) &= (12345), \\
(b) \theta((123)) &= (425)(361), \\
(c) \theta((12)(34)) &= (14)(56).
\end{align*}
\]

(2.15)
2.5. The associated great icosahedron $\mathcal{J}^*$

Associated with the icosahedron $\mathcal{J}$, with Schläfli symbol $\{3,5\}$, is the great icosahedron $\mathcal{J}^*$, with Schläfli symbol $\{3,\frac{5}{2}\}$, see e.g. [4], which shares the same twelve vertices $V_1, \ldots, V_6, V_{\bar{1}}, \ldots, V_{\bar{6}}$ as $\mathcal{J}$. We refer to the (thirty) edges and (twenty) faces of $\mathcal{J}^*$ as great edges and great faces. If $A, B$ are two vertices, then $AB$ is a great edge if and only if $AB$ is an edge, and conversely. A pair of opposite great edges will be called a great Edge; such pairs are the longer edges of the golden rectangles of Remark 3. In Fig. ?? we will refer to the great face $DEF$ as the one paired with the face $ABC$. Occasionally we will refer to such a pair $\{ABC, DEF\}$ as a Face. As far as the triples of diagonals $T$ are concerned, the pairs of opposite great faces correspond to the orbit $\Omega^*$ in Eq. (2.6).

3. Icosahedral construction of a double-five

Let us fix a basis $\mathcal{B} = \{b_1, \ldots, b_6\}$ for the 6-dimensional vector space $V(6,2)$ over the field $\mathbb{F}_2 = \{0,1\}$, and set $u = \sum_{i=1}^{6} b_i$ and $\bar{b}_i = b_i + u$. Then $\bar{\mathcal{B}} = \{\bar{b}_1, \ldots, \bar{b}_6\}$ is also a basis. We also set

$$
\begin{align*}
    b_{ij} &= b_i + b_j = \bar{b}_i + \bar{b}_j, \\
    b_{ijk} &= b_i + b_j + b_k, \\
    \bar{b}_{ij} &= b_{ij} + u = \bar{b}_i + b_j, \\
    \bar{b}_{ijk} &= b_{ijk} + u = b_{lmn},
\end{align*}
$$

(3.1)

and so on, where here, and in the ensuing, $ijklmn$ denotes a permutation of $123456$.

In terms of the basis $\{b_1, \ldots, b_6\}$ the 63 points of the projective space $PG(5,2) = V(6,2) - \{0\}$ are as follows:

(a) 6 points $b_j$ of “weight” 1; 6 points $\bar{b}_i = b_{jklmn}$ of weight 5;
(b) 15 points $b_{ij}$ of weight 2; 15 points $\bar{b}_{ij} = b_{klmn}$ of weight 4;
(c) 20 points $b_{ijk}$ of weight 3;
(d) 1 point $u (= \text{the unit point})$ of weight 6.

Now we can pick out 63 elements relating to our pair $\mathcal{J}, \mathcal{J}^*$ as follows

(a) 12 vertices: 6 arctic ones $V_i$ and 6 antarctic ones $V_i$;
(b) 15 Edges and 15 great Edges;
(c) 20 Faces;
(d) the centre $U$ of $\mathcal{J}$ (and of $\mathcal{J}^*$).
3.1. A bijection

We now define a bijection $b$ mapping these last $12 + 30 + 20 + 1$ elements onto the former $12 + 30 + 20 + 1$ points of $PG(5, 2)$ in the following obvious/naive manner:

(a) $b(V_i) = b_i$, $b(V^i) = b^i$;
(b) $b\{XY, X^iY^i\} = b(X) + b(Y)$;
(c) $b\{ABC, DEF\} = b(A) + b(B) + b(C)$;
(d) $b(U) = u$. \tag{3.2}

Concerning this definition of the bijection $b$, the following features should be noted:

(a) The vertices of the arctic cap of $\mathcal{I}$ are mapped onto the basis $\mathcal{B}$, and those of the antarctic cap are mapped onto the basis $\overline{\mathcal{B}}$.

(b) Here $XY$ denotes the edge, or great edge, joining vertices $X, Y$, and so $X^iY^i$ is the opposite edge or great edge. Since, by Eq. (3.2a), we have $b(\overline{X}) = b(X) + u$, and hence $b(X) + b(Y) = b(X) + b(Y)$, Eq. (3.2b) does successfully define $b$ on the Edges and great Edges. Note that 10 Edges map onto vectors of weight 2, namely the 5 tropical and 5 polar Edges of Eq. (2.10), while the 5 equatorial Edges map onto vectors of weight 4: $b(E_{r-1, r+1}) = b_{r-1, r+1} = b_{r, r-2, r+2}$. Similarly 5 great Edges map onto vectors of weight 2, while 10 great Edges map onto vectors of weight 4.

(c) As far as Eq. (3.2c) is concerned, despite appearances it treats a face $ABC$ equally with the paired great face $DEF$, on account of the result

$$b(A) + b(B) + b(C) = b(D) + b(E) + b(F).$$ \tag{3.3}

For in Fig. ?? note that $ABCDEF$ is a cap (with $C$ as pole); consequently Eq. (3.3) follows immediately from the following lemma.

Lemma 3.1. Let $X_1, \ldots, X_6$ be the vertices of any one of the twelve pentagonal caps of $\mathcal{I}$, and suppose that $\mathcal{I}$ is polar-labelled. Then $\{b(X_1), \ldots, b(X_6)\}$ is a basis for $V(6, 2)$ and $\sum_{i=1}^6 b(X_i) = u$.

Proof. By the definition of $u$, the result holds for the arctic cap. But the polar labelling scheme entails, see Fig. ?? or Eq. (2.9), that any cap shares an even number of vertices with the arctic cap, and an even number with the antarctic cap. Hence, since $b(V_i) = b_i + u$, the result holds for all twelve caps. \(\blacksquare\)

Lemma 3.2. If $\mathcal{I}$ is polar-labelled, then $b$ maps any three Edges of the same colour into collinear points of $PG(5, 2)$; that is, if $E, E', E''$ are any three perpendicular Edges of $\mathcal{I}$, then $b(E) + b(E') + b(E'') = 0$. 

For in Fig. ?? note that $ABCDEF$ is a cap (with $C$ as pole); consequently Eq. (3.3) follows immediately from the following lemma.
Proof. The result holds for the Edges coloured $p$ (and also for those coloured $q$) in Fig. ?? on account of Eq. (3.3). ■

Note: we will often, as immediately below, find it more convenient to deal with the 20 faces, rather than the 20 Faces, with $b(ABC)$ defined of course to be $b(A)+b(B)+b(C)$. Observe that if $\bar{f}$ is the face opposite to $f$, then

$$b(\bar{f}) = b(f).$$

3.2. A double-five of planes in $PG(5,2)$

Consider a configuration $\psi$ of $15 + 20 = 35$ points in $PG(5,2)$ corresponding under the bijection $b$ to the 15 Edges and 20 faces of $\mathcal{I}$. For the face $f_{rs}$ of $\mathcal{I}$ having colour label $rs$ in Fig. ?? we denote the corresponding point $b(f_{rs})$ in $PG(5,2)$ by $n_{rs}$. We also put $\lambda_r = b(A_r)$. Then we can display the 35 points of $\psi$ as an array

$$\psi = \begin{pmatrix}
\lambda_1 & n_{12} & n_{13} & n_{14} & n_{15} \\
n_{21} & \lambda_2 & n_{23} & n_{24} & n_{25} \\
n_{31} & n_{32} & \lambda_3 & n_{34} & n_{35} \\
n_{41} & n_{42} & n_{43} & \lambda_4 & n_{45} \\
n_{51} & n_{52} & n_{53} & n_{54} & \lambda_5
\end{pmatrix}.$$ (3.5)

Note that the 7-point set, $\alpha_r$ say, in the $r$th row arises from the 7 elements (three Edges and four faces) coloured $r$ in the left-handed system, and the 7-point set, $\beta_s$ say, in the $s$th column arises from the 7 elements coloured $s$ in the right-handed colouring system of the icosahedron.

Theorem 3.3. The configuration $\psi$ in Eq. (3.5), arising from the fifteen Edges and twenty faces of the icosahedron $\mathcal{I}$, is a double-five of planes in $PG(5,2)$. The nucleus of the double-five $\psi$ is the unit point $u = \sum_{i=1}^{6} b_i$ of the basis $B = \{b_1, \ldots, b_6\}$, and the even hyperplane $\varpi$ of $\psi$ consists of the vectors of even weight relative to this basis.

Proof. We have already seen, in Lemma 3.2, that the three Edges of colour $r$ are mapped onto a line, namely $\lambda_r$, of $PG(5,2)$. One way to see that each $\alpha_r$ and $\beta_s$ is indeed a plane is to display all 35 points of $\psi$ explicitly in terms of the basis $B = \{b_1, \ldots, b_6\}$. Using Eqs. (2.10), (2.11) and (3.2), we see that $\psi$ is explicitly

$$\psi = \begin{pmatrix}
345\overline{2}16 & 124 & 123 & 236 & 246 \\
356 & 45\overline{1}\overline{3}26 & 235 & 234 & 346 \\
456 & 416 & 51\overline{2}\overline{1}36 & 341 & 345 \\
451 & 516 & 526 & 12\overline{3}546 & 452 \\
513 & 512 & 126 & 136 & 23\overline{1}\overline{3}56
\end{pmatrix}.$$ (3.6)
(Alternatively, simply use Fig. ?? and Eq. (3.2).) Here $ij$, $\overline{ij}$, $ijk$ are abbreviations for $b_{ij}$, $\overline{b_{ij}}$, $b_{ijk}$, respectively. It is now apparent that each row $\alpha_r$, and each column $\beta_s$, of $\psi$ is indeed a plane, and that the five $\alpha$-planes, and also the five $\beta$-planes, are mutually skew. Hence $\psi$ is a double-five. From the form of the fifteen diagonal points in Eq. 3.6 it is clear that the even hyperplane $\varpi$ of $\psi$ consists of the vectors of even weight relative to the basis $\mathcal{B}$. Moreover, upon recalling that the face $f_{rs}$ is opposite to the face $f_{sr}$, note that we have, from Eq. (3.4),

\[ n_{rs} + n_{sr} = u, \quad (r \neq s), \]  

whence the unit point $u$ is the nucleus of the double-five $\psi$. $\blacksquare$

(See Secs. 3.4 and 4 below for further pleasing $PG(5, 2)$ manifestations of the colouring.)

### 3.3. Symmetries of a double-five

The double-five $\psi$ of Eq. (3.6) has a manifest $Z_5$ symmetry. For upon defining $A \in GL(6, 2)$ upon the preceding basis by $Ab_i = b_{\zeta(i)}$, with $\zeta = (12345)$, we see that

\[ An_{rs} = n_{r+s+1}, \quad A(\lambda_r) = \lambda_{r+1}, \]  

where $r$ takes the values $1, 2, 3, 4, 5$ modulo $5$. Thus the entries in the last four rows of $\psi$ follow immediately from those in the first row. In fact the full $A_5 \times Z_2$ symmetry of $\psi$ can be dealt with, in the above basis, equally explicitly and in a delightfully easy manner, by appeal once again to the coloured icosahedron in Fig. ??, as we now demonstrate.

#### Theorem 3.4.

Consider any direct symmetry $g \in G_0(\mathfrak{I})$ of the icosahedron. Then there exists an element $A(g) \in GL(6, 2)$ such that $A(g)$ permutes the twelve vectors $b_1, \ldots, b_6, \overline{b_1}, \ldots, \overline{b_6}$ in the “same” way that $g_{ss}$, cf. Eq. (2.12), permutes the twelve vertex labels 1, ..., 6, 1, ..., 6. Moreover $A(g) \in G_0(\psi)$. Similarly, corresponding to the central inversion symmetry $i \equiv \overline{1}$ of the icosahedron, the generator $J$ of the $Z_2$ symmetry of $\psi$ is given by

\[ Jb_i = \overline{b_i}, \quad i = 1, 2, \ldots, 6. \]  

**Proof.** Of course $g$ maps the arctic cap $\{V_1, \ldots, V_6\}$ onto another cap $\{gV_1, \ldots, gV_6\}$. By lemma 3.1 the image of the latter under the bijection $b$ is a another basis for $V(6, 2)$, and so we can define an element $A(g) \in GL(6, 2)$ by $A(g)b_i = b(gV_i)$. By lemma 3.1 again, $\sum_i b(gV_i) = u$, and so $A(g)u = u$. Consequently $A(g)b_i = b_j$ implies that $A(g)\overline{b_i} = \overline{b_j}$, and $A(g)b_i = \overline{b_j}$ implies that $A(g)\overline{b_i} = b_j$. It follows that the “over-definition” of $A(g)$ in the theorem, namely upon the “double basis” $\mathcal{B} \cup \overline{\mathcal{B}}$, is justified. Writing $\rho$ for the
colour permutation \( g_c \in A_5^\text{colour} \) induced by \( g \in G_0(\mathcal{I}) \), we see that \( A(g) \) is indeed a symmetry of \( \psi \), since by our arrangements it necessarily satisfies

\[
A(g)_{rs} = n_{\rho(r)\rho(s)}, \quad \text{and} \quad A(g)(\lambda_r) = \lambda_{\rho(r)}. \quad (3.10)
\]

Since \( G_0(\mathcal{I}) \cong A_5 \) is simple, the homomorphic image formed by the \( A(g) \) must be the \( A_5 \) subgroup \( G_0(\psi) \) of \( G(\psi) \). Finally, by Eq. (1.5), \( J \) is as in Eq. 3.9 since all the basis vectors lie off the even hyperplane \( \varpi \). (Note that \( Ju = u \), and hence that \( Jb_i = b_i \); so that indeed \( J^2 = I \).)

Let us give a couple of illustrations of Eq. 3.10. Firstly, consider a rotation \( g \) of the icosahedron through an angle \( \frac{2\pi}{3} \) about the line joining the mid-points of the faces \( f_{54}, f_{45} \); for which, see Eq. (2.12b), we have

\[
g_v = (4\bar{2}5)(4\bar{2}5)(3\bar{6}1)(3\bar{6}1). \quad (3.11)
\]

Then \( A(g) \in GL(6,2) \) is defined on the (ordered) basis \( \mathcal{B} \) by

\[
A(g)(\{b_1, b_2, b_3, b_4, b_5, b_6\}) = \{b_3, b_5, b_6, b_2, b_4, b_1\}. \quad (3.12)
\]

Now, as in Eq. (2.2), \( \rho = g_c = (123) \), and we can check from the explicit form of \( \psi \) in Eq. (3.6) that Eq. (3.10) does indeed hold in this case. For example, Eq. (3.10) asserts in this case that \( A(g)n_{14} = n_{24} \), i.e. that \( A(g)b_{236} = b_{234} \); but, from Eq. (3.12),

\[
A(g)b_{236} = \bar{b}_5 + \bar{b}_6 + \bar{b}_1 = \bar{b}_{561} = b_{234}.
\]

As a second illustration, consider the rotation \( g \) through an angle \( \frac{2\pi}{5} \) about the diagonal \( D_1 \); for which, see Fig. ??, \( g_c = (13254) \) and \( g_v = (26534)(\bar{2}6\bar{5}34) \). The associated symmetry \( A(g) \) of \( \psi \) is thus that given on the (ordered) basis \( \mathcal{B} \) by

\[
A(g)(\{b_1, b_2, b_3, b_4, b_5, b_6\}) = \{b_1, b_6, b_4, \bar{b}_2, \bar{b}_3, b_5\}, \quad (3.13)
\]

and Eq. (3.10) holds with \( \rho = g_c = (13254) \).

**Remark 7.** Recall that in our polar labelling of the icosahedron the 6th diagonal held a privileged position. Consequently a rotational symmetry \( g \in G_0(\mathcal{I}) \) about the diagonal \( D_6 \) gives rise to a symmetry \( A(g) \in G_0(\psi) \) with the property that it merely permutes the elements of the basis \( \mathcal{B} \), that is \( \mathcal{B} \cap A(g)\mathcal{B} = \emptyset \). Hence the above-mentioned manifest \( Z_5 \) symmetry of the double-five \( \psi \) of Eq. (3.6), in contrast to the five somewhat more hidden \( Z_5 \) symmetries, cf. Eq. (3.13), arising from the 5-fold rotational symmetries about the other five diagonals.

### 3.4. The parabolic quadric \( \mathfrak{P}_4(\psi) \), and its six spreads

Recall from Sec. 1.1, remark 1, that the fifteen diagonal points of a double-five \( \psi \) constitute a parabolic quadric \( \mathfrak{P}_4 = \mathfrak{P}_4(\psi) \) lying in the \( PG(4,2) \) of the even hyperplane
This parabolic quadric is in fact the only $P_4$ lying on $\psi$, and its nucleus is the privileged point $u = u(\psi)$. Now, see e.g. [8], there are precisely fifteen lines lying on a $P_4$, the 15 points and 15 lines on a $P_4$ forming a 153 configuration.

**Definition 3.5.** A partition of the fifteen points of $P_4$ into five mutually skew lines is termed a spread.

In the case of $P_4(\psi)$ one spread is uniquely determined by the double-five $\psi$, namely, see theorem 1.2, the set of five lines $\Sigma(\psi) = \{\lambda_1, ..., \lambda_5\}$. In fact a parabolic quadric $P_4$ possesses precisely six spreads, and it will prove of help in section 4.1 if we now exhibit these six spreads (in icosahedral terms!) in the case of $P_4(\psi)$, for $\psi$ as in Eq. (3.6).

To this end, consider the three edges, say $e, e', e''$, bounding the face $f_{rs}$ of $I; let $E, E', E''$ be the corresponding Edges, and let $\mu_{rs}$ denote the associated set $\{b(E), b(E'), b(E'')\}$ of three points in $PG(5, 2)$. Clearly $\mu_{rs}$ is a line; moreover $\mu_{rs} = \mu_{sr}$, since the opposite face $f_{sr}$ yields the same three Edges. Now note that the four lines $\{\mu_{rs}, s \neq r\}$ are mutually skew, since the four faces coloured $r$ in the left-handed scheme share no edge. (Indeed, they share no vertex, these four faces in fact accounting for all twelve vertices of $I$.) Note further that each of the lines $\mu_{rs} (= \mu_{sr})$ is skew to the line $r$ (and to $\lambda_s$), since a face $f_{rs}$ has its edges coloured using the three colours other than $r, s$; we therefore know that the six spreads of lines on our “icosahedral” $P_4$ are as in the next lemma.

**Lemma 3.6.** The six spreads of lines on $P_4(\psi)$ are $\Sigma_1, ..., \Sigma_6$, where

$$
\begin{align*}
\Sigma_r &= \{\lambda_r, \mu_{rs}; s \neq r\} \\
\Sigma_6 &= \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} = \Sigma(\psi) \quad (r = 1, ..., 5)
\end{align*}
$$

(3.14)

Explicitly, in terms of our basis $B$, the six spreads are as in the six rows, or equally the six columns, of the symmetric array

$$
\begin{array}{ccccccc}
\Sigma_1 & \Sigma_2 & \Sigma_3 & \Sigma_4 & \Sigma_5 & \Sigma_6 \\
\Sigma_1 & - & 12 \bar{4} 4 \bar{4} & 45 \bar{6} 64 & 23 \bar{6} 62 & 51 \bar{3} \bar{3} 35 & 34 \bar{5} \bar{2} 16 \\
\Sigma_2 & 12 \bar{4} 4 \bar{4} & - & 23 \bar{6} 62 & 51 \bar{1} 66 & 34 \bar{4} \bar{6} 63 & 45 \bar{1} \bar{3} 26 \\
\Sigma_3 & 45 \bar{5} 664 & 23 \bar{5} 52 & - & 34 \bar{1} \bar{3} \bar{3} & 12 \bar{6} 61 & 51 \bar{3} \bar{6} 26 \\
\Sigma_4 & 23 \bar{6} 62 & 51 \bar{1} \bar{6} 65 & 34 \bar{1} \bar{3} \bar{3} & - & 45 \bar{5} \bar{2} \bar{2} & 12 \bar{3} \bar{5} 46 \\
\Sigma_5 & 51 \bar{1} \bar{3} \bar{3} & 34 \bar{6} 63 & 12 \bar{6} 61 & 45 \bar{5} \bar{2} \bar{2} & - & 23 \bar{1} \bar{5} 56 \\
\Sigma_6 & 34 \bar{5} \bar{2} 16 & 45 \bar{1} \bar{3} \bar{3} & 51 \bar{3} \bar{6} 26 & 12 \bar{3} \bar{5} 46 & 23 \bar{1} \bar{5} 56 & - \\
\end{array}
$$

(3.15)

where, as previously, $ij$ and $ij$ are abbreviations for $b_{ij}$ and $\overline{b}_{ij}$. \[\blacksquare\]

**Remark 8.** From the point of view of the symmetry group $G(\psi)$ note that the 6th spread is privileged, being stable under the action of the $A(g)$ in Eq. (3.10), while $A(g)$ sends $\Sigma_r$ to $\Sigma_{p(r)}$, for $r = 1, ..., 5$. 

18
3.5. Classification of points and lines

Lemma 3.7. Under the action of $G(\psi)$ the 63 points of $\text{PG}(5, 2)$ fall into five orbits of lengths 20, 12, 15, 15, 1:

$$\text{PG}(5, 2) = \{n_{rs}\} \cup \mathcal{B} \cup \mathcal{P}_4 \cup \mathcal{P}_4 \cup \{u\},$$

(3.16)

where $\mathcal{B} = \mathcal{B} \cup \mathcal{B} = \{b, \overline{b}\}$ and $\mathcal{P}_4 = u + \mathcal{P}_4$. In particular the double-five $\psi$ consists of two orbits, namely the 15 points of $\mathcal{P}_4$ and the 20 off-diagonal points $\{n_{rs}\}$. ■

There are 651 lines in the projective space $\text{PG}(5, 2)$. In the rest of this section we will be particularly concerned with those 105, see lemma 3.8, which lie on the double-five $\psi$. Given $\psi$ as in Eq. (3.5), then, since $\alpha_p$ and $\beta_p$ are planes, we must have

$$\sum_{q \neq p} n_{pq} = 0, \quad \text{and} \quad \sum_{q \neq p} n_{qp} = 0.$$  (3.17)

In fact the second equality follows from the first upon recalling Eq. (3.7). Let $pqrst$ denote a permutation of 12345. Then we have the relations

$$(n_{pq} =) n_{pr} + n_{ps} + n_{pt} = n_{rq} + n_{sq} + n_{tq}$$

(N$_{pq}$)

— since $n_{pq}$ lies in both of the planes $\alpha_p$ and $\beta_q$. Now the four points in each of the equations (3.17) can be split into two pairs in three ways, giving rise to the three points on the line $\lambda_p$. Consequently it will prove helpful to introduce the following notation $d_p(qr, st)$ for the points on the line $\lambda_p$:

$$d_p(qr, st) = n_{pq} + n_{pr} = n_{ps} + n_{pt}$$

(A$_p$)

$$= n_{qp} + n_{rp} = n_{sp} + n_{tp}.$$  (B$_p$)

Again the second set of equalities follows from the first on account of Eq. (3.7). Of course the three points on $\lambda_p$, which arise from the three splits $qr|st, rs|qt, sq|rt$ of $qrst$ into an (unordered) pair of (unordered) duads, sum to zero:

$$d_p(qr, st) + d_p(rs, qt) + d_p(sq, rt) = 0.$$  (D$_p$)

(Indeed the relations (A$_p$) imply (D$_p$).)

For the rest of this section let $pqrst$ denote any even permutation of 12345. Then the points on the double-five $\psi$ of Eq. (3.6) satisfy the following further sets of relations

$$d_p(qr, st) = n_{qr} + n_{st} = n_{rq} + n_{ts},$$

(M$_p$)

and

$$d_p(qs, rt) + d_q(rs, pt) + d_r(ps, qt) = 0.$$  (D$_{st}$)

19
The relation \((D_{st})\) follows from the relations \((N_{pq}), (A_p), (B_q)\) upon using the relation \(n_{ps} + n_{tq} = d_r(ps, tq)\) from \((M_r)\). It should be noted on the other hand that in \((N_{pq})\) we cannot similarly pair off \(n_{ps}\) with \(n_{rq}\) — because \(tprqs\) is not an even permutation of \(pqrst\). In the same vein note that the double-five \(\psi\) of Eq. (3.6) satisfies, in conformity with \((M_p)\),

\[
n_{q5} + n_{r4} \in \lambda_p, \quad \text{but} \quad n_{q4} + n_{r5} \notin \lambda_p, \quad \text{for } pqr = 123, 231, 312. \quad (3.18)
\]

**Remark 9.** Given any double-five arranged in the form of the array (3.5) the relations \((M_p)\) and \((D_{st})\) hold either for all even permutations \(pqrst\); or for all odd permutations. We can always arrange for the former to hold by, if necessary, interchanging the numbering of, for example, \(\lambda_4\) and \(\lambda_5\).

It is not hard to check that there are no lines on \(\psi\) other than those given by the foregoing collinearities. Upon taking into account theorem 1.2, we can summarize the situation in respect of the internal lines of \(\psi\) as in the next lemma.

**Lemma 3.8.** There are precisely 105 lines lying on the double-five \(\psi\). These fall into four \(G(\psi)\)-orbits of lengths 5, 10, 60, 30 as follows:

1. the 5 lines \(\lambda_r\), given by the relations \((D_r)\);
2. the 10 other internal lines of \(\Psi_4\), given by the relations \((D_{st})\);
3. those \((2 \times 30 =)60\) tangents to \(\Psi_4\) which lie in one or other of the 10 planes \(\alpha_r, \beta_s\), as given by the relations \((A_p), (B_p)\);
4. the 30 other tangents to \(\Psi_4\) which lie on \(\psi\), as given by the relations \((M_p)\).

**Remark 10.** It is intriguing to note from the foregoing collinearities that each of the 35 points on \(\psi\) lie on precisely 9 of these 105 internal lines — despite the fact, lemma 3.7, that the 35 points split into two \(G(\psi)\)-orbits. One similarly sees that there are 56 lines external to \(\psi\), precisely 6 of which pass through each of the 28 external points — despite the fact that the 28 points lying off \(\psi\) fall into three \(G(\psi)\)-orbits. So in this respect a double-five is indistinguishable from a hyperbolic quadric \(H_5\)! A similar situation holds for the 210 lines tangent to \(\psi\), and also for the 280 bisecants of \(\psi\).

Notice that in icosahedral terms the 10 lines defined by the relations \((D_{st})\) are the 10 lines \(\mu_{st} = \mu_{ts}\) which, as in Sec. 3.4, arise from the three Edges defined by each of the 10 pairs of opposite faces \(\{f_{st}, f_{ts}\}\). The notation \(d_p(qr, st)\) for the 15 points of \(\Psi_4\) can also be given an icosahedral interpretation. For there are 15 involutions \(g_c \in A_5^{\text{volar}}\), all of the kind \(g_c = (qr)(st)\) (and so forming a single conjugacy class). The associated direct symmetry \(g \in G_0(\mathfrak{g})\) is a rotation through an angle \(\pi\) about some 2-fold axis of symmetry, and \(g\) keeps fixed precisely two of the six diagonals of \(\mathfrak{g}\), namely those two diagonals \(D, D'\) perpendicular to the axis of \(g\). Denote the Edge \(E\) determined by \(D, D'\) by \(E(qr, st)\), and set \(b(E) = d(qr, st)\). Then one sees that \(d(qr, st) = d_p(qr, st)\).
3.6. $G(\psi)$-orbits of hyperplanes

As noted in Sec. 1.3, a hyperplane $\pi \subset PG(5,2)$ either intersects each plane $\alpha_r$, (and also each plane $\beta_s$), of a double-five $\psi$ in a line, or else $\pi$ contains one of the planes $\alpha_r$ and intersects the other four $\alpha$-planes in lines, (and similarly also contains one of the planes $\beta_s$ and intersects the other four $\beta$-planes in lines). In particular a double-five $\psi$ is a set of hyperbolic type: $|\psi \cap \pi| \in \{15, 19\}$, for any hyperplane $\pi \subset PG(5,2)$. For $r \neq s$, there is a unique hyperplane containing $\alpha_r$ and $\beta_s$, namely the hyperplane $\text{join}(\alpha_r, \beta_s)$, and so all these 20 hyperplanes have intersection number 19. Also each of the five solids $\sigma_r = \text{join}(\alpha_r, \beta_r)$ lies inside three hyperplanes, thus yielding $5 \times 3 = 15$ further hyperplanes with intersection number 19. Consequently there are precisely $15 + 20 = 35$ hyperplanes with intersection number 19.

On a related tack, consider how the 31 solids $\sigma \subset \varpi = PG(4,2)$ intersect with the parabolic quadric $\mathcal{P}_4(\psi) \subset \varpi$. The section $\sigma \cap \mathcal{P}_4$ is of size 9, 7 or 5 according as it is a hyperbolic quadric $\mathcal{H}_3$, a parabolic cone $\Pi_0 \mathcal{P}_2$, or an elliptic quadric $\mathcal{E}_3$, respectively. (The notation is as in [6]; in particular $\Pi_0 \mathcal{P}_2$ denotes a cone joining a point vertex $\Pi_0$ to a conic $\mathcal{P}_2$ — the latter being merely three non-collinear points.) The 15 solids $\sigma$ which contain the nucleus $u$ of $\mathcal{P}_4(\psi)$ yield the parabolic cone sections (each of the 15 points of $\mathcal{P}_4$ serving once as the vertex $\Pi_0$ of the cone), and of the 16 solids not containing $u$, 10 yield hyperbolic sections and 6 yield elliptic sections. Now each solid $\sigma \subset \varpi$ lies inside three hyperplanes $\pi \subset PG(5,2)$, one of these hyperplanes of course being $\varpi$. Thus the 63 hyperplanes $\pi \subset PG(5,2)$ may be classified as follows:

- type I$: 20$ hyperplanes intersecting $\mathcal{P}_4(\psi)$ in the 9 points of a $\mathcal{H}_3$;
- type I$: 12$ hyperplanes intersecting $\mathcal{P}_4(\psi)$ in the 5 points of a $\mathcal{E}_3$;
- type II: $30$ hyperplanes intersecting $\mathcal{P}_4(\psi)$ in the 7 points of a $\Pi_0 \mathcal{P}_2$;
- type III: the even hyperplane $\varpi(\psi)$.

The intersection of the hyperplane $\text{join}(\alpha_r, \beta_s)$ with $\mathcal{P}_4(\psi)$ is a 9-point set consisting of the lines $\lambda_r$ and $\lambda_s$ together with three points $p_t \in \lambda_t, t \neq r, t \neq s$. (Incidentally the three points $p_t$ are necessarily collinear, since the 9 points of a hyperbolic quadric $\mathcal{H}_3$ lie in threes upon two sets of three skew lines.) Consequently type I$^+$ consists precisely of the 20 hyperplanes $\text{join}(\alpha_r, \beta_s), r \neq s$, all having intersection number 19 with $\psi$. The hyperplanes of type I$^-$ intersect $\mathcal{P}_4$ in 5 points, one on each of the lines $\lambda_r$; since they do not contain any $\lambda_r$, they intersect each $\alpha_r$, (and each $\beta_s$), in lines and so have intersection number 15 with $\psi$. A hyperplane which contains the solid $\sigma_r = \text{join}(\alpha_r, \beta_r)$ intersects $\mathcal{P}_4(\psi)$ in a 7-point set consisting of the line $\lambda_r$ together with four points $p_t \in \lambda_t, t \neq r$; such hyperplanes have intersection number 19 with $\psi$, and account for 15 out of the 30 hyperplanes of type II.

We will refer to these last as type II$_a$ hyperplanes, and to the remaining 15 hyperplanes of type II as type II$_b$. 

21
Since types $I^+$ and $\Pi_a$ have already accounted for the 35 hyperplanes having intersection number 19 with $\psi$, the hyperplanes of type $\Pi_b$ have intersection number 15 with $\psi$. Clearly, through $\lambda_r$ there pass 8 hyperplanes (namely $\text{join}(\alpha_r, \beta_s)$ and $\text{join}(\alpha_s, \beta_r)$, $s \neq r$) of type $I^+$, 3 of type $\Pi_a$ and 1 of type $\Pi_b$; so the 15 hyperplanes of type $\Pi_b$ arise from the remaining 3 hyperplanes through each of the 5 lines $\lambda_r$.

**Theorem 3.9.** (i) Under the action of the group $G(\psi)$ the 63 hyperplanes of $PG(5, 2)$ fall into five orbits $I^+$, $I^-$, $\Pi_a$, $\Pi_b$, $\Pi_c$ (as defined above), of respective lengths 20, 12, 15, 15, 1. The 35 hyperplanes belonging to the orbits $I^+$ and $\Pi_a$ have intersection number 19 with $\psi$, and the 28 hyperplanes belonging to the orbits $I^-$, $\Pi_b$, and $\Pi_c$ have intersection number 15.

(ii) In detail, in terms of their coordinate equations with respect to the basis $B$, the allocation of the 63 hyperplanes of $PG(5, 2)$ to the five $G(\psi)$-orbits is as follows:

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Number of Hyperplanes</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I^+$</td>
<td>5</td>
<td>$x_r = 0$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(\ldots) (x_r = 0)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(\ldots) (x_r + x_{r+1} = x_{r+2})</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(\ldots) (x_r + x_{r+1} = x_6)</td>
</tr>
<tr>
<td>$I^-$</td>
<td>1</td>
<td>$x_6 = 0$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>(\ldots) (x_6 = 0)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(\ldots) (x_r + x_{r+1} = x_{r+3})</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(\ldots) (x_r + x_{r+2} = x_6)</td>
</tr>
<tr>
<td>$\Pi_a$</td>
<td>15</td>
<td>$x_i = \overline{x_j}$</td>
</tr>
<tr>
<td>$\Pi_b$</td>
<td>15</td>
<td>$x_i = x_j$</td>
</tr>
<tr>
<td>$\Pi_c$</td>
<td>1</td>
<td>(\sum_{i=1}^{6} x_i = 0)</td>
</tr>
</tbody>
</table>

Where $r$ takes the values 1, 2, 3, 4, 5 (with addition mod 5), and $i, j$ (with $i \neq j$) take the values 1, 2, 3, 4, 5, 6; we have also introduced the abbreviation $\overline{x_j} = \sum_{i \neq j} x_i$.

**Proof.** From the lead-in to the theorem it is clear that each of $I^+$, $I^-$, $\Pi_a$, $\Pi_b$, $\Pi_c$ is a $G(\psi)$-set. Part (i) will thus follow once we show that $G(\psi)$ is transitive on each of these five sets. To this end consider the orbits of hyperplanes under the action of that $Z_5$ subgroup of $G(\psi)$ which is generated by $A \in GL(6, 2)$ where $Ab_i = b_{\zeta(i)}$, $\zeta = (12345)$. Clearly the sets $I^+$, $I^-$ comprise 8 $Z_5$-orbits, as given by the first 8 rows of the display (3.19). On passing to the larger subgroup $Z_5 \times Z_2$ of $G(\psi) = G_0(\psi) \times Z_2$, observe that these 8 $Z_5$-orbits coalesce in pairs to form 4 $(Z_5 \times Z_2)$-orbits. For note that $J(\Sigma_i x_i b_i) = \Sigma_i x_i \overline{b_i} = \Sigma_i x_i b_i$; so, for example, the hyperplane $x_1 + x_2 = x_3$ lies on the
same $Z_2$-orbit as the hyperplane $x_1 + x_2 = x_3$, i.e. as the hyperplane $x_4 + x_5 = x_6$. To show that these four $(Z_5 \times Z_2)$-orbits coalesce further to produce the two $G(\psi)$-orbits \( \Gamma^+, \Gamma^- \) of the theorem, it suffices to consider the element $A(g) \in G(\psi)$ of Eq. (3.12), for which $A(g)(\Sigma_i x_i b_i) = \Sigma_i x'_i b_i$, where, upon setting $\xi = x_2 + x_3 + x_4 + x_6$,

\[
\begin{align*}
  x'_1 &= x_6 + \xi, \\
  x'_2 &= x_4 + \xi, \\
  x'_3 &= x_1 + \xi, \\
  x'_4 &= x_5 + \xi, \\
  x'_5 &= x_2 + \xi, \\
  x'_6 &= x_3 + \xi.
\end{align*}
\] (3.20)

For note therefore that the hyperplane $x_1 = 0$ lies on the same $G(\psi)$-orbit as the hyperplane $x_6 + \xi = 0$, i.e. as the hyperplane $x_2 + x_4 = x_6$. Similarly, the hyperplane $x_6 = 0$ lies on the same $G(\psi)$-orbit as the hyperplane $x_2 + x_4 = x_6$. Similar considerations apply to the sets \( \Pi_a, \Pi_b \). For example, consider the set \( \Pi_b \) which consists of two \( (Z_5 \times Z_2) \)-orbits, one orbit consisting of the 10 hyperplanes $x_r = x_s$; and the other consisting of the 5 hyperplanes $x_r = x_6$. But these two orbits form a single $G(\psi)$-orbit, since it follows from Eq. (3.20) that the hyperplane $x_1 = x_2$ lies on the same $G(\psi)$-orbit as the hyperplane $x_4 = x_6$. Thus, in the course of proving part (i), we have also derived the more detailed results of part (ii).

How the 31 points of a hyperplane are distributed amongst the five orbits of points, see lemma 3.7, will be of help in Sec. 4.2. The distribution is as follows:

<table>
<thead>
<tr>
<th>Hyperplanes</th>
<th>( 20 \Gamma^+ )</th>
<th>( 12 \Gamma^- )</th>
<th>( 15 \Pi_a )</th>
<th>( 15 \Pi_b )</th>
<th>( 1 { \varpi } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points ↓</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 20 { n_{rs} } )</td>
<td>10</td>
<td>10</td>
<td>12</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>( 12 \overline{BB} )</td>
<td>6</td>
<td>6</td>
<td>4</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>( 15 \overline{P}_4 )</td>
<td>9</td>
<td>5</td>
<td>7</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>( 15 \overline{P}_4 )</td>
<td>6</td>
<td>10</td>
<td>7</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>( 1 { u } )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

(3.21)

For since a hyperplane $\pi \neq \varpi$ intersects the even hyperplane $\varpi = \overline{P}_4 \cup \overline{P}_4 \cup \{ u \}$ in 15 points, and its complement $\varpi^c = \{ n_{rs} \} \cup \overline{BB}$ in 16 points, this table follows from our results earlier in this section.

4. More icosahedral sets

4.1. More double-fives

Each of the twenty faces $f_{rs}$ of $\mathcal{I}$ gives rise to a plane $\gamma_{rs} \subset PG(5,2)$, namely

\[
\gamma_{rs} = \{ n_{rs} \} \cup \mu_{rs} \cup \tau_{rs},
\] (4.1)

where $\tau_{rs}$ denotes the image under the bijection $b$ of the three vertices of the face $f_{rs}$, and, as in Sec. 3.4, the line $\mu_{rs} = \mu_{sr}$ is the image of the three Edges defined by $f_{rs}$.
Taken together with the five planes $\alpha_r$ and the five planes $\beta_r$ defined by the rows and columns of $\psi$ in Eq. (3.5), we have thirty planes, which we can display in the form of the $6 \times 6$ array (with missing main diagonal)

$$\{\gamma_{ij} : i \neq j, \ i,j \in \{1,2,3,4,5,6\}\}$$

upon setting

$$\gamma_{6r} = \beta_r, \ \ \gamma_{6r} = \alpha_r, \ \ r \in \{1,2,3,4,5\}.$$ (4.3)

**Theorem 4.1.** Let $A_i$ denote the set of five planes in the $i$th row, and $B_j$ the set of five planes in the $j$th column, of the array $\{\gamma_{ij}\}$. Then

(i) the five planes in any one of the twelve sets $A_1, \ldots, A_6, B_1, \ldots, B_6$ are mutually skew;

(ii) the 35-point set $\psi_i$ defined by the five planes of $A_i$ coincides with the 35-point set defined by the five planes of $B_i$ for each $i = 1, \ldots, 6$;

(iii) each of the six 35-point sets $\psi_i$ is a double-five (with $\psi_6 = \psi$);

(iv) these six double-fives share the same set $P_4$ of 15 diagonal points (arising from the 15 Edges), but a particular double-five makes use of a particular one out of the six spreads of lines on $P_4$.

**Proof.** As in Sec. 3.2, we already know these results for $A_6, B_6$, and so we need only to consider $A_r, B_r$ for $r \in \{1,2,3,4,5\}$.

(i) For distinct $r, s, t \in \{1,2,3,4,5\}$, $\gamma_{rs}$ is skew to $\gamma_{rt}$ since the faces $f_{rs}$ and $f_{rt}$ share no vertices, and since no edge of $f_{rs}$ is opposite to an edge of $f_{rt}$. Moreover $\beta_r$ is skew to the four planes $\gamma_{rs}$, since the face $f_{rs}$ has no edges of colour $r$ and is distinct from the four faces $f_{qr}$ having $\beta$-colour $r$.

(ii) The 35 points for both $A_r$ and $B_r$ arise from the 12 vertices, 15 Edges and 8 faces of colour $r$.

(iii) Clearly $\gamma_{rs}$ and $\gamma_{sr}$ intersect in the line $\mu_{rs} = \mu_{sr}$, since the opposite faces $f_{rs}, f_{sr}$ have distinct vertices but yield the same three Edges; moreover $\gamma_{rs} \cap \gamma_{st}$ is a point, since the faces $f_{rs}, f_{st}$ have just one common vertex.

(iv) The set $\Sigma(\psi_i)$ of five skew lines determined, cf. theorem 1.2, by $\psi_i$, from the intersections of corresponding pairs of its $\alpha$- and $\beta$-planes, is the spread $\Sigma_i$ in Eqs. (3.14), (3.15):

$$\Sigma(\psi_i) = \Sigma_i.$$ (4.4)

\[\blacksquare\]

**Remark 11.** By way of illustration, using our usual display as a $5 \times 5$ array, the double-
five $\psi_2$ is explicitly

\[
\psi_2 = \begin{pmatrix}
12 & 24 & 41 & 416 & 23 & 63 & 6
\end{pmatrix}.
\]

Observe, as in part (iv) of the theorem, that the entries down the diagonal make use of the spread $\Sigma_2$ in Eqs. (3.14), (3.15). Note also, for example, that the points in the 4th row constitute the plane $\gamma_{42}$, and those in the 5th column constitute the plane $\gamma_{25}$; these two planes intersect in the point $b_6$, since the faces $f_{42}$ and $f_{25}$ share the one vertex $V_6$.

### 4.2. Icosahedral sets of elliptic type

Our original double-five $\psi$ was constructed from the 15 Edges and 20 faces of $\mathfrak{J}$:

\[
\psi = \mathfrak{P}_4 \cup \{n_{rs}\}.
\]

Use of the 12 vertices instead of the 20 faces gives rise to another interesting subset of $PG(5,2)$.

**Definition 4.2.** The configuration $\phi$ of $15 + 12 = 27$ points in $PG(5,2)$ corresponding under the bijection $b$ to the 15 Edges and 12 vertices of $\mathfrak{J}$ will be termed an icosahedral twenty-seven. Thus

\[
\phi = \mathfrak{P}_4 \cup \mathcal{B} \cup \overline{\mathcal{B}}.
\]

**Theorem 4.3.** (i) The icosahedral twenty-seven $\phi$ of Eq. (4.7), arising from the fifteen Edges and twelve vertices of the icosahedron $\mathfrak{J}$, is a set of elliptic type.

(ii) The symmetry group of $\phi$ coincides with that of $\psi$: $G(\phi) = G(\psi)$.

(iii) The configuration $\phi$ lies on each of the five double-fives $\psi_r$, $r = 1, \ldots, 5$ of theorem 4.1.

(iv) The double-five $\psi$ of Eq. (4.6) contains as subsets five icosahedral twenty-sevens, namely the five subsets

\[
\phi_r = \psi \setminus (\alpha_r \Delta \beta_r), \quad r = 1, \ldots, 5,
\]

which arise from the fifteen Edges together with those twelve faces which are not of colour $r$ in either colouring. (Here $\alpha_r \Delta \beta_r$ denotes the symmetric difference $(\alpha_r \cup \beta_r) \setminus \lambda_r$ of the sets $\alpha_r, \beta_r$.)

25
Proof. (i) From the table 3.21, the intersection numbers of \( \psi \) and \( \phi \) with the five kinds of hyperplane \( \pi \) are as follows:

\[
\begin{array}{cccccc}
\text{Orbit of } \pi & 201^+ & 121^- & 15 \Pi_a & 15 \Pi_b & 1 \{ \varpi \} \\
\text{Representative} & x_1 = 0 & x_6 = 0 & x_1 = \overline{x}_2 & x_1 = x_2 & \Sigma_i x_i = 0 \\
|\psi \cap \pi| & 19 & 15 & 19 & 15 & 15 \\
|\phi \cap \pi| & 15 & 11 & 11 & 15 & 15 \\
\end{array}
\]

Hence \( \phi \) is a set of elliptic type: \( |\phi \cap \pi| \in \{11, 15\} \).

(ii) Each symmetry of the double-five \( \psi \) stems from certain permutations of the twelve vectors of \( \mathcal{B} \cup \overline{\mathcal{B}} \), and stabilizes separately the subsets \( \mathcal{B} \cup \overline{\mathcal{B}}, \mathcal{P}_4 \) and \( \{n_{rs}\} \). So each symmetry of the double-five \( \psi \) in Eq. (4.6) is also a symmetry of the icosahedral twenty-seven \( \phi \) in Eq. (4.7). One way to see that the two symmetry groups coincide is to check that the only hyperplane section of \( \phi \) which is a parabolic quadric is the section by \( \varpi \), which yields the quadric \( \mathcal{P}_4(\psi) \). (For example the 15-point section by a hyperplane \( \pi \in \Pi_b \), see Eq. (4.9), meets \( \mathcal{P}_4(\psi) \) in a cone \( \Pi_0 \mathcal{P}_2 \), and one finds that the vertex \( \Pi_0 \) lies on \( > 3 \) interior lines of the 15-set, which last can not therefore be a \( \mathcal{P}_4 \).) Knowing now that each \( A \in G(\phi) \) stabilizes the privileged \( \mathcal{P}_4(\psi) \), each \( A \in G(\phi) \) must permute the twelve vectors of \( \mathcal{B} \cup \overline{\mathcal{B}} \), and so \( A \) lies in \( G(\psi) \).

(iii) If we adjoin to \( \phi \) the 8 further points corresponding to the 8 faces of colour \( r \), that is the 4 points \( n_{rs}, s \neq r \), and the 4 points \( n_{sr}, s \neq r \), we thereby obtain the double-five \( \psi_r \) of the preceding section:

\[
\psi_r = \phi \cup (\alpha_r \triangle \beta_r).
\]

(iv) By Eq. (4.10), deleting from the double-five \( \psi_r \) the 8-point set \( \alpha_r \triangle \beta_r \), consisting of the symmetric difference of an \( \alpha \)-plane and the corresponding \( \beta \)-plane of the double-five, results in an icosahedral twenty-seven

\[
\phi = \psi_r \setminus (\alpha_r \triangle \beta_r).
\]

The same must therefore be true of any double-five, for any choice of a corresponding pair of \( \alpha \)- and \( \beta \)-planes, whence \( \phi_r \) in Eq. (4.8) is an icosahedral twenty-seven.

Remark 12. According to Tonchev, see [15, Table I], there are in the space \( \text{PG}(5, 2) \) seven inequivalent kinds of sets of hyperbolic type and five inequivalent kinds of sets of elliptic type.

Theorem 4.4. (i) Tonchev’s hyperbolic set of type 3b is a double-five.

(ii) Tonchev’s elliptic set of type 3b is an icosahedral twenty-seven.
Proof. (i) The 35 points of the Tonchev set can be arranged in the requisite form:

\[
\psi_{\text{tonchev}} = \begin{pmatrix}
6, 10, 12 & 13 & 11 & 7 & 1 \\
15 & 24, 36, 60 & 51 & 23 & 43 \\
9 & 49 & 20, 44, 56 & 37 & 29 \\
5 & 21 & 39 & 16, 34, 50 & 55 \\
3 & 41 & 31 & 53 & 28 \\
\end{pmatrix}.
\] (4.11)

Here the points of \( PG(5, 2) \) are as labelled in [15]: \( 1 = 000001 \equiv (0, 0, 0, 0, 0, 1) \), \( 2 = 000010 \), ... , \( 32 = 100000 \), \( 63 = 111111 \). (Incidentally one sees that \( u(\psi_{\text{tonchev}}) \) is the point 2 and that \( \varpi(\psi_{\text{tonchev}}) \) has equation \( x_6 = 0 \), relative to the ordered basis \( \{32, 16, 8, 4, 2, 1\} \).)

(ii) The 27 points of the Tonchev set can be arranged as follows:

\[
\phi_{\text{tonchev}} = \begin{pmatrix}
6, 34, 36 & 7 & 37 & 1 (35) \\
5 & 14, 22, 24 & 29 & 19 (11) \\
39 & 31 & 8, 48, 56 & 47 (23) \\
3 & 45 & 18, 46, 60 & 63 (63) \\
(33) & (9) & (21) & (61) & 28, 40, 52 \\
\end{pmatrix}.
\] (4.12)

Here the points in parentheses when added to the Tonchev set are seen to convert it into a double-five, whence by theorem 4.3, the Tonchev set is an icosahedral twenty-seven. ■

5. Coordinate forms

5.1. Equations of the planes \( \alpha_r \) and \( \beta_r \)

Relative to the ordered basis \( B = \{b_1, \ldots, b_6\} \), the \( r \)th solid \( \sigma_r = \text{join}(\alpha_r, \beta_r) \) determined by the double-five \( \psi \) in Eq. (3.6) is seen to be given by the coordinate equations

\[
x_{r+2} + x_{r-2} = x_{r+1} + x_{r-1} = x_r + x_6.
\] (5.1)

Notice how this fits in with the fact that the \( r \)th line \( \lambda_r = \alpha_r \cap \beta_r \) is the set

\[
\lambda_r = \{b_{r+2-r-2}, b_{r+1-r-1}, b_{r6}\}.
\] (5.2)

In coordinate terms the line \( \lambda_r \) is given by the equations

\[
x_{r+2} = x_{r-2}, \quad x_r = x_6, \quad x_{r+1} = 0, \quad x_{r-1} = 0.
\] (5.3)

The planes \( \alpha_r \) and \( \beta_r \) are respectively given by the equations

\[
x_{r+2} + x_{r-2} = x_{r+1} + x_{r-1} = x_r + x_6, \quad x_{r-1} = 0,
\] (5.4)

\[
x_{r+2} + x_{r-2} = x_{r+1} + x_{r-1} = x_r + x_6, \quad x_{r+1} = 0.
\] (5.5)
Incidentally note that, on setting \( t = \frac{1}{2}(r+s) \), the twenty off-diagonal points \( \{n_{rs}\} \) of \( \psi \) are given by

\[
n_{rs} = \begin{cases} 
  \overline{b_{rst}}, & \text{if } s = r - 1, \text{ or } s = r - 2, \\
  b_{rst}, & \text{if } s = r + 1, \text{ or } s = r + 2.
\end{cases}
\] (5.6)

5.2. Cubic equations

Let \( \alpha \nabla \beta \) denote the complement \( (\alpha \triangle \beta)^c \) of the symmetric difference of two subsets \( \alpha, \beta \). Like the symmetric difference \( \triangle \), the binary operation \( \nabla \) is symmetric and associative. Moreover we have, for any three subsets, the result

\[
\alpha \nabla \beta \nabla \gamma = \alpha \triangle \beta \triangle \gamma.
\] (5.7)

Consequently the addition, using \( \nabla \), of an odd number of subsets coincides with the addition, using \( \triangle \), of the same subsets. We will need the following lemma. (This lemma is a re-phrase of (half of) lemma B in [10]. For the generalization to \( PG(m,p) \), for any prime \( p \), see theorem 4.2 of [7].)

**Lemma 5.1.** Suppose that a subset \( \gamma \) of \( PG(m,2) \) can be expressed in the form

\[
\gamma = \alpha_1 \nabla \alpha_2 \nabla ... \nabla \alpha_k,
\] (5.8)

where each \( \alpha_i \) is a subspace of \( PG(m,2) \) of (projective) dimension \( m - d \). Then \( \gamma \) has polynomial equation \( f = 0 \) where the polynomial \( f \) is homogeneous, i.e. satisfies \( f(0) = 0 \), and has (reduced) degree at most \( d \).

**Proof.** Any \((m - d)\)-dimensional subspace \( \alpha \) of \( PG(m,2) \) can be expressed as the intersection

\[
\alpha = \pi_1 \cap ... \cap \pi_d
\]
of \( d \) suitable hyperplanes. The equation of \( \alpha \) is thus \( h = 0 \), with \( h \) given by

\[
1 + h = (1 + g_1) ... (1 + g_d),
\] (5.9)

where \( g_j = 0 \) is the (linear) equation of the hyperplane \( \pi_j \). Note therefore that \( h \) is a polynomial (with no constant term) of degree \( d \). Now if in Eq. (5.8) the subspace \( \alpha_i \) has equation \( h_i = 0 \), then it follows that \( \gamma \) has equation \( f = 0 \) with \( f = h_1 + ... + h_k \). Hence the lemma since each \( h_i \) satisfies \( h_i(0) = 0 \) and has degree \( d \).

The following lemma is also relevant.

**Lemma 5.2.** A subset \( \psi \) of \( PG(m,2) \) has equation \( f = 0 \) with \( f \) a homogeneous polynomial of reduced degree \( \leq d \) if and only if \( \psi \) intersects every projective \( d \)-space in an odd number of points.
Proof. See [10].  ■

Theorem 5.3. Double-fives of planes, and also icosahedral twenty-sevens, have cubic equations.

Proof. Since a double-five \( \psi \) can be expressed in the form \( \psi = \alpha_1 \triangle \ldots \triangle \alpha_5 = \alpha_1 \nabla \ldots \nabla \alpha_5 \), with the \( \alpha_i \) planes, i.e. projective 2-spaces, lemma 5.1 implies that it has equation \( \Psi = 0 \) with the reduced degree of \( \Psi \) at most 3. Now it is easy to find planes which have even intersection with \( \psi \). For example the plane \( \text{join}(b_1, b_2, b_4) \) intersects \( \psi \) in the 2-set \( \{b_{12}, b_{124}\} \), and the plane \( \text{join}(b_4, b_5, b_6) \) intersects \( \psi \) in the 4-set \( \{b_{45}, b_{56}, b_{46}, b_{456}\} \). By lemma 5.2, \( \Psi \) has degree at least 3. (Alternatively the degree of \( \Psi \) can not be 2, since it is known, see e.g. [8], that quadrics in \( PG(5,2) \) do not possess spreads of planes.) So the reduced degree of \( \Psi \) is 3.

Since the icosahedral twenty-seven \( \phi_5 \) of Eq. (4.8) can be expressed in the form \( \phi_5 = \alpha_1 \triangle \alpha_2 \triangle \alpha_3 \triangle \alpha_4 \triangle \beta_5 = \alpha_1 \nabla \alpha_2 \nabla \alpha_3 \nabla \alpha_4 \nabla \beta_5 \), lemma 5.1 implies that it has equation \( f = 0 \) with the reduced degree of \( f \) at most 3. But certain planes have even intersection with \( \phi_5 \); for example, each of \( \alpha_1, \ldots, \alpha_4 \) (and also each of \( \beta_1, \ldots, \beta_4 \)) intersect \( \phi_5 \) in 6 points. By lemma 5.2 \( f \) has degree at least 3. So the reduced degree of \( f \) is 3.  ■

We now wish to display explicit coordinate expressions for these cubics. Still keeping to our basis \( \mathcal{B} \), let \( S_1 = \sum_{i=1}^{6} x_i, S_2 = \sum_{i<j} x_ix_j, \ldots \) denote the usual symmetric functions of the six coordinates \( x_1, \ldots, x_6 \), and let \( s_1, s_2, \ldots \) denote the corresponding symmetric functions of the first five coordinates \( x_1, \ldots, x_5 \). So \( S_1 = s_1 + x_6 \) and \( S_2 = s_2 + x_6 s_1 \). It will help to take advantage of the manifest invariance of both \( \phi \) in Eq. (4.7) and \( \psi \) in Eq. (3.6) under the \( Z_5 \) group generated by the basis transformation \( b_i \mapsto b_{\zeta(i)} \), with \( \zeta = (12345) \).

Now a basis for the set of \( Z_5 \)-invariant polynomials of degrees \( \leq 3 \) in the six coordinates is provided by the following nine polynomials:

1) the two linear polynomials \( x_6 \) and \( S_1 \);
2) the three quadratic polynomials \( Q, Q', Q'' \) defined by
\[
\begin{align*}
Q &= x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2 \\
Q' &= x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 \\
Q'' &= x_6s_1. \tag{5.10}
\end{align*}
\]
3) the four cubic polynomials \( C, C', C'^*, C''^* \) defined by
\[
\begin{align*}
C &= x_6(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) = x_6Q' \\
C' &= x_1x_2x_4 + x_2x_3x_5 + x_3x_4x_1 + x_4x_5x_2 + x_5x_1x_3 \\
C'^* &= x_6(x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2) = x_6Q \\
C''^* &= x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2. \tag{5.11}
\end{align*}
\]
We also define the cubic functions $C_{\Omega}, C_{\Omega^*}$ by

$$C_{\Omega} = \sum_{ijk \in \Omega} x_ix_jx_k, \quad C_{\Omega^*} = \sum_{ijk \in \Omega^*} x_ix_jx_k, \quad (5.12)$$

where $\Omega, \Omega^*$ denote the two $A_5^{\text{diag}}$-orbits of triples displayed in Eqs. (2.5), (2.6). Note therefore that

$$Q + Q' = s_2, \quad Q + Q' + Q'' = S_2,$$
$$C + C' = C_{\Omega}, \quad C^* + C'' = C_{\Omega^*},$$
$$C + C^* = x_6 s_2, \quad C' + C'' = S_3,$$
$$C_{\Omega} + C_{\Omega^*} = C + C' + C^* + C'' = S_3. \quad (5.13)$$

The following relations are also easily checked

$$C' = s_1 Q', \quad C'' = s_1 Q, \quad (5.14)$$

and hence we have the relations

$$C_{\Omega} = S_1 Q', \quad C_{\Omega^*} = S_1 Q. \quad (5.15)$$

**Theorem 5.4.** The icosahedral twenty-seven $\phi$ of Eq. (4.7) has cubic equation $\Phi = 0$, where

$$\Phi = Q + C_{\Omega} = Q + S_1 Q', \quad (5.16)$$

and the double-five $\psi$ of Eq. (3.6) has cubic equation $\Psi = 0$, where

$$\Psi = S_1 + Q + C_{\Omega} = S_1 + Q + S_1 Q'. \quad (5.17)$$

**Proof.** By their $Z_5$-invariance, the cubic polynomials $\Phi, \Psi$ are linear combinations of the nine $Z_5$-invariant polynomials listed above. Consider first of all $\Phi$. Since $\Phi (b_i) = 0$, the linear polynomials are absent in $\Phi$. Since $\Phi (b_{34}) = 0 = \Phi (b_{16})$ and $\Phi (b_{52}) = 1$, it follows that $Q', Q''$ are absent and $Q$ is present. Consequently

$$\Phi = Q + cC + dC' + e^* C^* + e'' C''$$

for some $c, c', c^*, e'' \in GF(2)$. Since $\Phi (b_{126}) = \Phi (b_{124}) = \Phi (b_{136}) = \Phi (b_{123}) = 1$, it follows that $c = 1, c' = 1, c^* = 0, e'' = 0$, respectively. Hence $\Phi = Q + C + C' = Q + C_{\Omega}$, as stated in the theorem. The result for $\Psi$ now follows upon noting that $S_1$ takes the value 0 on all vectors of even weight, and the value 1 on all vectors of odd weight. \hfill \blacksquare

Alternatively, cf. the proof of lemma 5.1, we can compute $\Psi$ as $\Sigma_{r=1}^5 h_r$, where $h_r = 0$ is the equation of $\alpha_r$. For, from Eq. (5.9), we have $h_r = g_1 + g_2 + g_3 + g_1 g_2 + g_2 g_3 + g_3 g_1 + g_1 g_2 g_3$, where, from Eq. (5.4), we may make the choice $g_1 = x_{r-1}, g_2 = x_{r-2} + x_{r+1} + x_{r+2}, g_3 = x_r + x_{r+1} + x_6$. 30
Remark 13. Observe that the restrictions of $\Phi$ and $\Psi$ to the even hyperplane $\overline{x}$ (whose equation is $S_1 = 0$), are equal and quadratic:

$$\Phi|_{\overline{x}} = \Psi|_{\overline{x}} = Q,$$

in conformity with the fact that $\Phi$ and $\Psi$ intersect $\overline{x}$ in the same $P_4$.

6. Miscellaneous remarks

1) The dual $V(6,2)^*$ of the vector space $V(6,2)$ gives rise to a dual projective space $PG(5,2)^*$ which may be identified with $V(6,2)^* \setminus \{0\}$. It is worth noting that a double-five $\psi \subset PG(5,2)$ automatically gives rise to a dual double-five $\psi^* \subset PG(5,2)^*$. For let $\alpha^o$ denote the subspace of $PG(5,2)^*$ which consists of those nonzero linear forms which are zero on each point of the subspace $\alpha$ of $PG(5,2)$. The 10 planes $\alpha_{r^o}, \alpha_{s^o}$ of $\psi$ yield 10 planes $\alpha_{r^o}, \alpha_{s^o}$ in $PG(5,2)^*$, and, by elementary linear algebra, these satisfy the defining incidence requirements for the 10 planes of a double-five $\psi^* \subset PG(5,2)^*$. In terms of the basis $\{x_1, ..., x_9\}$ dual to our usual basis $\{b_1, ..., b_9\}$, we find that the dual of the double-five $\psi$ of Eq. 3.6 is

$$\psi^* = \begin{pmatrix}
126 & 52 & 5 & 234 \\
345 & 54 & 13 & 236 \\
2 & 451 & 15 & 24 & 346 \\
5 & 3 & 512 & 46 & 21 & 35 & 456 \\
516 & 1 & 4 & 123 & 56 & 32 & 41
\end{pmatrix} .$$

Here $ij$, $i\overline{j}$, $ijk$ are now abbreviations for $x_i + x_j$, $\overline{x_i} = \sum_{j \neq i} x_j = x_i + S_1$, $x_i + x_j = x_i + x_j + S_1$, $x_i + x_j + x_k$, respectively.

2) Despite our use of Euclidean language when dealing with the coloured icosahedron in Sec. 2.1, it should be clear that we only really needed combinatorial considerations. We essentially dealt with the icosahedral graph $\Gamma(3)$, and our dealings with the great icosahedron $\Gamma^*$ could be replaced by ones with the distance-2 graph $\Gamma_2(3) = \Gamma(\Gamma^*)$ of $\mathbb{F}_2$. Presumably it would be worthwhile investigating whether interesting configurations in $PG(m,q)$ could arise from other regular polytopes (or maps, or graphs, or ... ).

One idea would be to start from some suitable, presumably simple, finite group $G$. After all, we could have adopted this approach for the double-five by using $G = A_5$. For we could have defined the set of 12 vertices of the icosahedron to be one of the conjugacy classes of elements of order 5 in $A_5$, two elements $A, B$ forming an edge if and only if $(AB)^5 = I$. Use of the group $G \cong L_3(2) \cong L_2(7)$ shows some promise. Here there is a set of 24 vertices, defined to be one of the conjugacy classes of elements of order 7 in $G$, two elements $A, B$ forming an edge if and only if $(AB)^7 = I$. There are then 24 vertices, 84 edges and 56 triangular faces, constituting the regular map $\{3, 7\}_8$.
on a surface of genus 3. In the case of $A_5$, the 12 vertices formed 6 antipodal pairs of the kind $\{A, A^{-1}\}$, and in the finite geometry these 6 pairs were mapped, see Sec. 3.1, into 6 three-point lines in the space $PG(5, 2)$, all passing through the unit point of our basis. Now in the case of $L_3(2) \cong L_2(7)$, the 24 vertices form 8 antipodal triads of the kind $\{A, A^2, A^4\}$. This suggests that in the finite geometry these 8 triads should be mapped into 8 four-point lines in the projective space $PG(7, 3)$ over the field $GF(3)$, all passing through the unit point of the basis. A preliminary investigation by the author shows promise that the regular map $\{3, 7\}_8$ will indeed guide one to some interesting finite geometry.

3) It is known that a “double-nine of solids” exists in $PG(7, 2)$. By this we mean a figure $\psi$ of 135 points which admits two decompositions

$$\psi = \bigcup_{r=1}^9 \alpha_r = \bigcup_{r=1}^9 \beta_r$$

into nine mutually skew solids such that $\alpha_r \cap \beta_r$ is a plane $\lambda_r$ and $\alpha_r \cap \beta_s, s \neq r$, is a point $n_{rs}$. Indeed a hyperbolic quadric $H_7 \subset PG(7, 2)$ admits such a decomposition, with the $\alpha$-solids belonging to one system of generators and the $\beta$-solids belonging to the other system. Of the 15 points of a solid $\alpha_r, 7$ lie on the intersection $\lambda_r$ of $\alpha_r$ with $\beta_r$, and the remaining 8 are accounted for by the 8 intersections $n_{rs}$ with the other 8 $\beta$-solids. (That this is so can be gleaned from [5, c. theorem 10].) Now the quadric $H_9$, like $H_5$, does not possess spreads. So one naturally enquires, does there exist a “double-seventeen of $PG(4, 2)$’s” in $PG(9, 2)$ which can be thought of as a generalization of the double-five of planes in $PG(5, 2)$? (Here $\alpha_r \cap \beta_r$ is a solid $\lambda_r, r = 1, 2, \ldots, 17$, and $\alpha_r \cap \beta_s, s \neq r$, is a point $n_{rs}$.) Clearly the incidence requirements are extremely tight and daunting! — all 31 points of each $PG(4, 2) \alpha_r$ being accounted for by the 15 points of $\lambda_r$ and the 16 points $n_{rs}, s \neq r$. If such a configuration does exist then it will have equation $f = 0$ where, by lemma 5.1, $\deg f \leq 5$, and where $\deg f > 2$, since quadrics in $PG(9, 2)$ do not possess spreads of $PG(4, 2)$’s.

4) Returning to projective dimension 5, clearly it would be nice to have a satisfying geometric construction of all of the Tonchev sets in $PG(5, 2)$ of hyperbolic and elliptic type.

A. Appendix: the “other” $A_5$ subgroups of $A_6$

Recall the following well-known exceptional property of the symmetric group $S_6$ acting on the six symbols $\{1, 2, 3, 4, 5, 6\}$, see e.g. [1, p. 209]. Not only does $S_6$ possess the class of six subgroups $\cong S_5$ consisting of the six stabilizers $\text{stab}(i), i \in \{1, 2, 3, 4, 5, 6\}$, of an individual symbol, it also possesses another class of six subgroups $\cong S_5$, namely the stabilizers $\text{stab}(T_i)$ of each of the six totals $T_1, \ldots, T_6$ which can be formed from the six symbols $1, \ldots, 6$. As was noticed by Sylvester in 1861, there exist precisely six totals,
with any two totals overlapping in precisely one syntheme. Consequently a syntheme \( \Lambda \) can be given a “duadic” labelling, via
\[
\Lambda_{ij} = T_i \cap T_j .
\]  
(A.1)

The six totals are displayed in the following \( 6 \times 6 \) array, the \( i \)th total \( T_i \) consisting of the five synthemes in the \( i \)th row, equally the \( i \)th column, of the array:

\[
\begin{array}{cccccc}
T_1 & T_2 & T_3 & T_4 & T_5 & T_6 \\
T_1 & - & 15 23 46 & 14 35 26 & 13 24 56 & 12 45 36 & 25 34 16 \\
T_2 & 15 23 46 & - & 12 34 56 & 14 25 36 & 24 35 16 & 13 45 26 \\
T_3 & 14 35 26 & 12 34 56 & - & 23 45 16 & 13 25 46 & 15 24 36 \\
T_4 & 13 24 56 & 14 25 36 & 23 45 16 & - & 15 34 26 & 12 35 46 \\
T_5 & 12 45 36 & 24 35 16 & 13 25 46 & 15 34 26 & - & 14 23 56 \\
T_6 & 25 34 16 & 13 45 26 & 15 24 36 & 12 35 46 & 14 23 56 & - \\
\end{array}
\]  
(A.2)

We should point out that the numbers \( 1, 2, 3, 4, 5, 6 \) labelling the six totals can be assigned in a quite arbitrary manner. In the preceding array we chose to number the totals so as to satisfy
\[
r6 \in \Lambda_{r6}, \quad \text{for} \ r = 1, 2, 3, 4, 5 .
\]  
(A.3)

In fact the array has the property that \( ab \in \Lambda_{cd} \) if and only if \( cd \in \Lambda_{ab} \) (where \( c, d \) are not necessarily distinct from \( a, b \)). (Thus the mapping \( ij \mapsto \Lambda_{ij} \), from duads to synthemes, is a duality of the \( 15_3 \) configuration formed by the 15 duads (“points”) and the 15 synthemes (“lines”).)

The foregoing exceptional property of \( S_n \) for \( n = 6 \) goes along with the fact that \( S_6 \) alone amongst the symmetric groups \( S_n \) possesses outer automorphisms (which map one class of \( S_5 \) subgroups on to the other class). Each permutation \( \sigma \in S_6 \) acts, via \( ij \mapsto \sigma(i)\sigma(j) \), on the fifteen duads, hence on the fifteen synthemes, and hence induces a permutation of the 6 totals \( T_1, T_2, ... \ T_6 \) (for any fixed numbering of the latter). Consequently there exists an automorphism \( \theta \) of \( S_6 \) which maps \( \sigma \) onto \( \theta(\sigma) = \rho \), where \( \rho \in S_6 \) is that permutation such that
\[
\sigma(T_i) = T_{\rho(i)}, \quad \text{and hence} \quad \sigma(\Lambda_{ij}) = \Lambda_{\rho(i)\rho(j)} .
\]  
(A.4)

Note that \( \theta \) maps \( \text{stab}(T_i) \) onto \( \text{stab}(i), i = 1, ..., 6 \). One can show that the effect of \( \theta \) on the transpositions \( (ab) \in S_6 \) is as follows:
\[
\text{if} \quad \Lambda_{ab} = \{ij, \ kl, \ mn\} \quad \text{then} \quad \theta(ab) = (ij)(kl)(mn) .
\]  
(A.5)

Using this last result one can show that the automorphism \( \theta \) of \( S_6 \), resulting from our chosen numbering system for the totals, is involutory: \( \theta^2 = \text{id} \). (See [11] for further details and references.)
REFERENCES

10. R. Shaw, A characterization of the primals in $PG(m, 2)$, *Designs, Codes and Crypt.* 2 (1992), 253-256.

34