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A FRACTIONAL ORDER APPROACH TO LOUDSPEAKER IDENTIFICATION

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ABSTRACT
With smartphones and tablets, loudspeakers play an ever important role in our everyday lives. Sound quality is paramount for clear communication and listening pleasure. In order to proper control the acoustic output of the loudspeaker a precise modeling and identification of the electromechanical part ('motor') is necessary. In this paper we show that the electrical impedance of the motor can be properly modeled and identified as a fractional order system. This is the main contribution of this paper. After showing the empirical evidence of fractional order behavior in actual loudspeaker measurement, we explain the related mathematical theory. We then describe a frequency domain subspace identification method adapted to fractional order systems. Finally by applying this identification method on the actual measured data, we show that the fractional order approach results in a better fit and a lower model complexity than the traditional integer order approach.

KEY WORDS
Fractional Order System, Frequency Domain, Subspace Identification, Loudspeaker

1 Introduction

Loudspeakers are more and more ubiquitous and important in our everyday life. The past few decades have witnessed the explosive development of cellphones, smartphones and tablets. This portable devices are in our hands all day long for communication, information and entertainment. Unfortunately due to constraints of size and integration, small loudspeakers are used to reproduce the audio content. These little devices are pushed to their limits to get enough sound level. The resulting distortion affects the quality of music and intelligibility of the speech. In order to reduce the distortion by equalization and linearization, a proper modeling and identification of the loudspeaker is necessary. Due to the wide range of frequencies involved (20 to 20kHz) and the complexity of the system the problem is not trivial. In the past 30 years considerable effort has been devoted to overcome the issues with no definitive answers [1, 2, 3].
In a previous paper [4] we have presented a technique based on polynomial state-space for nonlinear modeling of loudspeaker which can effectively be used in identification process. It is an overall approach that encompasses the whole system in one model which explains both linear and linear characteristics. In this paper we focus on the motor identification. As we will show in the following, the linear identification of the motor presents by itself specific challenges that requires innovative solutions. The contribution of this paper is the application of fractional calculus to loudspeaker identification. To the knowledge of the authors, this is a novel approach in this domain.
In a subsequent paper we will combine the advantages of both approaches, namely polynomial state space and fractional order, to obtain a full modeling of the loudspeaker.

2 Problem Description

2.1 Loudspeaker Mechanism

The most common type of driver is electro-dynamic. The driving part, the motor, is a moving coil into a static magnetic field. The audio signal goes through the coil and creates a variable magnetic field that interact with fixed magnets and generate a mechanical force that is roughly proportional to the electrical current. The acoustic radiation is insured by a lightweight cone (diaphragm) attached to the coil. An elastic suspension maintains the coil and the attached cone centered into the frame ("basket"). The cone is also mechanically connected to the basket by an elastic suspension called surround (see fig. 1). A simplified lin-

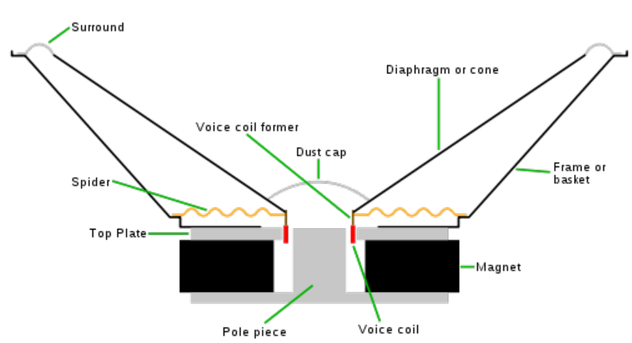

Figure 1. **Loudspeaker mechanism.**

ear model based on lumped parameters is traditionally used to describe the loudspeaker mechanism at low frequencies and small signal. It is composed of two differential equations.

$$u(t) = Ri(t) + L\frac{di}{dt} + Bl\frac{dx}{dt} \quad (1)$$

$$Bl.i(t) = m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx(t) \quad (2)$$

Where u(t) is the input voltage, i(t) the current, x(t) the cone displacement and R, L, Bl, m, r, k are electromechanical parameters of the loudspeaker. Bl is the product of the magnet field strength in the voice coil gap and the length of wire in the magnetic field. In the loudspeaker technology, it is a lumped parameter called the force factor.

This simplified model is used widely, however many studies have shown its insufficiencies. In particular it doesn't capture the influence of eddy currents in the iron pole structure [5]. Various improved models with added complexity have been proposed ([6] and references therein) but there is no general agreement about any of them.

2.2 Empirical Evidence of Fractional Order System

The loudspeaker can be studied in two parts: the electromechanical part (motor) and the acoustical part (diaphragm). The motor is responsible for the low frequencies behavior and most of the nonlinearities while the diaphragm produces high frequencies irregularities due to modal vibrations. Equations shows that the motor dynamics is fully observable through its electrical response i.e. its impedance curve. In this paper we will therefore focus on the modeling and identification of the impedance.

From (1) and (2), the electrical impedance can be expressed in the Laplace domain as:

$$Z(s) = \frac{mLs^3 + (rL + Rm)s^2 + (rR + Bl^2 + kL)s + kR}{ms^2 + rs + k} \quad (3)$$

The impedance curve (fig. 2) shows three important features, consistent with (3):

- The impedance tends toward a non-zero value at low frequencies, i.e. the voice-coil has a DC resistance.
- A strong mode stands out. This is due to the mechanical resonance of the moving part attached to the voice-coil combined with the spider and surround springs.
- At high frequencies there is a continuous rise of the impedance magnitude, which is consistent with an inductive behavior of the voice-coil.

However, a close inspection of the impedance magnitude shows something unusual: the asymptotic slope at high frequencies is lower that it should be. A linear regression in the log-log space confirms that the magnitude rises with the square-root of the frequency (see fig. 2).

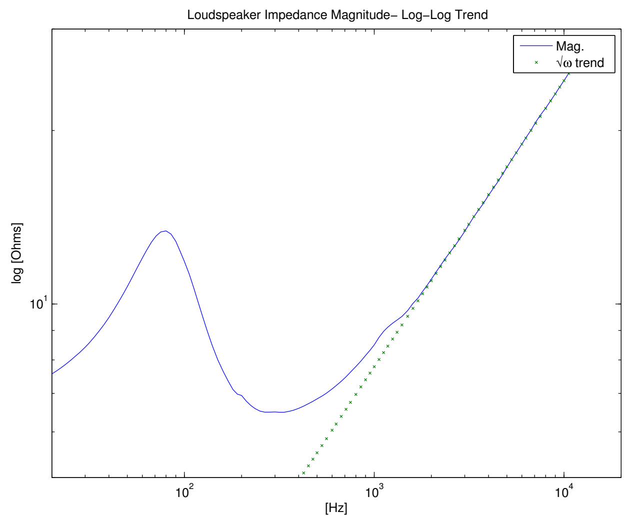

Figure 2. **Loudspeaker impedance (magnitude in log-log scale).**

This fact is well-known in the electroacoustic community and has been described in the aforementioned papers [5, 6]. The semi-inductive behavior is attributed to eddy currents in the iron pole structure of the motor and lossy coils have been successfully modeled using fractional derivative [7]. This fact made us believe that the loudspeaker motor is indeed a fractional order system and should be identified as such. Thus, we believe that the application of fractional order system identification on loudspeaker is one of the most comprehensive method to overcome the difficulties of previous approaches.

3 Mathematical Theory of Fractional Order System

The fractional order calculus has a long history and mathematicians have developed theoretical results for it. It is based on the generalization of the differential operator $D_t = \frac{\mathrm{d}}{\mathrm{d}t}$ to $D_t^\alpha = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$, where α could be a fractional number, and for that matter, any non-zero real number. The advantages of fractional derivatives become apparent in modeling of physical processes. Fractional derivatives and integrals also appear in the control of dynamic systems, when the system under control or/and the controller is described by a fractional differential equation. The mathematical modeling and simulation of systems described by fractional derivatives leads to differential equations of fractional order and the necessity of solving such equations. For a general exposition of fractional calculus and systems one may refer to [8].

There are three equivalent definitions most frequently used for fractional derivative of a function: Grunwald-Letnikov, Riemann-Liouville, and Caputo. The Riemann-Liouville approach starts with generalization of Cauchy's formulae for repeated integration to non-integer orders, defining the fractional order integration as:

$$J_t^{\alpha} f(t) = D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}\, \mathrm{d}\tau \quad (4)$$

where $\Gamma(.)$ is the Gamma function defined by the following expression known as factorial function

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt \quad (5)$$

for which, when α is an integer, it reduces to the conventional factorial i.e. $\Gamma(\alpha+1) = \alpha!$ The definition of fractional derivative can easily be derived by taking an n^{th} order derivative of an α^{th} order integral to obtain an $n - \alpha = q$ order derivative:

$$D_t^q f(t) = D_t^{n-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} D_t^n \int_0^t f(\tau)(t-\tau)^{\alpha-1}\, \mathrm{d}\tau \quad (6)$$

which is the Riemann-Liouville expression of fractional derivative. It should be noted that for $q = 1 (n = 2, \alpha = 1)$, the above expression becomes the usual first order derivative. In the general case, for non-integer q, n is chosen as the smallest integer bigger than q: $n = \lceil q \rceil$. Furthermore, most properties of conventional (integer) derivatives can be extended to the non-integer order case. Since in the analysis of dynamic systems one can take advantage of Laplace operator to represent the system by transfer function, it is possible to write:

$$L\{D_t^{\alpha} f(t)\} = s^{\alpha} L\{f(t)\} - \sum_{k=0}^{n-1} s^k D_t^{\alpha-1-k} f(0) \quad (7)$$

which becomes very simple if all the derivatives are zero at the time origin i.e.

$$L\{D_t^{\alpha} f(t)\} = s^{\alpha} L\{f(t)\} \quad (8)$$

Consequently, the Laplace transforms for various functions and their inverses can be derived.

3.1 Fractional Order System

A fractional order differential equation with commensurate orders can be described by:

$$\sum_{n=0}^{N} a_n D_t^{n\alpha}(y) = \sum_{m=0}^{M} b_m D_t^{m\alpha}(u) \quad (9)$$

The corresponding transfer function can be written as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{m=0}^{M} b_m s^{m\alpha}}{\sum_{n=0}^{N} a_n s^{n\alpha}} \quad (10)$$

with its state-space realization given by:

$$\begin{aligned} D_t^{\alpha} x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (11)$$

where $x(t) \in \mathbb{R}^N, u(t), y(t) \in \mathbb{R}$, are respectively the state, input, and output and $A \in \mathbb{R}^{N \times N}, B \in \mathbb{R}^{N \times 1}, C \in \mathbb{R}^{1 \times N}, D \in \mathbb{R}$, the state-space parameters matrices.
The state equation corresponding to (11) in Laplace domain can be written as:

$$\begin{aligned} s^{\alpha} X(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned} \quad (12)$$

Where: s is the Laplace complex frequency, α the fractional order ($\alpha \in \mathbb{R}^*$), U, X, Y respectively input, state, output, and A, B, C, D the state-space parameters matrices.

4 Identification Method

Frequency domain subspace identification has proven to be an efficient and flexible method for linear system identification [9]. For our purpose this method is extended to the parametric estimation of fractional order system mainly by using fractional frequency powers $(j\omega)^{\alpha}$.
Identification algorithm:

1. Frequency Response Function (FRF) estimation with periodic broadband stimulus (e.g. multitone) and frame-averaging
2. Frequency Domain Subspace Identification:
   (a) Frequency normalization to improve matrix conditioning
   (b) Estimate an extended observability matrix of $\hat{O}_r$ with sufficient rank r
      i. Construction of matrix Z as a frequency weighted input-output measurements compound matrix. Fractional power of frequencies: $(j\omega)^{\alpha}$ are used.
      ii. Elimination of input term by QR factorization
      of Z and estimation of $\hat{O}_r$by SVD
   (c) Estimation of matrices $\hat{A}, \hat{C}$ by least-squares solution of $\hat{O}_r$ recursive equations
   (d) Estimation of matrices $\hat{B}, \hat{D}$ by weighted linear least-square regression w.r.t measured FRF
   (e) Frequency de-normalization

5 Experimental Results

The identification algorithm has been applied on the measured impedance curve (fig. 2) with respectively: $\alpha = 1, z_k = j\omega_k$ and $\alpha = 1/2, z_k = \sqrt{j\omega_k}$ and for a range of system orders $n_a = 1, 2 \ldots, 30$. The resulting relative rms error $\epsilon[dB] = 20log(||G - \tilde{G}||/||G||)$, has been recorded for each and the results are displayed in fig. 3. We see that the half differential order yields lower error than the integer order and that the minimum is reached for a lower system order: 8 instead of 30. We observe also that the range of errors is greater for half differentiation order. The modeled impedance curves are displayed for the best case of integer differentiation order and half order along with the measured impedance in fig. 4 and fig. 5 respectively. The curves are graphed in magnitude along with the magnitude of the complex error curve $|G - \tilde{G}|$. The overall error obtained in case of integer order is -17dB and -24dB for half-order. It is interesting to see in the case of integer frequency order that the error is mainly concentrated in the low frequencies and that the high frequencies show signs of over-fitting. In he case of half-order the error curve is much flatter and the low frequencies show a better fit. There is little sign of over-fitting, the system order seems appropriate. A fine tuning with a nonlinear optimization procedure like Levenberg-Marquardt would likely reduce the error further.

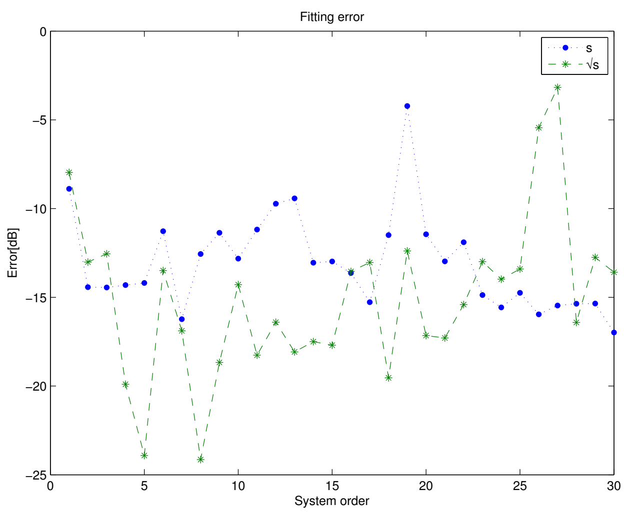

Figure 3. **Identification error vs. system order.**

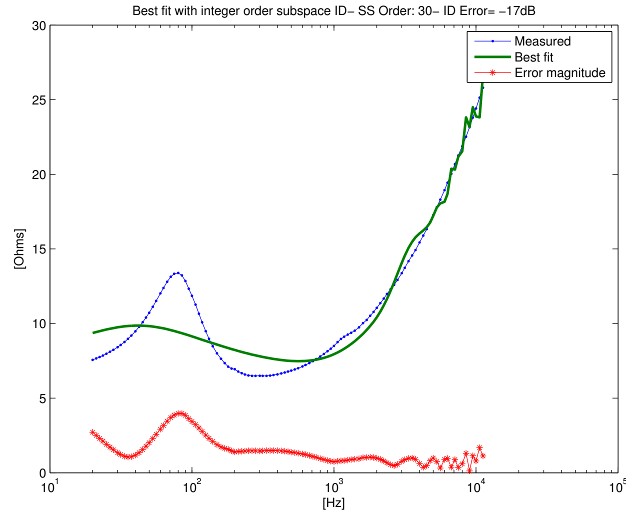

Figure 4. **Estimation based on integer order dynamics.**

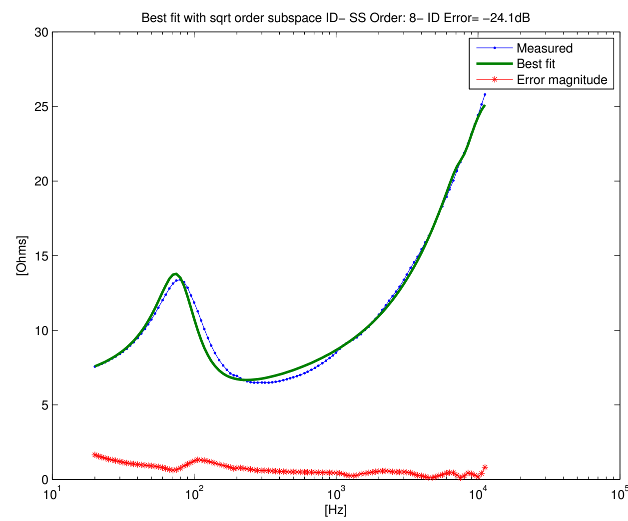

Figure 5. **Estimation based on half order ($\sqrt{j\omega}$) dynamics.**

6 Discussion and Conclusion

Fractional differential order opens new degrees of freedom with the choice of differential order step, commensurate vs. non commensurate orders, continuous vs. discrete order distribution. Recent study [10] shows that different orders and parameters combinations can result in very close transfer functions and the ambiguity grows rapidly with decreasing order values. For example, use of $\alpha = 0.1$ and a system order of 6 result in -30 dB error on the impedance fit. This ambiguity makes the black model approach impossible. On the other hand, the white model approach necessitate a complete understanding of the physical phenomenon and is unrealistic as we have seen. Therefore a grey model approach is the only practical one. The choice of grey model should be guided by the question: what are the useful parameters to ideneify? In our present case, the identification of the mode (resonance frequency, damping) and the DC resistance have physical pertinence and design consequences. On the other end, fractional order is necessary to model the behavior at frequencies above resonance. That suggest the use of a mixed model combining integer and fractional orders. Finally, the nonlinearities inherent to the operation of loudspeaker are not covered by a fractional order system which is inherently linear. The polynomial approach described in [4] could be used similarly with state-variables of fractional order. As mentioned in the introduction, this development will be subject of further studies and publications.

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