Finite Factor Representations of Higman-Thompson groups

Artem Dudko*

Department of Mathematics, Stony Brook University, NY

Konstantin Medynets†

Department of Mathematics, U.S. Naval Academy, Annapolis, MD

Abstract

We prove that the only finite factor-representations of the Higman-Thompson groups $\{F_{n,r}\}$, $\{G_{n,r}\}$ are the regular representations and scalar representations arising from group abelianizations. As a corollary, we obtain that any measure-preserving ergodic action of a simple Higman-Thompson group must be essentially free. Finite factor representations of other classes of groups are also discussed.

1 Introduction

The goal of this paper is to describe finite (in the sense of Murray-von Neumann) factor-representations of the Higman-Thompson groups (see Section 3 for the definition). The discussion of historical importance of these groups and their various algebraic properties can be found in [1], [2], and [11]. The following is the main result of the present paper.

Theorem. *Let G be a group from the Higman-Thompson families $\{F_{n,r}\}$, $\{G_{n,r}\}$ and π be a finite factor representation of G. Then either π is the regular representation or π has the form*

$$\pi(g) = \rho([g])\mathrm{Id},$$

where $[g]$ is the image of g in the abelianization G/G', G' is the commutator of G, $\rho : G/G' \to \mathbb{T}$ is a group homomorphism and Id *is the identical operator in some Hilbert space.*

*email: `artem.dudko@stonybrook.edu`

†email: `medynets@usna.edu`

Since finite factor representations are in one-to-one correspondence with positive definite class functions (termed *characters*, see Definition 2.1), this shows that the characters of any Higman-Thompson group G are convex combinations of the regular character and characters of its abelianization G/G'. The structure of group characters has implications on dynamical properties of group actions. Suppose that a group G admitting no non-regular/non-identity characters acts on a probability measure space (X, μ) by measure-preserving transformations. Setting $\chi(g) = \mu(Fix(g))$, $Fix(g) = \{x \in X : g(x) = x\}$, one can check that the function χ is a character. So our results imply that any non-trivial ergodic action of G on a probability measure space (X, μ) is essentially free, i.e. $\mu(Fix(g)) = 0$ for every $g \in G \setminus \{e\}$, see Theorem 2.11.

In our proofs, we mostly utilize the fact that the commutators of Higman-Thompson groups have no non-atomic invariant measures on the spaces where they are defined. This means that the orbit equivalence relations generated by their actions are compressible [8]. This observation allows us to state the main result in terms of dynamical properties of group actions (Theorems 2.9 and 2.10) — transformation groups whose actions are "compressible" (Definition 2.5) do not admit non-regular II_1-factor representations, except for possible finite-dimensional representations. This dynamical formulation allows us to apply the main result to other classes of transformation groups (Section 3).

In [13] Vershik suggested that the characters of "rich" groups should often arise as $\mu(Fix(g))$ for some invariant measure μ. Thus, this paper confirms Vershik's conjecture in the sense that the absence of non-trivial invariant measures implies the absence of non-regular characters. We also mention the paper [5], where Vershik's conjecture was established for full groups of Bratteli diagrams.

The structure of the paper is the following. In Section 2 we build the general theory of finite factor representations for groups admitting compressible actions. In Section 3, we apply our general results to the Higman-Thompson groups and to the full groups of irreducible shifts of finite type [9].

2 General Theory

In this section we show that if a group admits a compressible action on a topological space, then this group, under some algebraic assumptions, has no non-trivial factor representations. We will start with definitions from the representation theory of infinite groups.

Definition 2.1. A *character* of a group G is a function $\chi : G \to \mathbb{C}$ satisfying the following conditions

1) $\chi(g_1 g_2) = \chi(g_2 g_1)$ for any $g_1, g_2 \in G$;

2) the matrix $\left\{\chi\left(g_i g_j^{-1}\right)\right\}_{i,j=1}^{n}$ is nonnegatively defined for any n and $g_1, \ldots, g_n \in G$;

3) $\chi(e) = 1$. Here e is the group identity.

A character χ is called *indecomposable* if it cannot be written in the form $\chi = \alpha\chi_1 + (1-\alpha)\chi_2$, where $0 < \alpha < 1$ and χ_1, χ_2 are distinct characters.

For a unitary representation π of a group G denote by $\mathcal{M}_\pi$ the W^*-algebra generated by the operators of the representation π. Recall that the *commutant* S' of a set S of operators in a Hilbert space H is the algebra $S' = \{A \in B(H) : AB = BA \text{ for any } B \in S\}$.

Definition 2.2. A representation π of a group G is called a *factor representation* if the algebra $\mathcal{M}_\pi$ is a factor, that is $\mathcal{M}_\pi \cap \mathcal{M}'_\pi = \mathbb{C}\mathrm{Id}$.

The indecomposable characters on a group G are in one-to-one correspondence with the *finite type*[1] factor representations of G. Namely, starting with an indecomposable character χ on G one can construct a triple (π, H, ξ), referred to as the *Gelfand-Naimark-Siegal* (abbr. GNS) *construction.* Here π is a finite type factor representation acting in the space H, and ξ is a unit vector in H such that $\chi(g) = (\pi(g)\xi, \xi)$ for every $g \in G$, see, for example, [5, Sect. 2.3]. Note that the vector ξ is cyclic and separating for the von Neumann algebra $\mathcal{M}_\pi$. The latter means that if $A\xi = 0$ for some $A \in \mathcal{M}_\pi$, then $A = 0$.

Remark 2.3. We note that each character defines a factor representation up to *quasi-equivalence.* Two unitary representations π_1 and π_2 of the same group G are called *quasi-equivalent* if there is an isomorphism of von Neumann algebras $\omega : \mathcal{M}_{\pi_1} \to \mathcal{M}_{\pi_2}$ such that $\omega(\pi_1(g)) = \pi_2(g)$ for each $g \in G$. For example, all II_1 factor representations of an amenable group are hyperfinite [3, Corollary 6.9 and Theorem 6] and, hence, generate isomorphic algebras. At the same time, they might be not quasi-equivalent.

Suppose that G is an infinite conjugacy class (abbr. ICC) group. Then its left regular representation π generates a II_1-factor and the function $\chi(g) = (\pi(g)\delta_e, \delta_e)$ is an indecomposable character (termed the *regular character*).

Definition 2.4. We will say that a group H *has no proper characters* if χ being an indecomposable character of H implies that either χ is *identity character* given by

$$\chi(g) = 1 \text{ for every } g \in G$$

or the *regular character* defined as

$$\chi(g) = 0 \text{ if } g \neq e \text{ and } \chi(e) = 1.$$

We notice that for non-ICC groups the regular characters are decomposable.

[1] The classification of factors can be found in [12, Chapter 5].

Fix a *regular Hausdorff topological space* X. Notice that any two distinct points of X have open neighborhoods with disjoint closures. To exclude trivial counterexamples to our statements we assume that the set X is infinite. Suppose that a group G acts on X. For a group element $g \in G$, denote its *support* by $supp(g) = \overline{\{x \in X : g(x) \neq x\}}$.

Definition 2.5. We will say that the action of G on X is *compressible* if there exists a base of the topology $\mathfrak{U}$ on X such that

(i) for every $g \in G$ there exists $U \in \mathfrak{U}$ such that $\mathrm{supp}(g) \subset U$;

(ii) for every $U_1, U_2 \in \mathfrak{U}$ there exists $g \in G$ such that $g(U_1) \subset U_2$;

(iii) for every $U_1, U_2, U_3 \in \mathfrak{U}$ with $\overline{U}_1 \cap \overline{U}_2 = \varnothing$ there exists $g \in G$ such that $g(U_1) \cap U_3 = \varnothing$ and $\mathrm{supp}(g) \cap U_2 = \varnothing$.

(iv) for any $U_1, U_2 \in \mathfrak{U}$ there exists $U_3 \in \mathfrak{U}$ such that $U_3 \supset U_1 \cup U_2$.

Remark 2.6. Suppose that X is a Polish space. If an action of G on X is compressible, then the G-action has no probability invariant measure. The latter is equivalent to the G-orbit equivalence relation being compressible (see [8] and references therein). This observation motivates our terminology.

The following result relates dynamical properties of group actions to the functional properties of group characters.

Proposition 2.7. *Let G be a countable group admitting a compressible action by homeomorphisms on some regular Hausdorff topological space X. Then for every non-regular indecomposable character χ of G there exists $g \neq e$ such that $|\chi(g)| = 1$.*

Proof. Consider a proper indecomposable character χ of G. Assume that $|\chi(g)| < 1$ for all $g \neq e$. Let (π, H, ξ) be the GNS-construction associated to χ.

(1) We notice that the definition of the compressible action implies that χ has the multiplicativity property in the sense that if $U_1, U_2 \in \mathfrak{U}$ and $g, h \in G$ are such that

$$\mathrm{supp}(g) \subset U_1, \mathrm{supp}(h) \subset U_2 \text{ and } \overline{U}_1 \cap \overline{U}_2 = \varnothing$$

then

$$\chi(gh) = \chi(g)\chi(h). \tag{1}$$

Indeed, find an increasing sequence of finite sets $F_n \subset G$ with $\bigcup_n F_n = G$. Then by the conditions (i) and (iv) of Definition 2.5, we can find open sets $V_n \in \mathfrak{U}$ such that

$$V_n \supset \bigcup_{f \in F_n} \mathrm{supp}(f).$$

By the condition (iii) there exist elements $r_n \in G$ such that

$$r_n(U_1) \cap V_n = \varnothing \text{ and } \mathrm{supp}(r_n) \cap U_2 = \varnothing.$$

Then $r_n h r_n^{-1} = h$ and $supp(r_n g r_n^{-1}) \cap supp(f) = \varnothing$ for every $f \in F_n$. Passing to a subsequence if needed, we can assume that $\pi(r_n g r_n^{-1})$ converges weakly to an operator $Q \in \mathcal{M}_\pi$. Notice that $tr(Q) = \chi(g)$. Since the operator Q commutes with $\pi(F_n)$ for every n, we get that Q belongs to the center of $\mathcal{M}_\pi$. Therefor, Q is scalar and $Q = \chi(g)\mathrm{Id}$. Thus

$$\chi(gh) = \lim_{n\to\infty} (\pi(r_n g h r_n^{-1})\xi, \xi) = (Q\pi(h)\xi, \xi) = \chi(g)\chi(h).$$

(2) We claim that for any $\varepsilon > 0$ and any open set U there exists $g \in G$ with $supp(g) \subset U$ and $|\chi(g)| < \varepsilon$. Indeed, fix an element $h \neq e$ and $n \in \mathbb{N}$. Find n subsets $V_1, \ldots, V_n \in \mathfrak{U}$ such that $\overline{V}_j \cap \overline{V}_k = \varnothing$ for $j \neq k$. By assumptions (i) and (ii) we can choose elements $g_1, \ldots, g_n \in G$ such that $g_j(\mathrm{supp}(h)) \subset V_j$ for each j. Set

$$f = (g_1 h g_1^{-1})(g_2 h g_2^{-1}) \cdots (g_n h g_n^{-1}).$$

By multiplicativity, we obtain that $\chi(f) = \chi(h)^n$. Choosing n sufficiently large we get an element f with $|\chi(f)| < \varepsilon$. By assumptions (i) and (ii) we can find an element g conjugate to f with $\mathrm{supp}(g) \subset U$, which proves the claim.

(3) Consider an element $g \in G$, $g \neq e$. Find an open set U with $\overline{g(U)} \cap \overline{U} = \varnothing$. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Using the condition (ii) and (iv) of Definition 2.5, we can find subsets $U_1, \ldots, U_n, V_1, \ldots, V_n \in \mathfrak{U}$ with pairwise disjoint closures such that $g(V_i) \subset U_i \subset U$ for each i. Find $h_j \in G, j = 1, \ldots, n$ supported by U_j with $|\chi(h_j)| < \varepsilon$. Set $\xi_j = \pi(h_j g h_j^{-1})\xi$. Then for $i \neq j$, the multiplicativity of χ implies that

$$\begin{aligned}
(\xi_i, \xi_j) &= \chi(h_j g^{-1} h_j^{-1} h_i g h_i^{-1}) \\
&= \chi(h_j (g^{-1} h_j^{-1} g)(g^{-1} h_i g) h_i^{-1}) \\
&= \chi(h_j)\chi(g^{-1} h_j^{-1} g)\chi(g^{-1} h_i g)\chi(h_i^{-1}).
\end{aligned}$$

As $|\chi(h_j)| < \varepsilon$, we obtain that $|(\xi_j, \xi_i)| < \varepsilon$. Thus,

$$\|\xi_1 + \ldots + \xi_n\| \leq \big(n + n(n-1)\varepsilon\big)^{\frac{1}{2}}.$$

Since $(\xi_l, \xi) = \chi(g)$ for each l, we have

$$|\chi(g)| = \tfrac{1}{n}|(\xi_1 + \xi_2 + \ldots + \xi_n, \xi)| \leq \tfrac{1}{n}\big(n + n(n-1)\varepsilon\big)^{\frac{1}{2}}.$$

When n goes to infinity, we obtain

$$|\chi(g)| \leq \varepsilon^{\frac{1}{2}}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\chi(g) = 0$. Thus, χ is the regular character. □

Lemma 2.8. *Let G be a simple group and χ be a character. If $|\chi(g)| = 1$ for some $g \in G, g \neq e$, then*

$$\chi(s) = 1 \text{ for all } s \in G.$$

In particular, if χ is not the identity character, then $|\chi(s)| < 1$ for all $s \neq e$.

Proof. Let $c = \chi(g)$, $|c| = 1$. Consider the GNS construction (π, H, ξ) corresponding to χ. Using the Cauchy-Schwarz inequality and the fact that the vector ξ is separating, we obtain that

$$(\pi(g)\xi, \xi) = c \;\Rightarrow\; \pi(g)\xi = c\xi \;\Rightarrow \pi(g) = c\mathrm{Id}.$$

Take an arbitrary element $h \in G$ which does not commute with g and set $s = hgh^{-1}g^{-1}$. Then $\pi(s) = \mathrm{Id}$. It follows that $\pi(s_1) = \mathrm{Id}$ for all s_1 from the normal subgroup generated by s. Since G is simple, we get that $\pi(g) = \mathrm{Id}$ for every $g \in G$. Thus, χ is the identity character. □

As a corollary of Lemma 2.8 and Proposition 2.7 we immediately obtain the following result.

Theorem 2.9. *Let G be a simple countable group admitting a compressible action on a regular Hausdorff topological space X. Then G has no proper characters.*

Let G be a group. For a subgroup R of G and an element $g \in G$ set $C_R(g) = \{hgh^{-1} : h \in R\}$. Denote by $N(R)$ the normal closure of R in G, i.e., the subgroup of G generated by all elements of the form $grg^{-1}, g \in G, r \in R$.

Theorem 2.10. *Let G be a group and R be an ICC subgroup of G such that*

(i) R has no proper characters;

(ii) for every $g \in G \setminus \{e\}$, there exists a sequence of distinct elements $\{g_i\}_{i\geq 1} \subset C_R(g)$ such that $g_i^{-1}g_j \in R$ for any i, j.

Then each finite type factor representation π of G is either regular or has the form

$$\pi(g) = \omega([g]),$$

where ω is a finite factor representation of $G/N(R)$ and $[g] \in G/N(R)$ is the coset of the element g.

Proof. Consider an indecomposable character χ of G. Let (π, H, ξ) be the GNS-construction associated to χ.

(1) Consider the restriction of π onto the subgroup R. Set $H_R = \overline{Lin(\pi(R)\xi)}$. Since the restriction of χ on R is a character and the only indecomposable characters of the group R are the regular and the identity

characters, we can decompose the space H_R into R-invariant subspaces H_1 and H_2 (possibly trivial) such that $H_R = H_1 \bigoplus H_2$ with $\pi(R)|H_1$ being the identity representation and $\pi(R)|H_2$ being the regular representation.

The orthogonal projections $\{P_i\}$ onto H_i, $i = 1, 2$ belong to the center of the algebra generated by $\pi(R)$. In particular, P_i lies in the algebra $\mathcal{M}_\pi$. Furthermore,

$$\chi(g) = \alpha\chi_{id}(g) + (1 - \alpha)\chi_{reg}(g) \text{ for all } g \in R,$$

where χ_{id} is the identity character, χ_{reg} is the regular character, and $\alpha \in [0, 1]$. If $\alpha \neq 0, 1$, we can write down the vector ξ as

$$\xi = \alpha^{\frac{1}{2}}\xi_1 + (1 - \alpha)^{\frac{1}{2}}\xi_2, \tag{2}$$

where $\xi_1 \in H_1$, $\xi_2 \in H_2$ are unit vectors such that

$$(\pi(h)\xi_1, \xi_1) = \chi_{id}(h) = 1, \ \ (\pi(h)\xi_2, \xi_2) = \chi_{reg}(h) = \delta_{h,e} \text{ for all } h \in R.$$

For convenience, if $\alpha = 0$, we set $\xi_1 = 0, \xi_2 = \xi$, if $\alpha = 1$, we set $\xi_1 = \xi, \xi_2 = 0$. Observe that $H_i = \overline{Lin(\pi(R)\xi_i)}$, $i = 1, 2$.

(2) Assume that $H_2 \neq \{0\}$. Consider an arbitrary element $g \in G$, $g \neq e$. By our assumptions there exists a sequence of elements $\{h_n\} \in R \setminus \{e\}$ such that $h_m^{-1}g^{-1}h_m h_n^{-1}gh_n \in R$ for all n and m and elements $h_n^{-1}gh_n$ are pairwise distinct. Set $g_m = h_m^{-1}gh_m$. Since $g_m^{-1}g_n \in R \setminus \{e\}$, we get that

$$(\pi(g_n)\xi_2, \pi(g_m)\xi_2) = \chi_{reg}(g_m^{-1}g_n) = 0.$$

This shows that $\pi(g_m)\xi_2 \to 0$ weakly. Observe also that

$$\begin{aligned}(\pi(g_n)\xi_2, \xi_2) &= (\pi(g_n)P_2\xi, P_2\xi) = tr(P_2\pi(h_n^{-1}gh_n)P_2) \\ &= tr(\pi(h_n^{-1})P_2\pi(g)P_2\pi(h_n)) = tr(P_2\pi(g)P_2).\end{aligned}$$

Since the latter is independent of n and $\pi(g_n)\xi_2 \to 0$, we conclude that

$$(\pi(g)\xi_2, \xi_2) = tr(P_2\pi(g)P_2) = 0.$$

Set $H_0 = \overline{Lin(\pi(G)\xi_2)}$. Then $\pi(G)|H_0$ is quasi-equivalent to the regular representation. Since π is a factor representation, we conclude that π is the regular representation.

(3) Assume that $H_2 = \{0\}$. Then $\xi = \xi_1$ and $\pi(h) = \mathrm{Id}$ for every $h \in R$. Therefor, $\pi(g) = \mathrm{Id}$ for all $g \in N(R)$. This means that the representation π factors through the quotient $G/N(R)$ and defines a finite type factor representation ω of $G/N(R)$ such that $\pi(g) = \omega([g])$ for all $g \in G$. $\square$

Recall that a finite factor representation of a group G is of *type I* if the von Neumann algebra of the representation is isomorphic to the algebra of all linear operators in some finite-dimensional Hilbert space. We say that an action of group G on a measure space (Y, μ) is *trivial* if $g(x) = x$ for every $g \in G$ and μ-almost every $x \in X$. The following result shows that any ergodic action of a group admitting no characters must be *essentially free*, that is $\mu(Fix(g)) = 0$ for all $g \in G \setminus \{e\}$.

Theorem 2.11. *Assume that every finite factor representation of a countable ICC group G is either regular or of type I and that there is at most a countable number (up to quasi-equivalence) of finite factor representations of G. Then every faithful ergodic measure-preserving action of G is essentially free.*

Proof. Consider an ergodic action of G on a measure space (Y, μ). Set

$$\widetilde{Y} = \{(x, y) \in Y \times Y | x = g(y) \text{ for some } g \in G\}.$$

For a Borel set $A \subset \widetilde{Y}$ and a point $x \in Y$, set $A_x = \{(x, y) \in A\}$. Define a σ-finite measure $\widetilde{\mu}$ on $\widetilde{Y}$ by $\widetilde{\mu}(A) = \int_Y card(A_x) d\mu(x)$. Given a function $f \in L^2(\widetilde{Y}, \widetilde{\mu})$ and a group element $g \in G$, set

$$(\pi(g)f)(x, y) = f(g^{-1}x, y).$$

Then $\pi(g)$ is a unitary operator on the Hilbert space $L^2(\widetilde{Y}, \widetilde{\mu})$. Denote by ξ the indicator function of the diagonal of $Y \times Y$. Set $H = \overline{Lin\{\pi(G)\xi\}}$. We note the von Neumann algebra $\mathcal{M}_\pi$ generated by $\pi(G)$, restricted to H, is of finite type. We refer the reader to [4] for the details. Since the group G has at most a countable number of finite factor representations, the representation π decomposes into a direct sum (at most countable) of factor representations.

Our goal is to show that the representation π is regular. Then the uniqueness of the trace implies that $(\pi(g)\xi, \xi) = 0$ for every $g \neq e$. Using the identity $\mu(Fix(g)) = (\pi(g)\xi, \xi)$, we get that the action is essentially free.

Suppose to the contrary that the decomposition of π into factors contains a *non-regular* factor representation π_1, which, by our assumptions, generates a finite-dimensional von Neumann factor. Let P_1 be a projection from the center of $\mathcal{M}_\pi$ such that $\pi_1(g) = P_1\pi(g)$ for every $g \in G$. Set $\xi_1 = P_1\xi$.

Since for every $g \in G$ the unitary operator $(\pi'(g)f)(x, y) = f(x, g^{-1}y)$ belongs to $\mathcal{M}'_\pi$ and $\pi'(g)\xi = \pi(g^{-1})\xi$, we have that

$$\pi'(g)\pi(g)\xi_1 = \pi'(g)\pi(g)P_1\xi = P_1\pi'(g)\pi(g)\xi = P_1\xi = \xi_1$$

for all $g \in G$. This implies that the function $h(x) := |\xi_1(x, x)|$ is G-invariant and μ-integrable on Y. By the ergodicity, we get that $h(x) \equiv C$ on Y for some constant C. Note that if $C = 0$, then

$$0 = \int_{\widetilde{Y}} \xi_1(x, y)\xi(x, y) d\widetilde{\mu}(x, y) = (\xi_1, \xi),$$

which is impossible as the projection of ξ onto ξ_1 is non-trivial.

Fix an orthonormal basis $\eta_1, \ldots, \eta_n$ for $\overline{Lin\{\pi_1(G)\xi_1\}}$. For a given $g \in G$, write

$$\pi_1(g)\xi_1 = \sum_{j=1}^{n} \alpha_j(g)\eta_j$$

for some $\alpha_1(g), \ldots, \alpha_n(g)$ with $\sum |\alpha_j(g)|^2 = |\xi_1|^2 \leq 1$. Observe that

$$\sum_{j=1}^{n} \alpha_j(g)\eta_j(x, y) = (\pi_1(g)\xi_1)(x, y) = (\pi(g)\xi_1)(x, y) = \xi_1(g^{-1}x, y)$$

for every $(x, y) \in \widetilde{Y}$. Since $|\xi_1(g^{-1}x, y)| = C$ for $(x, y) \in \widetilde{Y}$ with $x = g(y)$, we conclude that $\sum_{j=1}^{n} |\eta_j(x, y)| \geq C > 0$ for (x, y) with $x = gy$ and, thus, for any $(x, y) \in \widetilde{Y}$. This implies that the function $\sum_{j=1}^{n} |\eta_j(x, y)|$ is not integrable with respect to $\widetilde{\mu}$. This contradiction yields that $\pi_1 = 0$ and, thus, the representation π is regular. □

We finish this section by giving examples of groups admitting no compressible actions. We observe that even though the following proposition yields a result to that of Theorem 2.10, the underlying assumptions are different and not mutually interchangeable.

Proposition 2.12. *Let G be a countable group with trivial center and such that every proper quotient is finite or abelian. Assume that the group G admits a compressible action on a regular Hausdorff space. Then all finite (Murray von Neumann) non-regular representations of G are of type I.*

Proof. Consider a non-regular indecomposable character χ of G. Let (π, H, ξ) be the GNS-construction associated to χ. By Proposition 2.7 there exists $g \neq e$ such that $|\chi(g)| = 1$. Choose $h \in G$ not commuting with g. Denote by N the normal subgroup of G generated by the element $ghg^{-1}h^{-1}$. Using the arguments from the proof of Lemma 2.8 we obtain that $\pi|N = \mathrm{Id}$. Thus, the representation π of the group G gives rise to the representation of G/N with the same von Neumann algebra. Recall that factor representations of abelian groups are scalar. □

If a group G as in the proposition above has a measure-preserving action on a measures space (X, μ) with $0 < \mu(Fix(g)) < 1$ for some $g \neq e$, then, in view of Theorem 2.11, such a group cannot have compressible actions. Examples of such groups are full groups of even Bratteli diagrams, commutators of topological full groups of Cantor minimal systems [5], and just infinite branch groups [6].

3 Applications

In this section we show that the results established in the previous section are applicable to the Hignam-Thompson groups and to the full groups of irreducible shifts of finite type.

3.1 The Higman-Thompson groups

Definition 3.1. Fix two positive integers n and r. Consider an interval $I_r = [0, r]$. Define the group $F_{n,r}$ as the set of all orientation preserving piecewise linear homeomorphisms h of I_r such that all singularities of h are in $\mathbb{Z}[1/n] = \{\frac{p}{n^k} : p, k \in \mathbb{N}\}$; the derivative of h at any non-singular point is n^k for some $k \in \mathbb{Z}$.

Observe that the commutator subgroup of $F_{n,r}$ is a simple group and the abelianization of $F_{n,r}$ is isomorphic to $\mathbb{Z}^n$ [1, Section 4]. Consider the subgroup $F^0_{n,r}$ of $F_{n,r}$ consisting of all elements $f \in F_{n,r}$ with $supp(f)$ being a subset of $(0, r)$. Observe that (the commutator subgroup) $F'_{n,r} = (F^0_{n,r})'$ [1, Section 4]. The following lemma shows that the commutator subgroup $F'_{n,r}$ satisfies the assumptions of Theorem 2.9.

Lemma 3.2. *The base of topology* $\mathfrak{U} = \{(a, b) : [a, b] \subset (0, r), a, b \in \mathbb{Z}[\frac{1}{n}]\}$ *satisfies the conditions (i)-(iv) of Definition 2.4 for the action of the group* $R = (F^0_{n,r})'$ *on* $(0, r)$*. Thus, the action of* R *is compressible.*

Proof. The conditions (i) and (iv) of Definition 2.4 are clearly satisfied.

To check the condition (ii), consider intervals $U_1 = (a, b)$ and $U_2 = (c, d)$ both in $\mathfrak{U}$. Replacing U_2 by a subinterval if necessary we may assume that $\frac{b-a}{d-c} = n^k$ for some $k \in \mathbb{Z}$. Since the function $\frac{a-x}{c-x}$ is continuous in x for $x \neq c$, we can find $x \in \mathbb{Z}[\frac{1}{n}]$ such that $0 < x < \min\{a, c\}$ and $\frac{a-x}{c-x} = n^k$ for some $k \in \mathbb{Z}$. Similarly, we can find $y \in \mathbb{Z}[\frac{1}{n}]$, $\max\{b, d\} < y < r$ such that $\frac{y-b}{y-d} = n^k$ for some $k \in \mathbb{Z}$. Let $g : [0, r] \to [0, r]$ be the function such that

$$g(0) = 0,\ g(x) = x,\ g(a) = c,\ g(b) = d,\ g(y) = y,\ g(r) = r,$$

and g is linear on each of the line segments $[0, x], [x, a], [a, b], [b, y], [y, r]$. Then $g \in R$ and $g(U_1) = U_2$.

To check the condition (iii), we consider intervals $U_i = (a_i, b_i)$, $i = 1, 2, 3$ from $\mathfrak{U}$ such that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. Without loss of generality, assume that $a_1 > b_2$. Set $a = a_1, b = b_1$ and fix $c, d \in \mathbb{Z}[\frac{1}{n}]$ with $\max\{b_2, b_3\} < c < d < r$ and $\frac{b-a}{d-c} = n^k$ for some $k \in \mathbb{Z}$. Construct g as above with $a_1 > x > b_2$ and $r > y > d$. Then $\text{supp}(g) \subset [x, y]$ and $g(U_1) = (c, d)$. Therefore, $g(U_1) \cap U_3 = \varnothing$ and $\text{supp}(g) \cap U_2 = \varnothing$. $\square$

Observe that all finite factor representations of abelian groups are scalar representations, i.e. $\pi(g) = c_g \text{Id}$, with $c_g \in \mathbb{T}$, the unit circle. In particular, the indecomposable characters of abelian groups are homomorphisms into $\mathbb{T}$.

Corollary 3.3. *(1) The group $F'_{n,r}$ has no proper characters. (2) If χ is an indecomposable character of $F_{n,r}$, then χ is either regular or $\chi(g) = \rho([g])$, where $[g]$ is the image of g in the abelianization of $F_{n,r}$ and $\rho : \mathbb{Z}^n \to \mathbb{T}$ is a group homomorphism.*

Proof. Statement (1) immediately follows from Lemma 3.2 and Theorem 2.9.

To establish the second result, we only need to check the condition (2) of Theorem 2.10. Fix $g \in F_{n,r}\backslash\{e\}$. Find an interval (a, b) with $g(a, b)\cap(a, b) = \emptyset$. Find a sequence of distinct elements $\{h_n\}_{n\geq 1} \subset (F_{n,r})'$ supported by (a, b). Then $(h_n^{-1}g^{-1}h_n)(h_m^{-1}gh_m) \in (F_{n,r})'$ for any $n \neq m$. This completes the proof. □

Definition 3.4. Let n and r be positive integers. Define Higman's group $G_{n,r}$ as the group of all right continuous bijections of $[0, r)$ which are piecewise linear, with finitely many discontinuities and singularities, all in $\mathbb{Z}[1/n]$, slopes in $\{n^k : k \in \mathbb{Z}\}$, and mapping $\mathbb{Z}[1/n] \cap [0, r)$ to itself.

Note that $F_{n,r} \subset G_{n,r}$. In fact, $F_{n,r}$ consists exactly of all continuous elements $g \in G_{n,r}$. In [7] Higman showed that the commutator subgroup $G'_{n,r}$ is simple and that the abelianization of $G_{n,r}$ is trivial for even n and is $\mathbb{Z}/2\mathbb{Z}$ for odd n.

Lemma 3.5. *The groups $R = F'_{n,r}$ and $G = G_{n,r}$ satisfy the conditions of Theorem 2.10.*

Proof. Corollary 3.3 shows that the group R has no proper characters. Consider an arbitrary element $g \in G, g \neq e$. Choose an open interval I such that $I \cap g^{-1}(I) = 0$ and g is continuous on both I and $g^{-1}(I)$. It follows that for any two elements $r_1, r_2 \in R$ with $\mathrm{supp}(r_1) \subset I$, $\mathrm{supp}(r_2) \subset I$ the element

$$h = r_2g^{-1}r_2^{-1}r_1gr_1^{-1} \neq e$$

is a continuous bijection of $[0, r)$. Observe that h acts identically near 0 and r. It follows that $h \in R$ and the elements $r_1gr_1^{-1}$ and $r_2gr_2^{-1}$ belong to the same coset of G/R. Since the group R has infinitely many elements supported by the set I, we immediately establish the condition (ii). □

The following result is an immediate corollary of Theorem 2.10 applied twice to the pairs $R = F'_{n,r}$, $G = (G_{n,r})'$ and $R = F'_{n,r}$, $G = G_{n,r}$.

Corollary 3.6. *(1) The group $G'_{n,r}$ has no proper characters.*

(2) If χ is an indecomposable character of $G_{n,r}$, then χ is either regular or $\chi(g) = \rho([g])$, where $[g]$ is the image of g in the abeliazation of $G_{n,r}$ and $\rho : G_{n,r}/G'_{n,r} \to \mathbb{T}$ is a group homomorphism.

3.2 Full groups of irreducible shifts of finite type

We refer the reader to [9, Section 6] for the comprehensive study of full groups of étale groupoids including the groups discussed below.

Let (V, E) be a finite directed graph. Suppose that the adjacency matrix of the graph is irreducible and is not a permutation matrix. For an edge $e \in E$, denote by $i(e)$ the initial vertex and by $t(e)$ its terminal vertex. Set

$$X = \{\{e_n\}_{n\geq 1} \in E^{\mathbb{N}} : t(e_k) = i(e_{k+1}) \text{ for every } k \in \mathbb{X}\}.$$

Equipped with the product topology, X is a Cantor set. We note that the space X along with the left shift is called a one-sided subshift of finite type, see [9] and references therein regarding relations with the symbolic dynamics.

An n-tuple $(e_1, \ldots, e_n) \in E^n$ is called *admissible* if $t(e_k) = i(e_{k+1})$ for every $1 \leq k \leq n-1$. Two admissible tuples $\overline{e} = (e_1, \ldots, e_n)$ and $\overline{f} = (f_1, \ldots, f_m)$ are called *compatible* if $t(e_n) = t(f_m)$. Each admissible tuple $\overline{e} = (e_1, \ldots, e_n)$ defines a clopen set $U(\overline{e}) = \{x \in X : x_i = e_i,\ i = 1, \ldots, n\}$. Such clopen sets form the base of topology. Given two compatible admissible tuples $\overline{e}_1$ and $\overline{e}_2$, define a continuous map $\pi_{\overline{e}_1,\overline{e}_2} : U(\overline{e}_1) \to U(\overline{e}_2)$ as

$$\pi_{\overline{e}_1,\overline{e}_2}(\overline{e}_1, x_{n+1}, x_{n+2}, \ldots) = (\overline{e}_2, x_{n+1}, x_{n+2}, \ldots).$$

Definition 3.7. Following [9], we define the *full group* of X, in symbols $[[X]]$, as the set of all homeomorphisms g of X for which there exists two clopen partitions $X = \bigsqcup_{i=1}^n U(\overline{e}_i) = \bigsqcup_{i=1}^n U(\overline{f}_i)$ with e_i and f_i being compatible admissible tuples (possibly of different lengths), $i = 1, \ldots, n$, such that $g|U_{\overline{e}_i} = \pi_{\overline{e}_i,\overline{f}_i}$ for every $i = 1, \ldots, n$.

For a clopen subset $Y \subset X$, set $[[X|Y]]$ as the set of all $g \in [[X]]$ with $\mathrm{supp}(g) \subset Y$.

The following result was established in [9, Lemma 6.1 and Theorem 4.16]

Proposition 3.8. *For any clopen set $Y \subset X$, the commutator group $[[X|Y]]'$ is simple.*

Fix an arbitrary point $x_0 \in X$. Find an increasing sequence of clopen sets $\{Y_n\}$ such that $X \setminus \{x_0\} = \bigcup_n Y_n$. Set $R = \bigcup_n [[X|Y_n]]'$. It follows from Proposition 3.8 that the group R is simple. Observe that the group R consists of all elements $g \in [[X]]'$ equal to the identity on some neighbourhood of x_0.

Denote by $\mathcal{F}$ the set of all admissible tuples which are *not prefixes* of x_0. Define $\mathfrak{U}$ as the family of all finite unions of sets from $\{U(\overline{e})\}_{\overline{e}\in\mathcal{F}}$. Notice that $\mathfrak{U}$ is a base of the topology on $X \setminus \{x_0\}$. One can check that $\mathfrak{U}$ satisfies conditions (i)-(iv) of Definition 2.5 for the action of R. Thus, using Theorem 2.9, we conclude that the group R has no characters. Considering R as a subgroup of $G = [[X]]$, one can check that the assumptions of Theorem 2.10 are satisfied. We leave the details to the reader.

Corollary 3.9. *If χ is an indecomposable character of $[[X]]$, then χ is either regular or $\chi(g) = \rho([g])$, where $[g]$ is the image of g in the abelianization of $[[X]]$ and $\rho : [[X]]/[[X]]' \to \mathbb{T}$ is a group homomorphism.*

To finish our discussion, we notice that the full group of the one-sided Bernoulli shift over the alphabet with n letters is isomorphic to $G_{n,1}$ [10].

Acknowledgements. We would like to thank R. Grigorchuk for the discussion of Higman-Thompson groups and for his valuable comments.

References

[1] K. Brown, *Finiteness properties of groups.* J. Pure Appl. Algebra **44** (1987), no. 1-3, 45-75.

[2] J.W. Cannon, J.W. Floyd, W.R. Parry, *Introductory notes on Richard Thompson's groups.* Enseign. Math. (2) 42 (1996), no. 3–4, 215–256.

[3] A. Connes, *Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$* Ann. of Math. (2) 104 (1976), no. 1, 73–115.

[4] J. Feldman and C. Moore, *Ergodic Equivalence Relations, Cohomology, and Von Neumann Algebras. I.* Trans. of the AMS, **234**, No. 2. (1977), 289–324.

[5] A. Dudko and K. Medynets, *On Characters of Inductive Limits of Symmetric Groups*, arXiv:1105.6325, (2012).

[6] R. Grigorchuk, *Some problems of the dynamics of group actions on rooted trees.* (Russian) Tr. Mat. Inst. Steklova **273** (2011), Sovremennye Problemy Matematiki, 72–191; translation in Proc. Steklov Inst. Math. **273** (2011), no. 1, 64–175

[7] G. Higman, *Finitely presented infinite simple groups.* Notes on Pure Mathematics, No. 8 (1974). Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974. vii+82 pp.

[8] R. Dougherty, S. Jackson and A.S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), 193-225.

[9] H. Matui, *Topological full groups of one-sided shifts of finite type*, (2012) arXiv:1210.5800.

[10] V. Nekrashevych, *Cuntz-Pimsner algebras of group actions.* J. Operator Theory **52** (2004), no. 2, 223-249.

[11] M. Stein, *Groups of piecewise linear homeomorphisms.* Trans. Amer. Math. Soc. **332** (1992), no. 2, 477-514.

[12] M. Takesaki, Theory of operator algebras I. Encyclopedia of Mathematical Sciences, vol. 124.

[13] A. Vershik, *Nonfree Actions of Countable Groups and their Characters*, arXiv:1012.4604 (2010).