Paths partition with prescribed beginnings in digraphs: a Chvátal-Erdős condition approach.

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Abstract

A digraph $D$ verifies the Chvátal-Erdős conditions if $\alpha(D) \leq \kappa(D)$, where $\alpha(D)$ is the stability of $D$ and $\kappa(D)$ is its vertex-connectivity. Related to the Gallai-Milgram Theorem ([5]), we raise in this context the following conjecture. For every set of $\alpha = \alpha(D)$ vertices $\{x_1, \ldots, x_\alpha\}$, there exists a vertex-partition of $D$ into directed paths $\{P_1, \ldots, P_\alpha\}$ such that $P_i$ begins at $x_i$ for all $i$. The case $\alpha(D) = 2$ of the conjecture is proved.

1 Introduction.

All topics of the paper deal with digraphs. Considered paths and circuits are directed ones. In our digraphs, circuits of length 2 are allowed, but not loops. We denote by $\kappa$ the vertex-connectivity, and by $\alpha$ the stability of a digraph. All partitions or coverings of digraphs mentioned in the paper are understood as vertex partitions or coverings.

The classical Gallai-Milgram Theorem (see [5]) states that every digraph admits a partition into $\alpha$ paths. In this paper, we are mainly concerned by finding conditions to prescribe the beginnings of paths in such a partition. This problem is motivated, in a remote way, by coverings of digraphs into circuits (for instance, see [1] or Conjecture 2).

The following definitions are given for a digraph $D$ with vertex set $V$ and arc set $E$. For a path $P$ of $D$, we denote by $b(P)$ and $e(P)$ respectively its beginning and its end. The internal vertices of $P$ are the vertices of $P \setminus \{b(P), e(P)\}$ (possibly empty). For two vertices $x$ and $y$ of $D$, an $(x, y)$-path is a path with beginning $x$ and end $y$. By extension, an $(X, Y)$-path $P$ is an $(x, y)$-path for
some $x \in X$ and $y \in Y$ such that the set of internal vertices of $P$ and $X \cup Y$ are disjoint.

For a path $P$ and a vertex $x$ of $P$, $xP$ (resp. $Px$) denote the maximal sub-path of $P$ which starts (resp. ends) at $x$. Moreover, if $y$ is a vertex of $xP$, $xPy$ denotes the maximal sub-path of $xP$ which ends in $y$ (i.e. the sub-path of $P$ which starts at $x$ and ends at $y$). We denote the concatenation of two paths by $PQ$ (is only used when there exists an arc from the end of $P$ to the beginning of $Q$).

Finally, for an arc $xy \in E$ we also denote by $xy$ the path of length 1 from $x$ to $y$.

A digraph $D$ verifies the Chvátal-Erdős conditions if we have $\alpha(D) \leq \kappa(D)$.

These were named from the following sufficient condition for a (non oriented) graph to have a hamilton cycle, given by Chvátal and Erdős in 1972.

**Theorem 1 ([4])** For a graph $G$, if $\alpha(G) \leq \kappa(G)$, then $G$ has a hamilton cycle.

For digraphs the condition $\alpha \leq \kappa$ ($\kappa$ is here the 'strong' vertex-connectivity) does not imply the existence of hamilton circuit. Infinite families of examples for $\alpha = 2$ and $\alpha = 3$ are given in [7]. However, according to the previous result for graphs, it could seem possible to ask for partitions into paths or circuits in digraphs which satisfy Chvátal-Erdős conditions. Several results and conjectures are stated in a survey of B. Jackson and O. Ordaz (see [7]). We present two new conjectures in this area.

The well-known result of Gallai-Milgram ([5]) asserts that every digraph $D$ admits a path partition into at most $\alpha(D)$ paths. If $D$ satisfies Chvátal-Erdős conditions, we would like to choose the beginnings of these paths.

**Conjecture 1** Let $D$ be a digraph with $\alpha(D) \leq \kappa(D)$. For every set of $\alpha = \alpha(D)$ vertices $\{x_1, \ldots, x_\alpha\}$, there exists $\{P_1, \ldots, P_\alpha\}$ a path partition of $D$ such that $P_i$ begins at $x_i$ for all $i$.

Note that the existence of a hamilton circuit in $D$ implies the result of Conjecture 1. In particular, Conjecture 1 is true for graphs and for digraphs with stability 1 (according to Camion’s theorem [3], strong digraphs with stability 1 have a hamilton circuit). The result of Conjecture 1 is also true if $D$ contains a hamilton path with a prescribed beginning. But, we cannot ask for such a path as a consequence of Chvátal-Erdős condition as seen in Figure 1. This example is derived from those given by B. Jackson and O. Ordaz in [7] to provide digraphs with $\alpha \leq \kappa$ and no hamilton circuit.

The second conjecture deals with partition into circuits. In [6], it is proved that if a digraph satisfies Chvátal-Erdős conditions, then it admits a partition into circuits. In addition, a recent result (see [1]) states that the vertices of every strongly connected digraph can be covered with at most $\alpha$ circuits. So, it could be possible to limit the number of circuits in a circuit partition of a digraph which satisfies Chvátal-Erdős conditions.

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Figure 1: Digraph with $\alpha \leq \kappa = 2$ and no hamilton path starting at $x$.

**Conjecture 2** Every digraph which satisfies Chvátal-Erdős conditions admits a circuit partition into at most $\alpha$ circuits.

Note that Conjecture 2 is true for $\alpha = 1$, according to Camion’s Theorem ([3]), and this seems the sole known case of resolution.

The two next sections give useful tools, Lemmas 2 and 3, for the proof of Conjecture 2 in the case $\alpha = 2$ which is detailed in Section 4.

### 2 Vertices reachable from a prescribed vertex.

Let $u_1, \ldots, u_p$ and $x$ be vertices of $D$. We say that a sequence of paths $(P_1, \ldots, P_p)$ satisfies $[u_1, \ldots, u_p \to x]$ if the $u_i$ are distinct, $b(P_i) = u_i$, $e(P_i) = x$ for every $i = 1, \ldots, p$ and if the paths $P_i$ are pairwise disjoint, except in $x$. We simply say $[u_1, \ldots, u_p \to x]$ to mean that such a sequence of paths exists. By extension, for $A_1, \ldots, A_p, p$ subsets of $V$, $[A_1, \ldots, A_p \to x]$ means that there exists a vertex $u_i$ in $A_i$, for $i = 1, \ldots, p$, such that $[u_1, \ldots, u_p \to x]$.

This will not be used here, but it is known that, for a fixed $x$, the sets of vertices $\{u_1, \ldots, u_k\}$ such that $[u_1, \ldots, u_k \to x]$ form a matroid (for instance, see [9], Chapter 39).

The following lemma is a corollary of Menger’s Theorem ([8]).

**Lemma 1** If, for $p \geq 1$, we have $[v_1, \ldots, v_{p-1}, y \to x]$ and $[v_p, \ldots, v_{2p-1} \to y]$, then there exists a sequence of integers $1 \leq i_1 < \ldots < i_p \leq 2p - 1$ such that $[v_{i_1}, \ldots, v_{i_p} \to x]$.

**Proof.** First, if $x = y$, we have $[v_p, \ldots, v_{2p-1} \to x]$. So, assume that $x \neq y$ and moreover suppose for the moment that none of the $v_i$ is equal to $x$. Denote $X = N_D(x)$, $D' = D \setminus x$ and $W = \{v_i : 1 \leq i \leq 2p - 1\}$. We prove that every separator from $W$ to $X$ has at least $p$ elements. Indeed, assume that there exists a set $S$ of vertices of $D'$ of size at most $p - 1$ which is a separator from $W$ to $X$. Denote $\mathcal{W} = \{z \in V \setminus x : \text{there exists a path from } W \text{ to } z \text{ in } D' \setminus S\}$ and
Theorem \( (\text{see} \ [3]) \) there exists a hamilton circuit of \( D' \). Moreover, it gives

\[ 3 \text{ Paths exchange.} \]

some control on beginnings and ends of the paths. We will refer later this result as 'paths exchange'.

Theorem 2 \((\text{see} \ [2])\) Paths exchange. Let \( D \) be a digraph and \( \{P_1, \ldots, P_k\} \) a path partition of \( D \). If \( k > \alpha(D) \), then there exists \( \{Q_1, \ldots, Q_{k-1}\} \) a path partition of \( D \) such that \( \{b(Q_i) : 1 \leq i \leq k-1\} \subset \{b(P_i) : 1 \leq i \leq k\} \) and \( \{e(Q_i) : 1 \leq i \leq k-1\} \subset \{e(P_i) : 1 \leq i \leq k\} \).

Finally, the next lemma is an easy corollary of the paths exchange and will be useful in next section.

Lemma 3 Let \( D \) be a digraph with stability 2 and two initial components \( M_1 \) and \( M_2 \). Then, for all \( x_1 \) in \( M_1 \) and \( x_2 \) in \( M_2 \) there exists two disjoint paths \( P_1 \) and \( P_2 \) which respectively begin in \( x_1 \) and \( x_2 \) and cover \( D \).

Proof. First, note that \( M_1 \) and \( M_2 \) have each stability 1. So by Camion’s Theorem (see [3]) there exists a hamilton circuit of \( M_1 \) and one of \( M_2 \) and then,
a hamilton path $Q_1$ of $M_1$ which starts at $x_1$ and a hamilton path $Q_2$ of $M_2$ which starts at $x_2$. Now, apply enough paths exchanges on the set of paths $\mathcal{P} = \{Q_1, Q_2\} \cup \{x : x \in V(D \setminus (M_1 \cup M_2))\}$ in order to obtain two disjoint paths $P_1$ and $P_2$ which cover $D$. The beginnings of $P_1$ and $P_2$ belong to $b(\mathcal{P})$, the beginnings of the paths of $\mathcal{P}$. But, in $D$ there is no path from any vertex of $b(\mathcal{P}) \setminus x_1$ to $x_1$, so $x_1$ is the beginning of $P_1$ or $P_2$. Similarly, $x_2$ is the beginning of the other path. $\square$

4 The main result.

This section presents a proof of Conjecture 1 for the case $\alpha = 2$.

**Theorem 3** Let $D$ be a digraph with $\alpha(D) \leq 2$ and $\kappa(D) \geq 2$. Then, for any distinct vertices $x$ and $y$ of $D$, there exists two disjoint paths which respectively start at $x$ and $y$ and cover $D$.

**Proof.** In fact, for a digraph $D$ with stability at most 2 and with at least two vertices, we prove the following stronger statement ($\ast$):

($\ast$) If $A$ and $B$ are non-empty subsets of $V$ such that $[A, B \rightarrow x]$ for all $x$ in $V$, then there exists two disjoint paths $P_1$ and $P_2$ which cover $D$ and such that $b(P_1) \in A$ and $b(P_2) \in B$.

For $\kappa(D) \geq 2$ and fixed $x$ and $y$, the condition of ($\ast$) holds with $A = \{x\}$ and $B = \{y\}$ and the conclusion of ($\ast$) gives the result of the theorem.

So, assume that ($\ast$) is not true and consider among all the counter-examples of ($\ast$) with minimum number of vertices, one with $|A| + |B|$ minimum. Denote by $D$ this extremal digraph. Note that $D$ has at least 3 vertices, this will be useful to apply induction.

We prove several facts on $D$, $A$ and $B$ to obtain a contradiction.

**Fact 1:** The digraph $D$ has a sole initial component. We denote it by $M$.

Indeed, if not, $D$ has two initial components. Denote it by $M_1$ and $M_2$. By hypothesis, $M_1$ has to contain a vertex $a$ of $A$ and $M_2$ has to contain a vertex $b$ of $B$. By Lemma 3, we obtain two disjoint paths which respectively start at $a$ and $b$ and cover $D$. This contradicts the choice of $D$.

**Fact 2:** We have $|A| \geq 2$ and $|B| \geq 2$.

By contradiction, assume that $|A| = 1$ and denote $A = \{a\}$. So, we consider $D' = D \setminus a$, $A' = N^{-1}_D(a)$ and $B' = B \setminus a$ (according that $a \in B$ is possible).

It is clear that $D'$ has stability at most 2 and that $D'$ has at least two vertices (as $D$ has at least 3 vertices). So, let us check that $D'$, $A'$ and $B'$ satisfy the hypothesis of ($\ast$). As $D$ has at least two vertices, $A'$ and $B'$ are not empty. Moreover, for every vertex $x$ of $D'$, as $[a, B \rightarrow x]$ in $D$, we have $[A', B' \rightarrow x]$.
in $D'$. So, ($\star$) applied to $D'$, $A'$ and $B'$ provides two disjoint paths $P$ and $Q$ which cover $D'$ and with $b(P) \in A'$ and $b(Q) \in B'$. Finally, the paths $a.P$ and $Q$ contradict the choice of $D$.

Fact 3: The sets $A$ and $B$ are disjoint.

First, let us prove that every vertex of $A \cap B$ has no in-neighbour in $D$. Otherwise, consider a vertex $x$ of $A \cap B$ and $y$ an in-neighbour of $x$. By hypothesis, there exists $(P, Q)$ a couple of paths which satisfies $[A, B \rightarrow y]$. As $P \setminus y$ and $Q \setminus y$ are disjoint, $x$ cannot belong to both. Assume that $x \notin P$ and consider $A' = A \setminus x$ (which is not empty because $d(P) \in A'$). Now, for every $z$ of $D$ we have $[A', B \rightarrow z]$. Indeed, if for a vertex $z$, $[A, B \rightarrow z]$ does not directly give $[A', B \rightarrow z]$, we have $[x, B \rightarrow z]$. Denote $b$ a vertex of $B$ such that $[x, b \rightarrow z]$. By Lemma 2, as $[d(P), x \rightarrow x]$, we have $[d(P), x \rightarrow z]$ or $[d(P), b \rightarrow z]$ and in both case, $[A', B \rightarrow z]$. Then, by minimality of $A$, there exists two disjoint paths which cover $D$ and start respectively in $A'$ and $B$. As $A' \subset A$, this contradicts the choice of $D$. Similarly, we obtain the same contradiction if $x \notin Q$, and conclude that $x$ cannot have an in-neighbour.

Now, we can prove that $A \cap B = \emptyset$. If not, consider a vertex $x$ of $A \cap B$. According to the previous remarks, $d_D(x) = 0$ and then, by Fact 1, \{x\} is the sole initial component of $D$. If $D \setminus x$ has two initial components $M_1$ and $M_2$, both intersect $N_D^+(x)$, and by hypothesis, both intersect $A \cup B$. So, pick $y \in M_1 \cap (A \cup B)$ and $z \in M_2 \cap N_D^+(x)$. By Lemma 3, there exists two disjoint paths $P$ and $Q$ which cover $D \setminus x$ with $b(P) = y$ and $b(Q) = z$. But now, $P$ and $x.Q$ are two disjoint paths which cover $D$ and respectively start in $A \cup B$ and $A \cap B$. This contradicts the choice of $D$.

Then, $D \setminus x$ has a sole initial component, $M$. By hypothesis, $M$ contains a vertex of $A \cup B$. Assume that there exists a vertex $a \in M \cap A$ and consider $D' = D \setminus x$, $A' = A \setminus x$ and $B' = (B \setminus x) \cup N_D^+(x)$. Note that $A' \neq \emptyset$, $B' \neq \emptyset$ and that $|D'| \geq 2$. Let us see that $[A', B' \rightarrow z]$ for every $z$ in $D'$. For $z$ a vertex of $D'$, consider $(P, Q)$ which realize $[A, B \rightarrow z]$ in $D$. Either $x$ does not belong to $P \cup Q$ and then $[A', B' \rightarrow z]$ in $D'$ is clear, or $x$ belongs to $P \cup Q$.

In this case, as $x$ is the initial component of $D$, we have either $x = d(Q)$, and through $Q \cap N_D^+(x) \neq \emptyset$, $[A', B' \rightarrow z]$ in $D'$ is clear again, or $x = d(P)$. Finally, if $x = d(P)$, denote by $x'$ the successor of $x$ along $P$ ($x' \in B'$). As $a \in M$, there exists a path from $a$ to $x'$. From $[a, x' \rightarrow x']$ and $[x', d(Q) \rightarrow z]$, we derive through the Lemma 2 that $[x', a \rightarrow z]$ or $[x', d(Q) \rightarrow z]$ and, in both cases, we have $[A', B' \rightarrow z]$. So, we apply ($\star$) to $A'$, $B'$ and $D'$ and obtain two disjoint paths $P'$ and $Q'$ which cover $D'$ with $d(P') \in A'$ and $d(Q') \in B'$. If $d(Q') \in N_D^+(x)$, then the paths $P'$ and $x.Q'$ cover $D$, start respectively in $A$ and $B$ and so, contradict the choice of $D$. So, $d(Q') \in (B' \setminus N_D^+(x)) \subset B$ and $d(P') \in A' \subset A$ and we apply a paths exchange on $P'$, $Q'$ and the path $x$. As $\{x\}$ is the initial component of $D$, we obtain two disjoint paths which cover $D$ whose the beginning of one of them is $x$. The beginning of the other path is $d(Q')$ or $d(P')$. We finally provide two disjoint paths which cover $D$ and start respectively in $A \cap B$ and $A \cup B$. This contradicts the choice of $D$. 

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Fact 4: The sole initial component of $D$ contains $A \cup B$.

By contradiction, assume that there exists $a \in M \setminus A$ and consider $A' = A \setminus a$ which is non empty (as $M \cap A \neq \emptyset$). Denote by $X$ the set \{x \in V : [A', B \rightarrow x]\}. If $X = V$, then, by minimality of $A$, we cover $D$ by two disjoint paths which start respectively in $A'$ and $B$, and as $A' \subset A$, this contradicts the choice of $D$. So, $Y$, the complementary of $X$ in $V$ is not empty. By hypothesis, there exists a path from $a$ to every vertex of $Y$. So, in particular, $Y \cap M = \emptyset$ and if $M_Y$ denotes an initial component of $Y$, there exists a vertex $x$ in $X$ which is the in-neighbour of a vertex of $M_Y$. Let us see that $u$ is a separator in $D$ from $X$ to $M_Y$. Indeed, if not, there exist two vertices $x \in X \setminus u$ and $y \in M_Y$ such that $xy$ is an arc of $D$. The existence of a path in $M_Y$ from $N^+_M(u)$ to $y$ assures that $[x, u \rightarrow y]$ in $D$. But this contradicts the fact that $y \notin X$. Indeed, there exists $a_1, a_2 \in A'$ and $b_1, b_2 \in B$ such that $[a_1, b_1 \rightarrow x]$ and $[a_2, b_2 \rightarrow u]$. By Lemma 2, $[a_1, b_1 \rightarrow x]$ and $[x, u \rightarrow y]$ give $[a_1, u \rightarrow y]$. Finally, through $(\star)$, we provide two disjoint paths which cover $D \setminus P$ and respectively start in $M_Y \cap N^+_M(u)$ and $M' \cap B$. So, the paths $Q'$ and $P' = P.Q'$ are disjoint, cover $D$ and respectively start in $B$ and $A$. This contradicts the choice of $D$.

We conclude similarly in the case $d(P) \in B$.

Fact 5: There exists no three distinct vertices $a, a' \in A$ and $b \in B$ such that $[a, b \rightarrow a']$ (and similarly, there exists no three distinct vertices $b, b' \in B$ and $a \in A$ such that $[a, b \rightarrow b']$).

If not, assume that we have three distinct vertices $a, a' \in A$ and $b \in B$ such that $[a, b \rightarrow a']$ and once again, we reduce $A$. We consider $A' = A \setminus a'$ and claim that $[A', B \rightarrow x]$ for every $x \in V$. Indeed, if for a vertex $x$ we have $[a', b' \rightarrow x]$ for some $b' \in B$, using Lemma 2 and that $[a, b \rightarrow a']$, we obtain that $[a, b \rightarrow x]$ or $[a, b' \rightarrow x]$. As $a \neq a'$, in both case, we have $[A', B \rightarrow x]$. Finally, through $(\star)$, we provide two disjoint paths which cover $D$ and respectively start in $A' \subset A$ and $B$. This contradicts the choice of $D$.

Fact 6: There exists no three distinct vertices $a, a', a'' \in A$ such that $[a, a' \rightarrow a'']$ (and similarly, there exists no three distinct vertices $b, b', b'' \in B$ such that $[b, b' \rightarrow b'']$).
Indeed, as \([a, b \rightarrow a]\) for some \(b \in B\), using Lemma 2 and that \([a, a' \rightarrow a'']\), we have \([a, b \rightarrow a'']\) or \([a', b \rightarrow a'']\). This is impossible by Fact 5 (note that \(a, a', a''\) and \(b\) are disjoint because \(A \cap B = \emptyset\) by Fact 3).

**Fact 7:** There exists no three distinct vertices \(a, a' \in A\) and \(b \in B\) such that \([a, a' \rightarrow b]\) (and similarly, there exists no three distinct vertices \(b, b' \in B\) and \(a \in A\) such that \([b, b' \rightarrow a]\)).

Indeed, if not, by the Fact 2, there exists a vertex \(b' \in B\) distinct of \(b\), and by the Fact 4, \(b'\) is in the initial component of \(D\). So, there exists a path from \(b'\) to \(a\) and we have \([a, b' \rightarrow a]\). By Lemma 2, and through \([a, a' \rightarrow b]\), we have \([a, b' \rightarrow b]\) or \([a', b' \rightarrow b]\), what contradicts in both case Fact 5.

**Fact 8:** If there exists two disjoint arcs \(xx'\) and \(yy'\) with \(x, x', y, y' \in A \cup B\), then there exists two disjoint paths with length at least 1, which cover \(d\) and whose first arcs are respectively \(xx'\) and \(yy'\). In particular, by choice of \(D\), we have \(x, y \in A\) or \(x, y \in B\).

Indeed, assume that there exists two disjoint arcs of \(D\), \(xx'\) and \(yy'\) with \(X = \{x, x', y, y'\} \subset A \cup B\). By Facts 6 and 7, \(x'\) and \(y'\) are not link by a path in \(D \setminus \{x, y\}\). In particular, \(\{x', y'\}\) is a stable set of \(D\). Now, we prove that every vertex \(z \in V \setminus X\) is the end of a \(\{x', y'\}\)-path in \(D \setminus \{x, y\}\). Indeed, fixed \(z \in V \setminus X\), as \(\alpha(D) \leq 2\), either there exists an arc from \(x'\) or \(y'\) to \(z\) and we are done, or there exists an arc from \(z\) to \(x'\) or \(y'\), for instance, say that \(zz' \in E\). By hypothesis, we have \([a, b \rightarrow z]\) for some \(a \in A\) and \(b \in B\). So, consider \(P\) and \(Q\) minimal such that \((P, Q)\) realizes \([A \cup B, A \cup B \rightarrow z]\). If \(x'\) does not belong to \(P \cup Q\), one this two paths does not contain \(x\), say \(P\). Then, \(P\) does not contain \(x\) and \(x'\) and starts in \(A \cup B\), but now, \(P, x'\) and \(xx'\) contradict one of the Facts 5, 6 or 7. So, \(x'\) belongs to \(P \cup Q\). By minimality of the paths \(P\) and \(Q\), \(x\) and \(y\) do not belong to \(P \cup Q\) and we have a path from \(x'\) to \(z\) in \(D \setminus \{x, y\}\).

Finally, \(x'\) and \(y'\) are respectively in two distinct initial components of \(D \setminus \{x, y\}\). By Lemma 3, there exists two disjoint paths which cover \(D \setminus \{x, y\}\) and respectively start at \(x'\) and \(y'\) what proves Fact 8.

**Fact 9:** The sets \(A\) and \(B\) have exactly two elements each.

By Fact 2, we have just to prove that \(|A| \leq 3\) and \(|B| \leq 3\). If not, assume that \(A\) has at least 3 vertices. First, note that there is no circuit in \(D[A]\), because a minimum path from \(B\) to such a circuit would provide three distinct vertices \(a, a' \in A\) and \(b \in B\) such that \([a, b \rightarrow a']\) what is impossible by Fact 5. So, pick three distinct vertices \(a, a', a'' \in A\). Through the previous remark, we can assume that \(a\) dominates \(a'\) (as \(\{a, a', a''\}\) is not a stable set) and \(a'\) does not dominate \(a''\). By Fact 5, \(a''\) does not dominate \(a'\), and then \(\{a', a''\}\) is a stable set of \(D\). Now, consider \(b\) and \(b'\) two distinct vertices of \(B\) (distinct from
By Fact 8, there is no arc from \( \{b, b'\} \) to \( \{a, a', a'', b, b'\} \) disjoint from \( a a' \). In particular, \( \{b, b'\} \) is a stable set of \( D \) and \( a'' \) dominates \( b \) or \( b' \), say \( b \). Then, by Facts 5 and 7, \( a' \) dominates \( b' \). To obtain a contradiction, we finally look at the vertices \( a, b \) and \( b' \). Indeed, \( ab' \in E \) and \( ab \in E \) are forbidden by Fact 7 and \( b'a \in E \) and \( ba \in E \) are forbidden by Fact 8 (respectively consider arcs \( a''b \) and \( a'\)). Then, \( \{a, b, b'\} \) should be a stable set. This contradicts \( \alpha(D) \leq 2 \).

By symmetry, we have \( |B| \leq 3 \).

**Fact 10:** There is no disjoint arcs \( ab \) and \( a'b' \) with \( a, a' \in A \) and \( b, b' \in B \) (and similarly, there is no disjoint arcs \( ba \) and \( b'a' \) with \( a, a' \in A \) and \( b, b' \in B \)).

Indeed, if the statement holds, by Fact 8, we provide \( P \) and \( P' \), two disjoint paths which cover \( D \setminus \{a, a'\} \) and respectively start at \( b \) and \( b' \). If \( aa' \in E \), the paths \( P \) and \( aa'.P' \) contradict the choice of \( D \). Similarly, \( a'a' \notin E \) and \( \{a, a'\} \) is a stable set of \( D \).

Now, as there exists a path from \( B \) to \( a \) for instance, there exists a vertex \( x \) of \( P \cup P' \) which dominates \( a \) or \( a' \). Without loss of generality, we can assume that \( x \) is a vertex of \( P \) and that \( x \) is the last vertex along \( P \) which dominates \( a \) or \( a' \). If \( x \) is the last vertex of \( P \), then the paths \( Pa \) and \( a'.P' \) (if \( xa \in E \)) or the paths \( aP.a'.P' \) (if \( xa' \in E \)) contradict the choice of \( D \). So, \( x \) is not the last vertex of \( P \) and we denote by \( x_+ \) the successor of \( x \) along \( P \). As \( \alpha(D) \leq 2 \), there exists an arc between \( x_+ \) and \( \{a, a'\} \) and by choice of \( x \), this arc ends in \( x_+ \). We discuss the different cases.

**Case 1:** We have \( xa \in E \) and \( ax_+ \in E \). In this case, we insert \( a \) in the path \( P \) and the paths \( Px.a.x_+P \) and \( a'.P' \) contradicts the choice of \( D \).

**Case 2:** We have \( xa' \in E \) and \( a'x_+ \in E \). As in Case 1, we insert \( a' \) in \( P \) to obtain a contradiction.

**Case 3:** We have \( xa' \in E \) and \( ax_+ \in E \). The paths \( a.x_+P \) and \( Px.a'.P' \) contradicts the choice of \( D \).

**Case 4:** We have \( xa \in E \) and \( a'x_+ \in E \). This case is not so straightforward as the previous ones. Consider the paths \( P_1 = Px.a \), \( P_2 = x_+P \) and \( P_3 = P' \) and, as \( \alpha(D) \leq 2 \), apply a paths exchange on them in order to obtain \( Q_1 \) and \( Q_2 \), two disjoint paths which cover \( D \setminus a' \). Now, whatever the beginning of \( P_1 \), \( P_2 \) or \( P_3 \) we lost in the path exchange, we can extend one of the paths \( Q_1 \) or \( Q_2 \) to obtain two disjoint paths which cover \( D \) and respectively start in \( A \) and \( B \). This contradicts the choice of \( D \).

Finally, we can conclude that such a digraph \( D \) does not exist. By Facts 3 and 9, \( A \) and \( B \) are disjoint and have exactly two elements each. By Facts 8 and 10, there is no two disjoint arcs in \( D[A \cup B] \). So, as \( \alpha(D) \leq 2 \), it is easy to check that \( D[A \cup B] \) contains a set \( X \) of 3 vertices pairwise linked. By Facts 5, 6 and 7, every vertex of \( A \cup B \) has in-degree at most 1 in \( D[A \cup B] \). So, \( D[X] \)
is a circuit of length 3, but by Fact 4, $A \cup B$ is in a sole component of $D$ and there exist a path from the vertex of $(A \cup B) \setminus X$ to $X$. However, as previously seen, this last remark contradicts one of the Facts 5, 6 or 7. \hfill \Box

References


