THE BIRTH OF A RANDOM MATRIX

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To A. A. Kirillov on his 70th birthday

ABSTRACT. We consider the behavior of a random stepped surface near a turning point, that is, a point at which the limit shape is not smooth. When the turning point is a smooth point of the frozen boundary, the resulting point process is identified with the standard Gaussian measure on infinite Hermitian matrices. A different point process appears if the turning point is a cusp of the frozen boundary.

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1. Matrices vs. Surfaces

1.1. Stepped surfaces

1.1.1. This note is mostly about a certain elementary connection between random matrices and random surfaces. Random matrices do not require an introduction — they permeate many branches of mathematics and physics. Extensive literature exists, in particular, on the relation between random matrices and various random surface models, see for example [7] and references therein.

1.1.2. Stepped surfaces is one of the simplest random surface models. A stepped surface $\pi$ is a continuous surface in $\mathbb{R}^3$ glued out of the sides of the unit cube, projecting one-to-one in the $(1, 1, 1)$ direction, and spanning a given boundary contour — see an example in Figure 1.

Stepped surfaces arise, for example, as zero-temperature interfaces in the 3D Ising model. There are elementary bijections between stepped surfaces and hexagonal dimers, rhombi tilings, etc., see [10] for an introduction.

Boundary conditions may be imposed at infinity, which means that we fix $\pi$ outside an arbitrarily large compact set. Three-dimensional partitions, and their skew generalizations, see Figure 1, are examples of such stepped surfaces.

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1.1.3. A natural family of probability measures on stepped surfaces \( \pi \) spanning given boundary is obtained by setting

\[
\text{Prob}(\pi) = \frac{1}{Z(q)} q^{\text{vol}(\pi)},
\]

where \( \text{vol}(\pi) \) is the volume enclosed by \( \pi \) (it suffices to define it up to an overall additive constant). Here \( q > 0 \) is a parameter, known as fugacity. The normalization constant

\[
Z(q) = \sum_{\pi} q^{\text{vol}(\pi)},
\]

where the summation ranges over all surfaces spanning given boundary, is traditionally called the partition function. When the region is infinite, boundary conditions at infinity and \( q < 1 \) insure the convergence of \( Z(q) \).

While for certain special boundary conditions the exact form of the partition function is of great interest in algebraic geometry and gauge theory (see [17] for an introduction), from a probabilistic viewpoint the limit \( q \to 1 \) is natural. This continuous, or thermodynamic, limit should be taken together with rescaling the boundary conditions by \( \varepsilon \), where

\[
\varepsilon = -\log q \to +0
\]

may be interpreted as the mesh size of our random surface.
1.1.4. The main feature of this limit is the formation of a nonrandom \textit{limit shape} — a manifestation of the law of large numbers. This phenomenon is clearly visible in the simulation in Figure 2 and may be proven on very general grounds, see [6]. Rather precise qualitative and quantitative information about the limit shape is available [12]. In particular, a basic and well-understood feature of limit shape is the existence of the \textit{frozen boundary} separating ordered phases (which may be interpreted as e.g. crystalline facets) from the disorder. Frozen boundary is an analytic plane curve. In fact, for polygonal boundary as in Figures 1 and 2, it is algebraic in suitable coordinates.

Finer properties of the random surface, such as correlation functions of local observables, are expected to be completely determined by the limit shape. Near a point inside the disordered region, one expects to see the unique translation-invariant ergodic Gibbs measure on stepped surfaces with given average slope (see e.g. [8], [11], [19] for theorems supporting these expectations). The uniqueness of such measure follows from a general theorem of S. Sheffield [23]. The correlations functions of this measure are determinantal (built from the Green function of Kasteleyn operator with appropriate boundary conditions). Explicit formulas may be found in [6] and also in [19].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{limit_shape.png}
\caption{A limit shape is forming}
\end{figure}

1.1.5. Translation invariance is maximally broken at special points of the frozen boundary, namely \textit{cusps} and \textit{turning points}. A turning point is a point of the frozen boundary where the disordered region meets two different frozen phases, that is,
a point where the frozen phase changes along the frozen boundary. These are precisely the points where the limit shape is not smooth, that is, where the normal to the limit shape is discontinuous. At such a point, the frozen boundary must be tangent to one of the lines forming the domain boundary (but perhaps not to the boundary itself).

By chance, such a point may simultaneously be a cusp of the frozen boundary, an example of which may be seen in Figure 2. We call such a point a cuspidal turning point, to distinguish it from cusps (at which, by our definition, the limit shape is $C^1$) and turning points (at which the frozen boundary is smooth).

In [19], [18] we gave an exact contour integral representation for all correlation function of local observables for a restricted, but representative, class of boundary conditions. The asymptotics of these integrals involves nothing but a careful steepest descent analysis, see e.g. [16] for a pedagogical account. We performed such analysis for points inside the disordered region, near a smooth point of the frozen boundary, and at a cusp, finding the so-called Pearcey process, see [1], [4], [5], [18], [24], in the latter case. The goal of this note is to treat the two remaining cases—an ordinary and a cuspidal turning point.

For unbounded domains, the so-called tentacles, which are the exponentially narrow channels of disorder separating unbounded frozen regions (two of these may be seen in Figure 2), may also be considered as a kind of turning point. In fact, they appear as a coalescence of two ordinary turning points at infinity. The local statistics inside a tentacle were analyzed by C. Boutillier in [3].

1.2. Turning point

1.2.1. The combinatorics of a stepped surface near a turning point may be described in very simple terms. Zooming in on a neighborhood of a turning point as in Figure 3 and viewing the stepped surface as a tiling by rhombi, one notices the special role of one kind of tiles—the ones without sides parallel to the boundary. In Figure 3 they are are the blue (or dark) ones. Following the language of [3], we will call them beads.

![Figure 3. The neighborhood of a turning point](image)

The following properties should be obvious from examining Figure 3:

1. a straight line running parallel to the boundary at distance $k$ intersects (or “threads”) exactly $k$ beads;

2. the points where these intersections occur interlace, that is, the beads on the $(k - 1)$st line lie between the beads on the $k$th one;
(3) (Gibbs property.) given the positions of the beads on the $k$th line, all interlacing configurations of beads below it are equally probable (as $q \to 1$).

More formally, letting
\[ b^{(k)} = (b_1^{(k)} > b_2^{(k)} > \cdots > b_k^{(k)}), \quad b_i^{(k)} \in \mathbb{Z} + \frac{k}{2}, \]  
be beads’ centers on the $k$th line, we have
\[ b^{(k)} \succ b^{(k-1)}, \]
where $a \succ b$ means that $a$ and $b$ interlace
\[ a_1 \geq b_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq b_{k-1} \geq a_k. \]
Such structures are, of course, well-known in representation theory under the name of Gelfand–Tsetlin tables or semistandard tableaux. See [13] for a panorama of interlacing sequences and the role that they play in many branches of mathematics.

1.2.2. Let the horizontal coordinate in Figure 3 be chosen so that zero is the turning point. Since the frozen boundary looks like a parabola near the turning point, we expect that the limits
\[ x^{(k)} = \sqrt{\varepsilon} b^{(k)}, \quad \varepsilon \to +0, \]
exist and are nontrivial. If so, the integrality condition in (1) becomes irrelevant and $x^{(k)}$ may be interpreted as eigenvalues of a $k \times k$ Hermitian matrix $X_k$, that is, as a (co)adjoint orbit for $U(k)$.

A probability measure on $x^{(k)}$, which is our object of interest, thus becomes a $U(k)$-invariant probability measure on the linear space $H(k)$ of $k \times k$ Hermitian matrices.

1.2.3. Next we note that Gibbs measures on sequences
\[ x^{(1)} \prec x^{(2)} \prec x^{(3)} \prec \cdots \]
are the same as $U(\infty)$-invariant measures on
\[ H(\infty) = \lim_{k \to \infty} H(k), \]
where $U(\infty) = \bigcup U(k)$ and the maps
\[ H(k) \ni X_k \mapsto X_{k-1} \in H(k-1) \]
cut out the top left corner. This equivalence is based on two very classical facts, namely:

1. the eigenvalues of a $k \times k$ matrix and its $(k-1) \times (k-1)$ corner interlace;
2. a $U(k)$-invariant measure on $k \times k$ matrices with eigenvalues $x^{(k)}$ induces a uniform (Lebesgue) measure on the $\binom{k}{2}$-dimensional convex set formed by eigenvalues
\[ x^{(1)} \prec x^{(2)} \prec \cdots \prec x^{(k-1)} \]
of $i \times i$ top left corners $X_i$, $i = 1, \ldots, k-1$. 

The second assertion is, perhaps, best explained in the framework of the orbit method — it is the quasiclassical limit of the branching rule for representations of unitary groups. It can be also seen without leaving the world of linear algebra, see, for example, [2].

1.2.4. The description of all $U(\infty)$-invariant measures on $H(\infty)$ may be found in the paper [20] by A. Vershik and G. Olshanski, see also [25], [21]. The ergodic ones are convolutions of the following elementary measures.

First, there is a Gaussian measure, that is, a measure with Fourier transform

$$\langle e^{i\text{tr} AX} \rangle = \exp \left( -\frac{\sigma^2}{2} \text{tr} A^2 \right),$$

where $\sigma^2$ is a parameter (variance). There are also $\delta$-measures on a scalar matrices and Wishart-like measures, namely

$$X = (\xi_i\bar{\xi}_j)_{i,j=1,2,...},$$

where $\xi_i$ are independent complex Gaussians.

Without loss of generality we may restrict ourselves to measures with zero mean.

1.2.5. For the Gaussian measure (3) we have

$$\lim \frac{\|X_k\|}{\sqrt{k}} = \lim \frac{x_k^{(1)}}{\sqrt{k}} = 2\sigma,$$

in probability. This is exactly the kind of behavior that we expect. Indeed, the shape of the frozen boundary near a turning point (a parabola) shows that the largest eigenvalue $x_k^{(1)}$ grows like a square root of $k$.

1.2.6. For Wishart measures, the norm $\|X_k\|$ grows linearly in $k$. If a Wishart component were present, the first bead would run off into the frozen region.

We believe that it should be possible to show very generally that the frozen region is completely frozen, that is, the probability to observe a single bead anywhere in its interior vanishes. However, it seems that no such general statement is at present available in the literature. Fortunately, various ad hoc arguments are available in the present setup. First, whenever exact formulas for correlation functions are available, such vanishing is immediate. In fact, the 1-point correlation function vanishes as $e^{-c/\varepsilon}$, with $c > 0$, at any point inside the frozen region.

Another argument ruling out Wishart components is the following. The rate at which $x_k^{(1)}$ grows, is determined by the variance $\text{Var}(\xi_i)$ of the random variables $\xi_i$ in (4). In any convolution of Wishart measures, there may be only finitely many components for which $\text{Var}(\xi_i)$ is maximal. This means that some finite number of beads will run off into the frozen region far ahead of everybody else. These tiles will follow, essentially, a simple random walk weighted by the area enclosed under its graph in space-time. It is well known, see e.g. [26] that the trajectory of such random walk follows, rescaled, a straight line in suitably chosen exponential coordinates in space-time. On the other hand, the frozen boundary is always convex in the same coordinates, see [12], thus pushing the random walk back to the edge of the disordered region.
1.2.7. Having excluded Wishart components, it remains to show that the limit law is not a nontrivial convex combination of Gaussian measures. But the nonrandomness of the frozen boundary implies the parameter $\sigma$ in (3) is not random. This proves the following,

**Theorem 1.** The beads are distributed as eigenvalues of corners of an infinite Gaussian Hermitian random matrix.

This eigenvalue distribution is discussed, for example, by Baryshnikov in [2]. In the context of the tiling problems, it first appears in the paper [8] by K. Johansson, see the remarks around formulas (2.26)–(2.29) there. The observation of [8] was further developed in the work of E. Nordenstam [15] and K. Johansson and E. Nordenstam [9].

1.2.8. It is tautological corollary of Theorem 1 that for any boundary conditions leading to at least one turning point the stepped surfaces partition function may be viewed as a perturbation of a standard Gaussian random matrix integral. It is a popular philosophy that such matrix integrals describe a sizable fraction of the known universe. Depending on one’s taste, one may view Theorem 1 as either supporting this point of view or suggesting that random surfaces may be a more fundamental object.

It is also amusing to notice that in the case when the disordered region is completely surrounded by the frozen boundary (as is the case e.g. for compact polygonal boundary conditions, see [12]), the random surface interacts with the boundary conditions only through turning points.

1.2.9. In Section 2.2 we consider the asymptotics of correlation functions near a cuspidal turning point. From the contour integral representation, we derive an formula for the limiting kernel, which may be viewed, for example, as a nonstationary deformation of the Airy process [22]. These formulas await a more conceptual understanding.

1.2.10. Acknowledgments. We had an opportunity to discuss the results presented here at several meetings and conferences, in particular, we are grateful to A. Borodin, R. Kenyon, K. Johansson, and G. Olshanski for their helpful comments.

We felt that a special volume dedicated to A. A. Kirillov would be the right place to present these results in print, as it makes a particularly close contact with several objects central to Kirillov-style representation theory. It is also a happy occasion to thank him for sharing his inspiration, wisdom, and advice.

2. Contour Integrals

2.1. Ordinary turning point

2.1.1. We now turn to a saddle point analysis of our formulas from [19], [18] in a neighborhood of a turning point. Let the boundary in Figure 3 be vertical and have horizontal (that is, time) coordinate $t_0$ as illustrated in Figure 4. The vertical coordinate in Figure 4 gives the positions $b_i$ of the beads.
2.1.2. Fluctuations around the limit shape are described by local correlation functions of beads

$$\langle \rho_{t_1,b_1} \cdots \rho_{t_n,b_n} \rangle = \frac{1}{Z(q)} \sum_\pi q^{\text{vol}(\pi)} \prod_k \rho_{t_k,b_k}(\pi), \quad (6)$$

where

$$\rho_{t,b}(\pi) = \begin{cases} 1 & \text{if } (t, b) \text{ is the center of a bead,} \\ 0 & \text{otherwise.} \end{cases}$$

The results of [19], [18] show these correlation functions are determinantal:

$$\langle \rho_{t_1,b_1} \cdots \rho_{t_n,b_n} \rangle = \det[K((t_i, h_i), (t_j, h_j))]_{1 \leq i,j \leq n}. \quad (7)$$

The kernel $K$ in (7) has the form

$$K((t_1, b_1), (t_2, b_2)) = \frac{1}{(2\pi)^2} \int \int \frac{1}{z-w} \frac{\Phi(w, t_2)}{\Phi(z, t_1)} \frac{w^{b_2+B(t_2)} z^{b_1+B(t_1)}}{\sqrt{zw}} \, dz \, dw, \quad (8)$$

where $\Phi(z, t)$ is given by a certain (possibly infinite) product, and the zig-zag line in Figure 4 is the graph of the function $-B(t)$. For $t$ near the turning point $t_0$, we have

$$B(t) = -t/2 + \text{const},$$

where the constant may be incorporated in the coordinate $b_i$. The contour of integrations go as shown in Figure 5. Note the time-ordering of contours.

2.1.3. The function $\Phi$ in (8) has the form

$$\Phi(z, t; q) = \exp \left( \frac{S(z; \varepsilon)}{\varepsilon} \right) \prod_{m=t_0+\frac{1}{2} \ldots t-\frac{1}{2}} (1 - z^{-1} q^{-m})^{-1} \quad (9)$$

with $S(z; \varepsilon)$ regular at $\varepsilon = 0$ for $z$ on the domain on integration.

Without loss of generality, we may assume that

$$t_0 = 0.$$
The second derivative of $S$ at $z = 1$ is nonvanishing and, in fact, negative
\[
\left(z \frac{\partial}{\partial z}\right)^2 S(z; 0) \bigg|_{z=1} = -\sigma^2 < 0,
\]
which reflects the fact that the frozen boundary in Figure 4 curves in a particular direction.

2.1.4. The asymptotic analysis of the integral (8) follows the standard route. As $\varepsilon \to 0$ the integral is dominated by an arbitrarily small neighborhood of $z = w = 1$. We zoom in on this point by introducing new variables
\[
z = e^{\sqrt{\varepsilon} \zeta}, \quad w = e^{\sqrt{\varepsilon} \omega}.
\]
Simultaneously, we rescale the vertical coordinate as in (2)
\[
b_1 = \frac{x}{\sqrt{\varepsilon}} + \frac{C_0}{\varepsilon}, \quad b_2 = \frac{y}{\sqrt{\varepsilon}} + \frac{C_0}{\varepsilon},
\]
where the constant $C_0$ is chosen so that to cancel the $S'(1)$ term in
\[
\varepsilon^{t_2-t_1} \frac{\Phi(w, t_2) w^{b_2+B(t_2)}}{\Phi(z, t_1) z^{b_1+B(t_1)}} \to \frac{\zeta^{t_1}}{\omega^{t_2}} \exp \left( \frac{\sigma^2}{2} (\zeta^2 - \omega^2) - x\zeta + y\omega \right), \quad \varepsilon \to 0. \quad (10)
\]
In other words, $C_0$ is the vertical coordinate of the turning point.

2.1.5. It is clear from (10) that the imaginary axis $\zeta \in i\mathbb{R}$ is the steepest descent contour for integration in $\zeta$, while the $\omega$-contour wants to be deformed toward the real axis. Due to the singularity at $\omega = 0$, this deformation amounts to encircling the origin. The required deformation may be performed without crossings of the contours in the case $t_1 \geq t_2$. We conclude that for $t_1 \geq t_2$,
\[
\varepsilon^{t_2-t_1-1} K((t_1, b_1), (t_2, b_2)) \to \frac{1}{(2\pi i)^2} \int \! \int d\omega \, d\zeta \frac{\zeta^{t_1}}{\zeta - e^{-\omega t_2}} \exp \left( \frac{\sigma^2}{2} (\zeta^2 - \omega^2) - x\zeta + y\omega \right), \quad (11)
\]
where the integration contours in $\zeta$ and $\omega$ may be seen in Figure 6.

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**Figure 5.** Contours of integration for $t_1 < t_2$. For $t_1 \geq t_2$ one takes $|w| < |z|$. 

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Figure 6. Integration contours for $t_1 \geq t_2$

Note that in Figure 6 we have $|\omega| < |\zeta|$ uniformly for $\omega$ and $\zeta$ on the contours of integration. Hence we may integrate term-wise the expansion

$$
\frac{1}{\zeta - \omega} \zeta^{t_1} = \sum_{n=-t_2}^{-1} \frac{\zeta^{t_1+n}}{\omega^{t_2+n+1}} + \cdots,
$$

where dots stand for terms regular at $\omega = 0$. Such terms integrate to zero. This term-wise integration may be performed using Hermite polynomials, see Section 2.1.7.

2.1.6. For $t_1 < t_2$, the $\omega$-contour has to cross the $\zeta$-contour before we may wrap it around $\omega = 0$. As we push the $\omega$-contour though the $\zeta$-contour we pick up a residue at $\zeta = \omega$ that equals

$$
-\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\zeta \zeta^{t_1-t_2} e^{(y-x)\zeta} = \begin{cases} 
\frac{(y-x)^{t_2-t_1-1}}{(t_2-t_1-1)!}, & y > x, \\
0, & y < x.
\end{cases}
$$

Here the $\zeta$-contour is as in Figure 6. Depending on the sign of $(y-x)$, it may be closed in the right or left half-pane, whence the evaluation.

This extra $\delta$-function term may be interpreted as the difference between two expansions

$$
\frac{1}{\zeta - \omega} \zeta^{t_1} = \begin{cases} 
\sum_{n \geq 0} \frac{\zeta^{t_1+n}}{\omega^{t_2+n+1}}, & t_1 < t_2, \\
\sum_{n=-t_2}^{-1} \frac{\zeta^{t_1+n}}{\omega^{t_2+n+1}} + \cdots, & t_1 \geq t_2,
\end{cases}
$$

where dots, as before, stand for terms regular at $\omega = 0$. This may be seen by pushing the $\omega$-contour to encircle the infinity of the $\omega$-plane.

Using this convention, the asymptotics of the correlation kernel may be summarized as follows:

**Theorem 2.** As $\varepsilon \to 0$,

$$
\varepsilon^{t_2-t_1-1} K((t_1, b_1), (t_2, b_2)) \to K_{\text{Gauss}}((t_1, x), (t_2, y)),
$$

where
\[
K_{\text{Gauss}}((t_1, x), (t_2, y)) = \frac{1}{(2\pi i)^2} \oint \frac{d\omega}{\omega} \int_{-\infty}^{\infty} d\zeta \frac{1}{\zeta - \omega} \exp \left( \frac{\sigma^2}{2} (\zeta^2 - \omega^2) - x\zeta + y\omega \right),
\]
and time ordering is as in (13).

2.1.7. To bring the kernel (14) into a more standard form found in [8], we may integrate expansion (13) term-wise using standard formulas for Hermite polynomials. We get
\[
\frac{1}{2\pi i} \oint \frac{d\omega}{\omega} \exp \left( -\frac{\sigma^2}{2} \omega^2 + y\omega \right) \frac{d\omega}{\omega^{n+1}} = \frac{\sigma^n}{2^{n/2} n!} H_n \left( \frac{y}{\sigma \sqrt{2}} \right),
\]
and similarly,
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left( \frac{\sigma^2}{2} \zeta^2 - x\zeta \right) \zeta^n d\zeta
= \left( -\frac{d}{dx} \right)^n \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left( \frac{\sigma^2}{2} \zeta^2 - x\zeta \right) d\zeta
= \frac{1}{\sqrt{\pi} 2^{(n+1)/2} \sigma^{n+1}} H_n \left( \frac{x}{\sigma \sqrt{2}} \right) \exp \left( -\frac{x^2}{2\sigma^2} \right).
\]
Hence, for example, at equal time \( t = t_1 = t_2 \), we get a kernel conjugate to the standard expression [14]
\[
\exp \left( \frac{x^2 - y^2}{4\sigma^2} \right) K_{\text{Gauss}}((t, x), (t, y)) = \sum_{n=0}^{t-1} \frac{\phi_n(x)\phi_n(y)}{\|\phi_n\|^2},
\]
with
\[
\phi_n(x) \propto H_n \left( \frac{x}{\sigma \sqrt{2}} \right) \exp \left( -\frac{x^2}{4\sigma^2} \right).
\]
The kernel (16) is orthogonal projection onto polynomials of degree < \( t \) with respect to the Hermite inner product. This may also be seen directly from (14) by manipulating the integrals.

We also note that while the expanded form of the kernel is easier to recognize, integral representations as in (14) are typically more convenient. For example, they are particularly suitable for asymptotic analysis.

2.2. Cuspidal turning point. At a cuspidal turning point, the frozen boundary has a \( x^2 = y^3 \) type of singularity pointing in the direction of edge in the limit shape, see Figure 7.

When the cusp points downwards, as in Figure 7, the function \( S \) from (9) satisfies
\[
\left( z \frac{\partial}{\partial z} \right)^2 S(z; 0) \bigg|_{z=1} = 0, \quad \left( z \frac{\partial}{\partial z} \right)^3 S(z; 0) \bigg|_{z=1} < 0.
\]
Here, as before, we assumed for simplicity that the cusp occurs at \( t_0 = 0 \), thus making the critical point in the integral equal \( z = w = 1 \).
The proper scaling of the vertical coordinate now takes the form

\[ b_1 = \frac{C_1 x}{\varepsilon^{1/3}} + \frac{C_0}{\varepsilon}, \quad b_2 = \frac{C_1 y}{\varepsilon^{1/3}} + \frac{C_0}{\varepsilon}, \]

(17)

where the constant \( C_0 \) is chosen so that to cancel the linear term in the expansion of \( S \). It controls the vertical position of the cusp. The constant \( C_1 \) serves to normalize the cubic term in \( S \)—it controls the width of the cusp.

With this rescaling, we arrive at the following

**Theorem 3.** The correlation kernel in the vicinity of a cuspidal turning point, rescaled as in (17) converges to

\[ K((t_1, x), (t_2, y)) = \frac{1}{(2\pi i)^2} \int d\omega \int d\zeta \frac{1}{\zeta - \omega} \frac{\zeta^{11}}{\omega^{12}} \exp(\zeta^3 - \omega^3 - x\zeta + y\omega), \]

(18)

where the integration contours are as in Figure 8 and the integrand is time-ordered as in (13).
This is an interesting deformation of the Airy process. Note, in particular, that at equal time $t_1 = t_2 = t$ this is a rank $|t|$ perturbation of the Airy kernel, due to 

$$
\frac{(\zeta/\omega)^t - 1}{\zeta - \omega}
$$

being a polynomial with $|t|$ terms.

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