

# The Invariants and Covariants of an $m$ -ary Form

Fengmei Zou<sup>1)</sup>

**Abstract.** This paper gives two necessary and sufficient conditions of invariants of an  $m$ -ary form and one necessary and sufficient condition of covariants of an  $m$ -ary form. By these criteria we compute some invariants and covariants of an  $m$ -ary form. At last, two examples of computing the syzygies of invariants by the characteristic set method are given.

**Keywords:** Form, General Linear Group, Invariant, Covariant

**Notation:** Throughout this paper,  $\mathcal{C}$  stands for the field of complex numbers,  $S_m$  for the symmetric group with  $m$  letters,  $GL(\mathcal{C}, m)$  for the general linear group, and  $SL(\mathcal{C}, m)$  for the special linear group.

## 1. Introduction

Invariant theory is both a classical and a new field of mathematics. It played a central role in 19th century algebra and geometry [?]. But Hilbert's success, namely, his three fundamental contributions to modern algebra—the Nullstellensatz, the Basis Theorem and the Syzygy Theorem also spelled the doom of 19th century invariant theory, which was left with no big problems to solve and soon faded into oblivion. In the 1930s, H. Weyl, E. Cartan, I. Schur, etc, (cf. [?]) prompted by the developments in the theory of Lie groups and their representations, went further by letting  $G$  be any group and defining the action via an arbitrary linear representation  $G \rightarrow GL(V)$ . In particular, the invariants of an arbitrary system of tensors, not only invariants and covariants of systems of forms, drew attention, which showed that the classical invariant theory was really a special case of that theory. But again these important developments failed to materialize after the publication of the book [?]. In the second half of this century, newly developed techniques from algebraic geometry, algebraic combinatorics, algebraic representations and algebraic groups have been applied with great success to some of its outstanding problems [?, ?, ?, ?, ?, ?, ?]. This has moved invariant theory, once again, to the forefront of mathematical research.

One of the main problems in invariant theory is to explicitly give the fundamental invariants and their syzygies. The classical results for this problem are mainly for  $GL(\mathcal{C}, 2)$  [?, ?, ?, ?, ?, ?]; the modern results are mainly concerned with singling out the whole classes of  $G$ -modules of reductive groups  $G$  for which the problem is solved [?, ?, ?, ?, ?, ?]. However, the success in each particular case was conditioned by some special properties of the modules under consideration. How to compute the explicit invariants and covariants of  $GL(\mathcal{C}, m)$  ( $m \geq 3$ ) still remained open, besides the case of the 3-ary form of degree 3. This is because the computation according to the known methods becomes too large for  $m \geq 3$ .

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<sup>1)</sup>Institute of Systems Science, Academia Sinica, Beijing 100080, P.R. China

But this problem is very important in some mathematical fields. For example, an explicit description of the algebra of invariants is usually followed by a meaningful geometric interpretation [?, ?, ?, ?, ?, ?]; In certain situations, the algebra of invariants helped to clarify the structure of other algebras(e.g., the cohomology algebras of homogeneous spaces [?, ?] and the algebras of modular forms [?]); Finally, an explicit description of the algebra of invariants is important for applications in coding theory and combinatorics [?, ?, ?, ?], and some special problems in differential equations [?].

The authors of [?] wrote “It might be worthwhile to push the XIXth Century computations of invariants a little further along, with the help of modern computers”. After a careful study of the method of computing the given degree invariants of a binary form in [?], we find that this method can be generalized to computing invariants and covariants of an arbitrary  $m$ -ary form ( $m \geq 3$ ). Using this method to compute the given degree invariants, we need only solve the linear equations obtained by acting  $(m - 1)$  differential operators on the invariants of  $\text{diag}(a_1, \dots, a_m)$ , as will be stated in Theorem ???. The same is true for computing covariants, as stated in Theorem ???. Furthermore, we only need to solve the linear equations obtained by acting only one differential operators on the invariants of  $\text{diag}(a_1, \dots, a_m)$  and  $S_m$ , as stated in Theorem ???.

In this chapter, we will only consider the invariants and covariants of the general linear group  $GL(\mathcal{C}, m)$  acting on the polynomial algebra  $\mathcal{C}[V]$ , where  $V$  is a finite-dimensional vector space over  $\mathcal{C}$ .

An  $m$ -ary form  $f(x_1, \dots, x_m; n) \in \mathcal{C}[x_1, \dots, x_m]$  of degree  $n$  in the variables  $x_1, \dots, x_m$  is a homogeneous polynomial of degree  $n$  in the variables  $x_1, \dots, x_m$ . Thus, we can write  $f(x_1, \dots, x_m; n)$  as follows:

$$f = f(x_1, \dots, x_m; n) = \sum_{r_1 + \dots + r_m = n} \frac{n!}{r_1! \dots r_m!} a_{r_1, \dots, r_m} x_1^{r_1} \dots x_m^{r_m}$$

Let

$$X = (x_1, \dots, x_m)', \quad Y = (y_1, \dots, y_m)'$$

and a linear transformation  $X = GY$  with  $\det(G) \neq 0$ . Under the linear transformation  $X = GY$ , the  $m$ -ary form  $f(x_1, \dots, x_m; n)$  is changed into another  $m$ -ary form in the variables  $y_1, \dots, y_m$ . After expanding and regrouping terms, we obtain an  $m$ -ary form

$$f'(y_1, \dots, y_m; n) = \sum_{r_1 + \dots + r_m = n} \frac{n!}{r_1! \dots r_m!} b_{r_1, \dots, r_m} y_1^{r_1} \dots y_m^{r_m}$$

in the variables  $y_1, \dots, y_m$  whose coefficients  $b_{r_1, \dots, r_m}$  are polynomials in the  $a_{r_1, \dots, r_m}$  and the entries of  $G$ .

Let  $g$  be a nonnegative integer. A non-constant polynomial  $C(A_{r_1, \dots, r_m}; x'_1, \dots, x'_m)$  in the variables

$$A_{n, \dots, 0}, \dots, A_{r_1, \dots, r_m}, \dots, A_{0, \dots, n} \text{ and } x'_1, \dots, x'_m$$

is said to be a covariant of index  $g$  of  $m$ -ary forms of degree  $n$  if for all  $m$ -ary forms  $f(x_1, \dots, x_m; n)$  of degree  $n$  and all non-degenerated linear transformations  $G$  of the variables, the following identity holds:

$$C(b_{r_1, \dots, r_m}; y_1, \dots, y_m) = G^g C(a_{r_1, \dots, r_m}; x_1, \dots, x_m).$$

A covariant in which the variables  $x'_1, \dots, x'_m$  do not occur is said an invariant.

- Example 1.1** 1. For an arbitrary  $m$ -ary form  $f(x_1, \dots, x_m; n)$ ,  $f$  itself is a covariant; the Hessian polynomial  $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$  of  $f$  is a covariant.
2. For an arbitrary  $m$ -ary quadratic form  $f(x_1, \dots, x_m; 2)$ , its discriminant is an invariant.

**Remark 1.2** In modern language, the invariant defined here is called a relative invariant.

A covariant  $C(A_{r_1, \dots, r_m}; x'_1, \dots, x'_m)$  is said to be homogeneous if it is homogeneous both as a polynomial in the variables  $\{A_{r_1, \dots, r_m}\}$  and as a polynomial in the variables  $x'_1, \dots, x'_m$ . By the definitions, every covariant can be written as a linear combination of homogeneous covariants. If  $C$  is a homogeneous covariant, the degree of  $C$  is the total degree of  $C$  as a polynomial in  $\{A_{r_1, \dots, r_m}\}$ ; while the order of  $C$  is the total degree of  $C$  as a polynomial in  $x'_1, \dots, x'_m$ .

In the following, invariants and covariants are supposed to be homogeneous.

Hilbert [?, ?] proved that the algebra of invariants is finitely generated. A set of generators of the algebra is called a system of fundamental invariants. There are three methods for computing the fundamental invariants, namely, the  $\Omega$ -process, solving linear equations arising from the Lie algebra action and generating invariants in symbolic representation.

We will find that the second method is effective in computing invariants (see Theorem ??). In fact, this method can be used effectively to compute the degree 9 invariants of the 3-ary form of degree 4 in variables  $x'$ s.

## 2. Invariants and Covariants of an $m$ -ary Form

Let  $A = \text{diag}(k_1, k_2, \dots, k_m)$ ,  $B_{ij}(u) = \text{Id} + uE_{ij}$ ,  $u \in \mathcal{C}$ , where  $\text{Id}$  is the identity matrix and  $E_{ij}$  is the matrix whose  $(i, j)$ <sub>th</sub> entry is 1 and whose other entries are zero. We have the following [?, ?]:

**Proposition 2.1** For every  $G \in GL(\mathcal{C}, m)$ ,  $G = BD$ , where

$$B \in SL(\mathcal{C}, m), D = \text{diag}(1, \dots, 1, d).$$

**Proposition 2.2** For  $m \geq 3$ ,  $SL(\mathcal{C}, m)$  is equal to its commutator group. More precisely,

$$B_{i,j}(u) = B_{i,k}(u)B_{k,j}(1)B_{i,k}(-u)B_{k,j}(-1).$$

From these two propositions, we have

**Corollary 2.3**  $SL(\mathcal{C}, m)$ , ( $m \geq 3$ ), is generated by the elements

$$\{B_{i,i+1}(u), B_{i+1,i}(u) | 1 \leq i \leq m-1, u \in \mathcal{C}\}.$$

**Proof** We know from [?] that  $SL(\mathcal{C}, m)$  is generated by the elements

$$\{B_{i,j}(u) | i \neq j, 1 \leq i, j \leq m\}.$$

Hence we only need to prove that these elements can be generated by

$$\{B_{i,i+1}(u), B_{i+1,i}(u) | 1 \leq i \leq m - 1\}.$$

By Proposition ??,

$$B_{i,i+2}(u) = B_{i,i+1}(u)B_{i+1,i+2}(1)B_{i,i+1}(-u)B_{i+1,i+2}(-1),$$

thus

$$B_{i,i+k}(u) = B_{i,i+k-1}(u)B_{i+k-1,i+k}(1)B_{i,i+k-1}(-u)B_{i+k-1,i+k}(-1).$$

Similarly  $B_{i,j}(u)$  ( $i > j + 1$ ), can be generated by  $\{B_{i+1,i}(u) | 1 \leq i \leq m - 1\}$ . □

Without loss of generality, we only consider the case of 3-ary forms.

Let

$$f = f(x_1, x_2, x_3; n) = \sum_{r_1+r_2+r_3=n} \frac{n!}{r_1!r_2!r_3!} a_{r_1,r_2,r_3} x_1^{r_1} x_2^{r_2} x_3^{r_3}$$

be a 3-ary form of degree  $n$  in variables  $x$ 's, a homogeneous polynomial  $I$  of degree  $g$  of the coefficients of  $f$  be

$$I = I(a_{r_1,r_2,r_3}) = \sum Z_{s_{n00} \dots s_{r_1 r_2 r_3} \dots s_{00n}} a_{n,0,0}^{S_{n00}} \dots a_{r_1,r_2,r_3}^{S_{r_1 r_2 r_3}} \dots a_{0,0,n}^{S_{00n}} \tag{3.1}$$

where all  $S_{r_1 r_2 r_3} \geq 0$ . Then

**Proposition 2.4**  $I$  is an  $A$ -invariant if and only if the following system of equations are satisfied:

$$\begin{aligned} nS_{n00} + (n-1)(S_{(n-1)10} + S_{(n-1)01}) + \dots + r_1(S_{r_1(n-r_1)0} + \dots + S_{r_1 0(n-r_1)}) \\ + \dots + S_{1(n-1)0} + \dots + S_{10(n-1)} = p; \end{aligned} \tag{1}$$

$$\begin{aligned} nS_{0n0} + (n-1)(S_{1(n-1)0} + S_{0(n-1)1}) + \dots + r_2(S_{(n-r_2)r_2 0} + \dots + S_{0r_2(n-r_2)}) \\ + \dots + S_{(n-1)10} + \dots + S_{01(n-1)} = p; \end{aligned} \tag{2}$$

$$\begin{aligned} nS_{00n} + (n-1)(S_{10(n-1)} + S_{01(n-1)}) + \dots + r_3(S_{(n-r_3)0r_3} + \dots + S_{0(n-r_3)r_3}) \\ + \dots + S_{(n-1)01} + \dots + S_{0(n-1)1} = p, \end{aligned} \tag{3}$$

where  $p$  is the power of the transformation determinant.

**Proof** Under the transformation of type  $A$ , the form  $f(x_1, x_2, x_3; n)$  is changed into  $f'(y_1, y_2, y_3; n)$ , where  $x_i = k_i y_i$ . The coefficients  $b_{r_1,r_2,r_3}$  of  $f'(y_1, y_2, y_3; n)$  is  $k_1^{r_1} k_2^{r_2} k_3^{r_3} a_{r_1,r_2,r_3}$ . After substituting the  $b_{r_1,r_2,r_3}$  by  $k_1^{r_1} k_2^{r_2} k_3^{r_3} a_{r_1,r_2,r_3}$  in  $I$  and comparing the power of  $k_i$ , we obtain the above equations by the definition of invariant. □

**Definition 2.1** The left-hand side of Equation (1) ((2), (3), resp.) is called the weight of  $I$  w.r.t.  $x_1(x_2, x_3, \text{resp.})$ . Denote it by  $p_{x_1}(p_{x_2}, p_{x_3}, \text{resp.})$ .

Hence,  $I$  is an  $A$ -invariant if and only if all the weights of  $I$  w.r.t.  $x_1, x_2, x_3$  are equal to the power  $p$  of the transformation determinant, in which anyone of the weights is called the weight of  $I$ . The weight  $p$  and the degree  $g$  of  $I$  satisfy  $ng = 3p$ .

For the linear substitution  $X = B_{12}(u)Y$ ,  $f(x_1, x_2, x_3; n)$  is changed to

$$f'(y_1, y_2, y_3; n) = \sum_{r_1+r_2+r_3=n} \frac{n!}{r_1!r_2!r_3!} b_{r_1r_2r_3} y_1^{r_1} y_2^{r_2} y_3^{r_3},$$

where

$$\begin{aligned} b_{r_1, r_2, r_3} &= a_{r_1, r_2, r_3}, & \text{if } r_2 = 0, \\ b_{r_1, r_2, r_3} &= \sum_{k=0}^{r_2} \frac{r_2!}{(r_2-k)!k!} a_{r_1+r_2-k, k, r_3} u^{r_2-k}, & \text{if } r_2 \neq 0. \end{aligned}$$

Note that if  $r_2 \neq 0$ , then

$$\frac{\partial b_{r_1, r_2, r_3}}{\partial u} = r_2 b_{r_1+1, r_2-1, r_3}.$$

If  $I$  is a  $B_{12}(u)$ -invariant, then because  $\det(B_{12}(u)) = 1$ .

$$I(b_{r_1, r_2, r_3}) = I(a_{r_1, r_2, r_3}).$$

Hence

$$\frac{\partial I(b_{n,0,0}, \dots, b_{0,0,n})}{\partial u} = \sum_{r_2 \neq 0} \frac{\partial I}{\partial b_{r_1, r_2, r_3}} \cdot (r_2 \sum_{r_1+r_3=n-r_2} b_{r_1+1, r_2-1, r_3}) = 0.$$

On the other hand, if  $I$  satisfies (4), then after the linear substitution  $B_{12}(u)$ ,  $I$  is still independent of  $u$ , so  $I(b_{r_1, r_2, r_3}) = I'(a_{r_1, r_2, r_3})$ . Let  $u = 0$ , then  $I'(a_{r_1, r_2, r_3}) = I(a_{r_1, r_2, r_3})$ , thus,  $I$  is an  $B_{12}(u)$ -invariant. Similarly, for the other  $B_{ij}(u)$ . Hence for matrices  $B_{ij}(u)$ , define the related differential operators  $D_{ij}$  as:

$$D_{12} = \sum_{r_2=1}^n (r_2 \sum_{r_1+r_3=n-r_2} a_{r_1+1, r_2-1, r_3}) \frac{\partial}{\partial a_{r_1, r_2, r_3}}$$

$$D_{23} = \sum_{r_3=1}^n (r_3 \sum_{r_1+r_2=n-r_3} a_{r_1, r_2+1, r_3-1}) \frac{\partial}{\partial a_{r_1, r_2, r_3}}$$

$$D_{13} = \sum_{r_3=1}^n (r_3 \sum_{r_1+r_2=n-r_3} a_{r_1+1, r_2, r_3-1}) \frac{\partial}{\partial a_{r_1, r_2, r_3}}$$

$$D_{21} = \sum_{r_1=1}^n (r_1 \sum_{r_2+r_3=n-r_1} a_{r_1-1, r_2+1, r_3}) \frac{\partial}{\partial a_{r_1, r_2, r_3}}$$

$$D_{32} = \sum_{r_2=1}^n (r_2 \sum_{r_1+r_3=n-r_2} a_{r_1, r_2-1, r_3+1}) \frac{\partial}{\partial a_{r_1, r_2, r_3}}$$

$$D_{31} = \sum_{r_1=1}^n (r_1 \sum_{r_2+r_3=n-r_1} a_{r_1-1, r_2, r_3+1}) \frac{\partial}{\partial a_{r_1, r_2, r_3}}.$$

The next proposition is a criterion for determining an  $SL(\mathcal{C}, 3)$ -invariant, which simplifies a result of [?] (cf. [?], Theorem 4.5.2).

**Proposition 2.5** A homogeneous function  $I$  of the coefficients of a form  $f$  is an  $SL(\mathcal{C}, 3)$ -invariant if and only if

$$D_{ij}I = 0, (i \neq j).$$

More precisely,  $I$  is a  $B_{ij}(u)$ -invariant if and only if  $D_{ij}I = 0$ .

**Proof** This is a direct consequence of the above discussion. □

The next proposition describes the relations among these differential operators.

**Proposition 2.6** If  $E$  is a homogeneous function in the coefficients  $a_{r_1, r_2, r_3}$ , with weights  $p_{x_1}, p_{x_2}, p_{x_3}$  w.r.t.  $x_1, x_2, x_3$  resp., then

$$\begin{aligned} (D_{12}^k D_{21} - D_{21} D_{12}^k)E &= k(p_{x_1} - p_{x_2} + (k-1))D_{12}^{(k-1)}E \\ (D_{12} D_{21}^k - D_{21}^k D_{12})E &= k(p_{x_1} - p_{x_2} - (k-1))D_{21}^{(k-1)}E \\ (D_{23}^k D_{32} - D_{32} D_{23}^k)E &= k(p_{x_2} - p_{x_3} + (k-1))D_{23}^{(k-1)}E \\ (D_{23} D_{32}^k - D_{32}^k D_{23})E &= k(p_{x_2} - p_{x_3} - (k-1))D_{32}^{(k-1)}E. \end{aligned}$$

**Proof** We will only prove the first identity, the proof of others are similar. We use induction on  $k$ .

For  $k = 1$ :

To prove this case, use induction on the degree  $g$  of  $E$ . For  $g = 1$ , we have

$$\begin{aligned} D_{12} D_{21} a_{r_1, r_2, r_3} &= D_{12}(r_1 a_{r_1-1, r_2+1, r_3}) = r_1(r_2 + 1)a_{r_1, r_2, r_3}, \\ D_{21} D_{12} a_{r_1, r_2, r_3} &= D_{21}(r_2 a_{r_1+1, r_2-1, r_3}) = r_2(r_1 + 1)a_{r_1, r_2, r_3}. \end{aligned}$$

Hence

$$(D_{12} D_{21} - D_{21} D_{12})a_{r_1, r_2, r_3} = (r_1 - r_2)a_{r_1, r_2, r_3} = (p_{x_1} - p_{x_2})a_{r_1, r_2, r_3}.$$

Suppose that it is true for  $g < n$ . Since  $D_{12}$  and  $D_{21}$  are linear differential equations, we get

$$\begin{aligned} D_{12}(E_1 + E_2) &= D_{12}E_1 + D_{12}E_2, \\ D_{12}(E_1 E_2) &= E_1 D_{12}E_2 + E_2 D_{12}E_1, \\ D_{21}(E_1 + E_2) &= D_{21}E_1 + D_{21}E_2, \\ D_{21}(E_1 E_2) &= E_1 D_{21}E_2 + E_2 D_{21}E_1, \end{aligned}$$

for any functions  $E_1$  and  $E_2$  of the coefficients of an  $m$ -ary form. In particular,  $D_{12}E_1 = 0$  if the weight  $p_{x_2}$  of  $E_1$  is zero. Hence suppose that  $E$  is a monomial, and then let  $E = E_1 E_2$ , with degrees of  $E_1$  and  $E_2$  more than one, then

$$\begin{aligned} (D_{12} D_{21} - D_{21} D_{12})E &= (D_{12} D_{21} - D_{21} D_{12})(E_1 E_2) \\ &= E_1 (D_{12} D_{21} - D_{21} D_{12})E_2 + E_2 (D_{12} D_{21} - D_{21} D_{12})E_1 \\ &= E_1 (p_{x_1 E_2} - p_{x_2 E_2})E_2 + E_2 (p_{x_1 E_1} - p_{x_2 E_1})E_1 \\ &= (p_{x_1 E_1} + p_{x_1 E_2} - (p_{x_2 E_1} + p_{x_2 E_2}))E_1 E_2 \\ &= (p_{x_1} - p_{x_2})E. \end{aligned}$$

Thus the statement is true for  $k = 1$ .

Suppose that the statement is true for  $k$ , then

$$\begin{aligned} (D_{12}^{(k+1)} D_{21} - D_{12} D_{21} D_{12}^k)E &= D_{12}(D_{12}^k D_{21} - D_{21} D_{12}^k)E \\ &= k(p_{x_1} - p_{x_2} + (k-1))D_{12}^k E, \\ (D_{12} D_{21} - D_{21} D_{12})D_{12}^k E &= (p_{x_1} + k - (p_{x_2} - k))D_{12}^k E, \end{aligned}$$

since the weight  $p'_{x_1}$  ( $p'_{x_2}$ , resp.) of  $D_{12}^k E$  is the weight  $p_x$  ( $p_{x_2}$ , resp.) of  $E$  plus  $k$  (minus  $k$ , resp.). Adding up the above two identities to see that

$$\begin{aligned} & (D_{12}^{(k+1)} D_{21} - D_{21} D_{12}^{(k+1)}) E \\ &= ((k+1)(p_{x_1} - p_{x_2}) + (k+1)k) D_{12}^k E \\ &= (k+1)(p_{x_1} - p_{x_2} + k) D_{12}^k E. \end{aligned}$$

□

Now, we prove one of the main theorems of this chapter:

**Theorem 2.7** A homogeneous function  $I$  of the coefficients of an arbitrary 3-ary form  $f$  is an invariant of  $f$  if and only if the following two conditions are satisfied:

1. The weights of  $I$  w.r.t.  $x_1, x_2, x_3$  are equal;
2.  $D_{i,i+1} I = 0$  for  $i = 1, 2$ , where  $D_{i,i+1}$  are the corresponding differential operators related to the matrices  $B_{i,i+1}(u)$ .

**Proof** “ $\implies$ ” is obvious.

We will prove “ $\impliedby$ ”:

By Corollary ?? and Proposition ??, we only need to prove that

$$D_{i+1,i} I = 0 \quad (i = 1, 2).$$

In the following, we prove that  $D_{21} I = 0$ . The proof of  $D_{32} I = 0$  is similar.

We know from the proof of Proposition ?? that in the sequence:

	$I$	$D_{21} I$	$D_{21}^2 I$	$\dots$	$D_{21}^k I$	$\dots$
weight $p_{x_1}$	$p_{x_1}$	$p_{x_1} - 1$	$p_{x_1} - 2$	$\dots$	$p_{x_1} - k - 1$	$\dots$
degree	$g$	$g$	$g$	$\dots$	$g$	$\dots$

there exists an integer  $k$  such that  $D_{21}^j I = 0$  for all  $j \geq k$ , while  $D_{21}^{k-1} I \neq 0$ . Thus

$$0 = D_{12} D_{21}^k I - D_{21}^k D_{12} I = k(-k+1) D_{21}^{k-1} I,$$

since  $k \neq 0$ ,  $D_{21}^{k-1} I \neq 0$ , so  $k = 1$ , i.e.  $D_{21} I = 0$ .

Here we let  $D_{ij}^0 E = E$  for any polynomial  $E$  of the coefficients of  $f$ . □

For the determination of given degree covariants of a 3-ary form, we have the following main theorem

**Theorem 2.8** A function

$$C(a_{n,0,0}, \dots, a_{r_1, r_2, r_3}, \dots, a_{0,0,n}; x_1, x_2, x_3) = C_{t,0,0} x_1^t + \dots + C_{0,0,t} x_3^t,$$

homogeneous in the  $a'_{r_1, r_2, r_3}$ s as well as the  $x'_i$ s, is a covariant of  $f(x_1, x_2, x_3; n)$  if and only if the following are satisfied:

1. The weights of  $C_{t,0,0}$  w.r.t.  $x_2, x_3$  equal  $p$ , while the weight of  $C_{t,0,0}$  w.r.t.  $x_1$  equals  $p + t$ , where  $p$  is the power of the transformation determination.

2.  $C_{t,0,0}$  satisfies the two differential operators

$$D_{12}C_{t,0,0} = 0, D_{23}C_{t,0,0} = 0,$$

and the  $C_{r_1,r_2,r_3}$  are derived from  $C_{t,0,0}$  via the formulas:

$$\begin{aligned} D_{21}C_{0,r_2,t-r_2} &= 0; \\ D_{21}C_{r_1,r_2,r_3} &= r_1C_{r_1-1,r_2+1,r_3} \quad \text{if } r_1 \neq 0. \\ D_{32}C_{r_1,0,t-r_1} &= 0; \\ D_{32}C_{r_1,r_2,r_3} &= r_2C_{r_1,r_2-1,r_3+1} \quad \text{if } r_2 \neq 0. \end{aligned}$$

More intuitively, we have the diagram:

$$\begin{array}{ccccccc} C_{t,0,0} & \xrightarrow{D_{21}} & tC_{t-1,1,0} & \xrightarrow{D_{21}} & \cdots & \xrightarrow{D_{21}} & t!C_{0,t,0} \\ & & \downarrow D_{32} & & \cdots & & \downarrow D_{32} \\ & & tC_{t-1,0,1} & & \cdots & & t!tC_{0,t-1,1} \\ & & & & \cdots & & \downarrow D_{32} \\ & & & & \cdots & & \vdots \\ & & & & & & \downarrow D_{32} \\ & & & & & & t!t!C_{0,0,t} \end{array}$$

From this diagram, we know that all the other coefficients of a covariant are determined by  $C_{t,0,0}$ , so  $C_{t,0,0}$  is also called the source of the covariant  $C$  (cf. [?]).

The proof of this theorem is a consequence of the following proposition

**Proposition 2.9** Every function

$$C(a_{n,0,0}, \dots, a_{r_1,r_2,r_3}, \dots, a_{0,0,n}; x_1, x_2, x_3) = C_{t,0,0}x_1^t + \cdots + C_{0,0,t}x_3^t$$

of the coefficients  $a_{r_1,r_2,r_3}$  and the variables  $x_1, x_2, x_3$  that is homogeneous in the  $a'_{r_1,r_2,r_3}$ s as well as the  $x_i$ 's is a covariant of  $f(x_1, x_2, x_3; n)$  if and only if

1. The weights of  $C_{t,0,0}$  w.r.t.  $x_2, x_3$  equal to  $p$ , while the weight of  $C_{t,0,0}$  w.r.t.  $x_1$  equals to  $p + t$ , and  $ng = 3p + t$ , where  $g$  is the degree of  $C$ .
2.  $C$  satisfies the following equations:

$$\begin{aligned} D_{12}C &= x_2 \frac{\partial C}{\partial x_1}, & D_{23}C &= x_3 \frac{\partial C}{\partial x_2}, \\ D_{21}C &= x_1 \frac{\partial C}{\partial x_2}, & D_{32}C &= x_2 \frac{\partial C}{\partial x_3}. \end{aligned}$$

This is because Condition 1 holds if and only if  $C$  is an  $A$ -covariant, while Condition 2 holds if and only if  $C$  is a  $B_{i,i+1}(u), B_{i+1,i}(u)$ -covariant. Since the precise proof of this proposition is similar to that of the binary form in [?], we omit it.

Using Theorems ?? and ??, we can calculate some invariants and covariants of 3-ary forms.



### 2.1. Some Invariants of 3-ary Forms

Let  $T(a_{r_1, r_2, r_3})$  be a polynomial of the coefficients  $\{a_{r_1, r_2, r_3}\}$  of a 3-ary form  $f$ . Define the action of  $S_3$  on  $\mathcal{C}[\{a_{r_1, r_2, r_3}\}]$  as:

$$\sigma T(a_{r_1, r_2, r_3}) = T(a_{\sigma(r_1), \sigma(r_2), \sigma(r_3)}), \sigma \in S_3,$$

and let

$$S_3 T = \frac{1}{3!} \sum_{\sigma \in S_3} \sigma T,$$

which is called the symmetrization of  $T$  [?]. Here  $S_3$  is the symmetric group.

**Lemma 2.10** Suppose that  $I$  of (3.1) is an invariant of  $GL(\mathcal{C}, 3)$ , then there exists a polynomial  $P$ , which has minimal terms, such that  $I = S_3 P$ .

**Proof** For each monomial  $M$  in  $I$ , let  $O_M = \{\sigma M | \sigma \in S_3\}$  be the  $S_3$ -orbit of  $M$ . Let

$$P = \sum_M \#(O_M) Z_M M,$$

of which any two monomials are not in the same  $S_3$ -orbit, and any orbit (of the monomials of  $I$ ) has a representative, where  $\#(O_M)$  is the number of elements in  $O_M$ , and  $Z_M$  is the coefficient of  $M$  in  $I$ . Then  $S_3 P = I$ . This is because  $\sigma I = I$ , implies  $Z_{\sigma M} = Z_M$ , for all  $\sigma \in S_3$ .  $\square$

Note that the polynomial  $P$  in the above lemma is not unique. If we define an order of monomials, such as the purely lexicographical order, and let the monomial of  $P$  be maximal in its orbit of the monomials  $M$  of  $I$ , then  $P$  is unique. The next theorem is another criterion for determining an invariant of a given form. We will use this theorem to compute some invariants in the following examples.

**Theorem 2.11** Consider  $S_3$  as a subgroup of  $GL(\mathcal{C}, 3)$ . Suppose that  $I$  is a homogeneous polynomial of the coefficients of  $f$ . Then  $I$  is an invariant of a 3-ary form  $f$  if and only if the following three conditions hold:

1.  $I$  is an  $A$ -invariant;
2.  $I$  is an  $S_3$ -invariant;
3.  $D_{12}I = 0$ .

**Proof** “ $\implies$ ” is obvious.

“ $\impliedby$ ”: We only need to prove that  $I$  is also an  $SL(\mathcal{C}, 3)$ -invariant.  $D_{12}I = 0$  implies that  $I$  is a  $B_{12}(u)$ -invariant. Note that

$$(1, 3)B_{12}(u)(1, 3) = B_{32}(u), \quad (2, 3)B_{32}(u)(2, 3) = B_{23}(u),$$

i.e.  $B_{23}(u)$  can be generated by the elements of  $S_3$  and  $B_{12}(u)$  as an element of  $GL(\mathcal{C}, 3)$ . Hence  $I$  is also a  $B_{23}(u)$ -invariant by Conditions 2 and 3. Thus, the theorem follows from Proposition ?? and Theorem ??.  $\square$

Note that this is true for an arbitrary  $m$ -ary form ( $m \geq 3$ ).

**Example 2.2** To compute the discriminant  $D$  of the 3-ary quadratic form  $f$ , solve the weight equations (1), (2), (3) with  $p = 2, n = 2$ . Let  $D$  be:

$$D = z_1 a_{2,0,0} a_{0,2,0} a_{0,0,2} + z_2 a_{2,0,0} a_{0,1,1}^2 + z_3 a_{0,2,0} a_{1,0,1}^2 + z_4 a_{0,0,3} a_{1,1,0}^2 + z_5 a_{1,1,0} a_{1,0,1} a_{0,1,1},$$

for which each power of the monomials is a solution of the weight equations. Now symmetrize  $D$  to get

$$S_3 D = \frac{1}{6} (6z_1 a_{2,0,0} a_{0,2,0} a_{0,0,2} + 6z_5 a_{1,1,0} a_{1,0,1} a_{0,1,1} + 2(z_2 + z_3 + z_4) (a_{2,0,0} a_{0,1,1}^2 + a_{0,2,0} a_{1,0,1}^2 + a_{0,0,3} a_{1,1,0}^2)).$$

Let

$$P = s_1 a_{2,0,0} a_{0,2,0} a_{0,0,2} + s_2 a_{1,1,0} a_{1,0,1} a_{0,1,1} + s_3 a_{2,0,0} a_{0,1,1}^2.$$

Then instead of operating the operator  $D_{12}$  on  $D$ , let it act on  $S_3 P$  to determine the coefficients  $s_i$ . By  $D_{12}(S_3 P) = 0$ , we get

$$s_1 = N_1, \quad s_2 = 2N_1, \quad s_3 = -3N_1.$$

Let  $N_1 = 1$ . Then

$$\begin{aligned} D &= S_3 (a_{2,0,0} a_{0,2,0} a_{0,0,2} + 2a_{1,1,0} a_{1,0,1} a_{0,1,1} - 3a_{2,0,0} a_{0,1,1}^2) \\ &= a_{2,0,0} a_{0,2,0} a_{0,0,2} - a_{2,0,0} a_{0,1,1}^2 - a_{0,2,0} a_{1,0,1}^2 - a_{0,0,3} a_{1,1,0}^2 + 2a_{1,1,0} a_{1,0,1} a_{0,1,1}. \end{aligned}$$

In the following examples, we compute the polynomials  $P$  such that  $S_3 P = I$  are linearly independent invariants over  $\mathcal{C}$ .

**Example 2.3** The degree  $g = 3$  invariants of the 3-ary form of degree  $n = 4$  in the variables  $x$ 's.

Solving the weight equations (1), (2), (3) with weight  $p = 4, n = 4$  to get 23 solutions. Let  $I$  be the polynomial of the form in (3.1), with indeterminate coefficients, of which the exponents of the monomials are these 23 solutions. There are 9 different  $S_3$ -orbits of these monomials. Thus let

$$\begin{aligned} P &= z_1 a_{4,0,0} a_{0,4,0} a_{0,0,4} + z_2 a_{4,0,0} a_{0,2,2}^2 + z_3 a_{4,0,0} a_{0,3,1} a_{0,1,3} \\ &\quad + z_4 a_{2,2,0} a_{1,1,2}^2 + z_5 a_{0,2,2} a_{3,1,0} a_{1,1,2} + z_6 a_{2,2,0} a_{2,0,2} a_{0,2,2} \\ &\quad + z_7 a_{2,1,1} a_{1,2,1} a_{1,1,2} + z_8 a_{3,1,0} a_{1,0,3} a_{0,3,1} + z_9 a_{3,1,0} a_{1,2,1} a_{0,1,3}. \end{aligned}$$

From  $D_{12}(S_3 P) = 0$ , we get 17 linear equations with 9 variables  $z_i$ . By "isolve" these equations in Maple 5.3, we obtain the solution

$$\begin{aligned} z_9 &= 36N_1, \quad z_8 = 8N_1, \quad z_7 = -12N_1, \quad z_6 = 6N_1, \quad z_5 = -72N_1, \\ z_4 &= 36N_1, \quad z_3 = -12N_1, \quad z_2 = 9N_1, \quad z_1 = N_1. \end{aligned}$$

Let  $Z_1 = 1$ , we have

$$\begin{aligned} P &= a_{4,0,0} a_{0,4,0} a_{0,0,4} + 9a_{4,0,0} a_{0,2,2}^2 - 12a_{4,0,0} a_{0,3,1} a_{0,1,3} \\ &\quad + 36a_{2,2,0} a_{1,1,2}^2 - 72a_{0,2,2} a_{3,1,0} a_{1,1,2} + 6a_{2,2,0} a_{2,0,2} a_{0,2,2} \\ &\quad - 12a_{2,1,1} a_{1,2,1} a_{1,1,2} + 8a_{3,1,0} a_{1,0,3} a_{0,3,1} + 36a_{3,1,0} a_{1,2,1} a_{0,1,3}. \end{aligned}$$

Hence  $S_3 P$  is the unique degree 3 invariant of the 3-ary form of degree  $n = 4$  in variables  $x$ 's (up to a constant).

**Example 2.4** The degree  $g = 6$  invariants of the form  $f(x_1, x_2, x_3; 4)$ .

Solving the weight equations (1), (2), (3) with weight  $p = 8$ , we obtain 561 solutions. Let  $I_1$  be a polynomial of the form in (3.1), such that the powers of the monomials are these solutions, and the coefficients  $Z_i$  are unknown variables that we want to solve. Since the number of different  $S_3$ -orbits of the monomials in  $I_1$  is 123, i.e. the number of the unknown variables turns out to be 123. Let  $P$  be the polynomial with 123 indeterminate coefficients, such that any two monomials of which are not in the same orbit, any orbit has a representative. Then the system of linear equations obtained from  $D_{12}(S_3P) = 0$  has 527 equations. Solve this linear equations by the “isolve” in Maple 5.3. Its solution is of dimension 2, which shows that there are exactly two linearly independent invariants of degree  $g = 6$  (up to a constant). For the explicit expressions, we omit here.

**Example 2.5** The degree  $g = 9$  invariants of the 3-ary form of degree 4 in variables  $x$ 's.

Similar to the above examples, we solve the weight equations (1), (2) and (3) with weight  $p = 12$  to get 6992 solutions. Let  $I$  be the general homogeneous polynomial with the solutions as the exponents of the monomials, then the number of different orbits of its monomials is 1281, hence, we obtain a polynomial  $P$  of 1281 terms, such that  $S_3P$  is an invariant if and only if  $D_{12}(S_3P) = 0$ . It is not difficult to solve the linear equations of the coefficients of this equation, while this expression is too long to be presented here.

**Remark 2.6** These examples are computed in Maple 5.3 on Dec Alpha-station 5/333.

## 2.2. Some Covariants of 3-ary Forms

In this section, the order  $t$  and degree  $g$  of a covariant satisfy  $ng = 3p + t$ , where  $p$  is the weight w.r.t.  $x_2, x_3$  of the source of the covariant, and  $n$  is the degree of  $x_1, x_2, x_3$  of the form.

1. For  $n = 2$

$$f(x_1, x_2, x_3; 2) = a_{2,0,0}x_1^2 + 2a_{1,1,0}x_1x_2 + 2a_{1,0,1}x_1x_3 + a_{0,2,0}x_2^2 + 2a_{0,1,1}x_2x_3 + a_{0,0,2}x_3^2.$$

It is easy to find that all covariants of degree  $g$  no more than 6 are of the form  $D^i f^j$ , where

$$D = a_{2,0,0}a_{0,2,0}a_{0,0,2} + 2a_{1,1,0}a_{1,0,1}a_{0,1,1} - a_{2,0,0}a_{0,1,1}^2 - a_{0,2,0}a_{1,0,1}^2 - a_{0,0,2}a_{1,1,0}^2$$

is the discriminant.

2. For  $n = 3$

$$\begin{aligned} f(x_1, x_2, x_3; 3) = & a_{3,0,0}x_1^3 + 3a_{2,1,0}x_1^2x_2 + 3a_{2,0,1}x_1^2x_3 + 3a_{1,2,0}x_1x_2^2 \\ & + 6a_{1,1,1}x_1x_2x_3 + 3a_{1,0,2}x_1x_3^2 + a_{0,3,0}x_0,3,0^3 \\ & + 3a_{0,2,1}x_2^2x_3 + 3a_{0,1,2}x_2x_3^2 + a_{0,0,3}x_3^3 \end{aligned}$$

We have the table:

Covariants of Degree  $g \leq 6$  of 3-ary Form of Degree  $n = 3$

$g \setminus p$	0	1	2	3	4	5	6
1	$f$	0	0	0	0	0	0
2	$f^2$	0	0	0	0	0	0
3	$f^3$	0	$H$	0	0	0	0
4	$f^4$	0	$Hf$	0	$S$	0	0
5	$f^5$	0	$Hf^2$	0	$Sf$	0	0
6	$f^6$	0	$Hf^3$	0	$Sf^2, H^2$	0	$T$

where  $g$  is the degree of a covariant and  $p$  is the weight of a covariant w.r.t.  $x_2$ ,  $H$  is the Hessian polynomial of  $f$  (up to some constant). The source of  $H$  is

$$S_H = a_{2,0,1}^2 a_{1,2,0} + a_{2,1,0}^2 a_{1,0,2} - a_{3,0,0} a_{1,2,0} a_{1,0,2} - 2a_{2,1,0} a_{2,0,1} a_{1,1,1} + a_{3,0,0} a_{1,1,1}^2,$$

and  $S$ , ( $T$ , resp.) is the unique invariant of degree 4, (6, resp.), (see [?]).

3. For  $n = 4$

$$f(x_1, x_2, x_3; 4) = \sum_{r_1+r_2+r_3=4} \frac{4!}{r_1!r_2!r_3!} a_{r_1,r_2,r_3} x_1^{r_1} x_2^{r_2} x_3^{r_3}.$$

We have the table:

The Number of Covariants of Degree  $g \leq 6$  of 3-ary Form of Degree  $n = 4$

$g \setminus p$	0	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0
3	1	0	1	0	1	0	0	0	0
4	1	0	1	0	2	0	0	0	0
5	1	0	1	0	3	0	2	0	0
6	1	0	1	0	4	0	5	0	2
7	1	0	1	0	4	0	7	0	?

where the covariants of weight ( $p = p_{x_2} =$ )0 are  $f, f^2, \dots, f^7$ , and the covariants of weight 2 are

$$H, Hf, Hf^2, Hf^3, Hf^4,$$

$H$  is the Hessian of  $f$  (up to some constant), its source is

$$S_H = a_{4,0,0} a_{2,1,1}^2 - a_{4,0,0} a_{2,2,0} a_{2,0,2} + a_{3,0,1}^2 a_{2,2,0} - 2a_{3,1,0} a_{3,0,1} a_{2,1,1} + a_{3,1,0}^2 a_{2,0,2}.$$

The unique invariant  $I_1$  of  $g = 3, p = 4$  is computed in Example ???. The two covariants of  $g = 4, p = 4$  are  $fI_1$  and  $C_1$ , where the source of  $C_1$  is

$$\begin{aligned} S_{C_1} = & a_{3,1,0} a_{3,0,1} a_{1,2,1} a_{1,1,2} + a_{4,0,0} a_{2,1,1} a_{1,3,0} a_{1,0,3} + a_{3,1,0} a_{2,2,0} a_{2,0,2} a_{1,1,2} \\ & + a_{3,0,1} a_{2,2,0} a_{2,0,2} a_{1,2,1} - a_{4,0,0} a_{2,0,2} a_{1,3,0} a_{1,1,2} - a_{3,1,0} a_{2,2,0} a_{2,1,1} a_{1,0,3} \\ & + a_{3,1,0}^2 a_{1,2,1} a_{1,0,3} + a_{4,0,0} a_{2,0,2} a_{1,2,1}^2 + a_{4,0,0} a_{2,2,0} a_{1,1,2}^2 + a_{3,0,1} a_{2,2,0}^2 a_{1,0,3} \\ & + a_{3,1,0} a_{2,0,2}^2 a_{1,3,0} - a_{2,2,0}^2 a_{2,0,2}^2 + 2 a_{3,1,0} a_{2,1,1}^2 a_{1,1,2} + 2 a_{2,2,0} a_{2,1,1}^2 a_{2,0,2} \\ & + 2 a_{3,0,1} a_{2,1,1}^2 a_{1,2,1} - a_{3,0,1}^2 a_{1,2,1}^2 + a_{3,0,1}^2 a_{1,3,0} a_{1,1,2} - a_{3,1,0} a_{3,0,1} a_{1,3,0} a_{1,0,3} \\ & - a_{4,0,0} a_{2,2,0} a_{1,2,1} a_{1,0,3} - a_{3,0,1} a_{2,1,1} a_{2,0,2} a_{1,3,0} - 3 a_{3,0,1} a_{2,2,0} a_{2,1,1} a_{1,1,2} \\ & - 3 a_{3,1,0} a_{2,1,1} a_{2,0,2} a_{1,2,1} - a_{4,0,0} a_{2,1,1} a_{1,2,1} a_{1,1,2} - a_{2,1,1}^4 - a_{3,1,0}^2 a_{1,1,2}^2. \end{aligned}$$

For other covariants, we only list the number of terms of the source obtained by “isolve” in Maple 5.3, of which the coefficients of the monomials are obtained by solving the linear equations of the coefficients of  $D_{12}S_C = 0, D_{23}S_C = 0$ , where  $S_C$  satisfies Condition 1 of Theorem ??.

The Number of Terms of the Source of a covariant

$g, p \setminus$	Number of Terms
$g = 5, p = 4$	25,50,49
$g = 5, p = 6$	179,151
$g = 6, p = 4$	25,36,49,15
$g = 6, p = 6$	15,268,268,268,268
$g = 6, p = 8$	438,438
$g = 7, p = 4$	25,47,15,49
$g = 7, p = 6$	428,428,428,428,428,436,428

For the case of  $g = 7, p = 8$ , there are 1133 unknown variables and 2003 equations.

4.  $n = 5$

$$f(x_1, x_2, x_3; 5) = \sum_{r_1+r_2+r_3=5} \frac{5!}{r_1!r_2!r_3!} a_{r_1,r_2,r_3} x_1^{r_1} x_2^{r_2} x_3^{r_3}.$$

We have the table:

Covariants of Degree  $d \leq 4$  of 3-ary Form of Degree  $n = 5$

$g \setminus p$	0	1	2	3	4	5	6
1	$f$	0	0	0	0	0	0
2	$f^2$	0	0	0	0	0	0
3	$f^3$	0	$H$	0	$C_2$	0	0
4	$f^4$	0	$Hf$	0	$fC_2, C_3$	0	$C_4$

where  $H$  is the Hessian(up to a constant), its source is

$$S_H = a_{5,0,0}a_{3,2,0}a_{3,0,2} - a_{5,0,0}a_{3,1,1}^2 - a_{4,1,0}^2a_{3,0,2} - a_{4,0,1}^2a_{3,2,0} + 2a_{4,1,0}a_{4,0,1}a_{3,1,1}.$$

The sources of  $C_2, C_3$  are

$$\begin{aligned} S_{C_2} = & 12 a_{2,2,1}^2 a_{3,0,2} + 3 a_{3,0,2}^2 a_{1,4,0} + 12 a_{3,1,1}^2 a_{1,2,2} + 3 a_{5,0,0} a_{1,2,2}^2 \\ & + 3 a_{3,2,0}^2 a_{1,0,4} + 12 a_{2,1,2}^2 a_{3,2,0} + a_{5,0,0} a_{1,4,0} a_{1,0,4} - 12 a_{2,1,2} a_{2,2,1} a_{3,1,1} \\ & - 12 a_{2,1,2} a_{2,3,0} a_{3,0,2} - 12 a_{2,0,3} a_{2,2,1} a_{3,2,0} + 12 a_{2,0,3} a_{2,3,0} a_{3,1,1} \\ & + 12 a_{4,0,1} a_{2,1,2} a_{1,3,1} - 12 a_{3,1,1} a_{3,0,2} a_{1,3,1} - 4 a_{4,0,1} a_{2,0,3} a_{1,4,0} \\ & + 4 a_{4,1,0} a_{2,0,3} a_{1,3,1} + 12 a_{4,1,0} a_{2,2,1} a_{1,1,3} - 12 a_{4,0,1} a_{2,2,1} a_{1,2,2} \\ & - 12 a_{4,1,0} a_{2,1,2} a_{1,2,2} + 6 a_{3,2,0} a_{3,0,2} a_{1,2,2} - 4 a_{5,0,0} a_{1,3,1} a_{1,1,3} \\ & - 4 a_{4,1,0} a_{2,3,0} a_{1,0,4} + 4 a_{4,0,1} a_{2,3,0} a_{1,1,3} - 12 a_{3,2,0} a_{3,1,1} a_{1,1,3}, \end{aligned}$$

$$\begin{aligned}
 S_{C_3} = & -a_{3,1,1}^4 - a_{5,0,0}a_{3,0,2}a_{2,3,0}a_{2,1,2} - a_{4,1,0}a_{4,0,1}a_{2,3,0}a_{2,0,3} \\
 & -a_{4,1,0}a_{3,2,0}a_{3,1,1}a_{2,0,3} - 3a_{4,1,0}a_{3,1,1}a_{3,0,2}a_{2,2,1} - a_{4,0,1}a_{3,1,1}a_{3,0,2}a_{2,3,0} \\
 & + a_{4,0,1}^2a_{2,3,0}a_{2,1,2} + a_{4,1,0}a_{4,0,1}a_{2,2,1}a_{2,1,2} + a_{5,0,0}a_{3,2,0}a_{2,1,2}^2 \\
 & + a_{4,1,0}a_{3,2,0}a_{3,0,2}a_{2,1,2} + a_{4,1,0}^2a_{2,2,1}a_{2,0,3} + a_{4,0,1}a_{3,2,0}^2a_{2,0,3} \\
 & + a_{5,0,0}a_{3,1,1}a_{2,3,0}a_{2,0,3} + a_{5,0,0}a_{3,0,2}a_{2,2,1}^2 + a_{4,0,1}a_{3,2,0}a_{3,0,2}a_{2,2,1} \\
 & + a_{4,1,0}a_{3,0,2}^2a_{2,3,0} - a_{5,0,0}a_{3,1,1}a_{2,2,1}a_{2,1,2} - 3a_{4,0,1}a_{3,2,0}a_{3,1,1}a_{2,1,2} \\
 & + 2a_{4,1,0}a_{3,1,1}^2a_{2,1,2} + 2a_{4,0,1}a_{3,1,1}^2a_{2,2,1} + 2a_{3,2,0}a_{3,1,1}^2a_{3,0,2} \\
 & - a_{4,0,1}^2a_{2,2,1}^2 - a_{4,1,0}^2a_{2,1,2}^2 - a_{3,2,0}^2a_{3,0,2}^2 - a_{5,0,0}a_{3,2,0}a_{2,2,1}a_{2,0,3}.
 \end{aligned}$$

The source of  $C_4$  has 130 terms.

### 3. A Note on the Computation of the Syzygies

In [?], the syzygies of the invariants are computed by Grobner Bases. In fact, they can be computed also by the characteristic set method. The following two examples are taken from [?].

**Example 3.1** It can be proved that the invariant ring  $C[x_1, x_2]^{Z_4}$  of the group

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

is generated by the invariants:

$$f_1 = x_1^2 + x_2^2; \quad f_2 = x_1^2x_2^2; \quad f_3 = x_1x_2^3 - x_1^3x_2$$

To compute the syzygy among them, we introduce three new variables  $a, b, c$  and compute the characteristic sets of the polynomials

$$p_1 = x_1^2 + x_2^2 - a; \quad p_2 = x_1^2x_2^2 - b; \quad p_3 = x_1x_2^3 - x_1^3x_2 - c$$

according to a variable order satisfied:  $a, b, c \prec x_1, x_2$ . There is only one characteristic set:

$$\begin{aligned}
 p_1 &= x_1^2 + x_2^2 - a \\
 q_2 &= -x_2^4 + ax_2^2 - b \\
 q_3 &= -c^2 + ba^2 - 4b^2.
 \end{aligned}$$

Hence there is a syzygy among  $p_1, p_2, p_3$ , which is  $q_3$ .

**Example 3.2** It can be proved that the invariant ring  $C[x, y, z]^{D_6}$  of the dihedral group

$$D_6 = \{1, \delta, \delta^2, \delta^3, \delta^4, \delta^5, \sigma, \sigma\delta, \sigma\delta^2, \sigma\delta^3, \sigma\delta^4, \sigma\delta^5\}$$

is generated by the invariants

$$\begin{aligned}
 g_1 &= x^2 + y^2; \\
 g_2 &= z^2; \\
 g_3 &= x^6 - 6x^4y^2 + 9x^2y^4; \\
 g_4 &= 3x^5yz - 10x^3y^3z + 3xy^5z
 \end{aligned}$$

where

$$\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We compute the characteristic sets of  $\{g_1 - a, g_2 - b, g_3 - c, g_4 - d\}$  according to a variable order satisfied:  $a, b, c, d, \prec x, y, z$ , and obtain the unique characteristic set:

$$\begin{aligned} q_1 &= 16xy^5z - d - 16xy^3za + 3xyza^2 \\ q_2 &= -c - 16y^6 + 24y^4a - 9y^2a^2 + a^3 \\ q_3 &= z^2 - b \\ q_4 &= -d^2 - bc^2 + bca^3 \end{aligned}$$

So there is a syzygy  $q_4$  among the fundamental invariants.

**Problem** From these two examples, we know that we can use the characteristic set method to compute some syzygies. Can we use the characteristic set method to determine the full system of syzygies?

*Acknowledges:* I thank Prof. Shi He and Dr. Liu Xianfang for their encouragement and discussions.

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