Common Fixed Point Theorems for Fuzzy Mappings in Quasi-Metric Spaces

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Abstract

In this paper we prove common fixed point theorems for a class of fuzzy mappings in Smyth-complete quasi-metric spaces. Well-known theorems are special case of our results.

Keywords: Fuzzy mapping; Fixed point; Quasi-metric space; Smyth-complete; Left $K$-complete.

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1. Introduction

Heilpern [5] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy mappings which is the generalization of the fixed point theorem for multi-valued mappings of Nadler [8]. Recently, Gregori and Romaguera [4] showed the importance of the Smyth completeness regarding many areas of research in theoretical computer science and proved some fixed
point theorems for fuzzy mappings in Smyth-complete and left $K$-complete quasi-metric spaces, respectively. Also Telci and Fisher [13] obtained a fixed point theorem for these mappings.

In this paper, we consider a generalized contractive type condition involving fuzzy mappings in Smyth-complete quasi-metric spaces and we establish a common fixed point theorem which extends many theorems obtained by many authors.

2. Basic notions and preliminary results

In the following, the letter $\Gamma$ denotes the set of positive integers. If $A$ is a subset of a topological space $(X, \tau)$, we will denote by $\text{cl} A$ the closure of $A$ in $(X, \tau)$.

A quasi-metric on a nonempty set $X$ is a nonnegative real valued function $d$ on $X \times X$, such that, for all $x, y, z \in X$:

(a) $d(x, y) = d(y, x) = 0 \iff x = y$, and
(b) $d(x, y) \leq d(x, z) + d(z, y)$.

A pair $(X, d)$ is called a quasi-metric space, if $d$ is a quasi-metric on $X$.

Each quasi-metric $d$ on $X$ induces a $T_0$ topology $\tau(d)$ on $X$, which has a base the family of all $d$-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

If $d$ is a quasi-metric on $X$, then the function $d^{-1}$ defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also a quasi-metric on $X$. By $d \wedge d^{-1}$ we denote $\min\{d, d^{-1}\}$ and also we denote $d^S$ the metric on $X$ by $d^S(x, y) = \max\{d(x, y), d(y, x)\}$, for all $x, y \in X$.

A sequence $(x_n)_{n \in \Gamma}$ in a quasi-metric space $(X, d)$ is called left $K$-Cauchy [10], if for each $\varepsilon > 0$ there is a $n_\varepsilon \in \Gamma$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \Gamma$ such that $m \geq n \geq n_\varepsilon$.

The quasi-metric space $(X, d)$ is said to be left $K$-complete [10] if each left $K$-Cauchy sequence in $(X, d)$ converges (with respect to the topology $\tau(d)$).

A quasi metric space $(X, d)$ is said to be Smyth-complete [7] if each left $K$-Cauchy sequence in $(X, d)$ converges in the metric space $(X, d^S)$. Clearly, every Smyth-complete quasi-metric space is left $K$-complete. In general the converse implication does not hold.

Let $(X, d)$ be a quasi-metric space and let $\mathcal{K}_0^S(X)$ be the collection of all nonempty compact subset of the metric space $(X, d^S)$. Then the Hansdorff distance $H_d \mathcal{K}_0^S(X)$ on $\mathcal{K}_0^S(X)$ is defined by
A fuzzy set on \( X \) is an element of \( I^X \) where \( I = [0,1] \). If \( A \) is a fuzzy set in \( X \), then the number \( A(x) \) is called the grade of membership of \( x \) in \( A \).

The \( \alpha \)-level set of \( A \), denoted by \( A_\alpha \), is defined by

\[
A_\alpha = \{ x \in X : A(x) \geq \alpha \} \quad \text{for each } \alpha \in (0,1], \text{ and } A_0 = \{ x \in X : A(x) > 0 \}
\]

where the closure is taken in \( (X,d^S) \).

**Definition 2.1**[4] Let \((X,d)\) be a quasi-metric space. A fuzzy set \( A \) in the quasi-metric space \((X,d)\) will be called an approximate quantity. The family 
\( \mathcal{A}(X) \) of all fuzzy sets on \((X,d)\) is defined by

\[
\mathcal{A}(X) = \{ A \in I^X : A_\alpha \text{ is } d^S \text{-compact for each } \alpha \in [0,1] \text{ and } \sup \{ A(x) : x \in X \} = 1 \}.
\]

**Definition 2.2**[5] Let \((X,d)\) be a quasi-metric space and let \( A,B \in \mathcal{A}(X) \) and \( \alpha \in [0,1] \). Then we define,

\[
p_{\alpha}(A,B) = \inf \{ d(x,y) : x \in A_\alpha, y \in B_\alpha \} = d(A_\alpha,B_\alpha)
\]

\[
D_{\alpha}(A,B) = H_d(A_\alpha,B_\alpha)
\]

\[
p(A,B) = \sup \{ p_{\alpha}(A,B) : \alpha \in [0,1] \}
\]

\[
D(A,B) = \sup \{ D_{\alpha}(A,B) : \alpha \in [0,1] \}.
\]

**Definition 2.3**[4] A fuzzy mapping on a quasi-metric space \((X,d)\) is a function \( F \) defined on \( X \), which satisfies the following two conditions:

1. \( F(x) \in \mathcal{A}(X) \) for all \( x \in X \)
2. If \( a,z \in X \) such that \( (F(z))(a) = 1 \) and \( p(a,F(a)) = 0 \), then \( (F(a))(a) = 1 \).

We need the following lemmas for our main result which was given [4].

**Lemma 2.4**[4] Let \((X,d)\) be a quasi-metric space. Then, for each \( A \in \mathcal{A}(X) \) there exists \( p \in X \) such that \( A(p) = 1 \).

**Lemma 2.5**[4] Let \((X,d)\) be a quasi-metric space and let \( A,B \in \mathcal{A}(X) \) and \( x \in A_1 \). There exists \( y \in B_1 \) such that \( d(x,y) \leq D_1(A,B) \).
Lemma 2.6[4] Let \((X, d)\) be a quasi-metric space and let \(A, B \in \mathcal{A}(X)\). Then \(p(A, B) = p_\alpha(A, B)\)

Lemma 2.7[4] Let \((X, d)\) be a quasi-metric space and let \(A \in \mathcal{A}(X)\) and \(y \in A\). Then \(p(x, A) \leq d(x, y)\) for each \(x \in X\).

Lemma 2.8[4] Let \((X, d)\) be a quasi-metric space and let \(A \in \mathcal{A}(X)\) and \(x \in X\). Then \(p_\alpha(x, A) \leq d(x, y)\) for each \(x, y \in X\) and \(\alpha \in [0, 1]\).

Lemma 2.9[4] Let \((X, d)\) be a quasi-metric space and let \(A \in \mathcal{A}(X)\) and \(x \in X\). Then \(p_\alpha(x, B) \leq D_\alpha(A, B)\) for each \(B \in \mathcal{A}(X)\) and each \(\alpha \in [0, 1]\).

Lemma 2.10[4] Let \((X, d)\) be a quasi-metric space and let \(A \in \mathcal{A}(X)\). If \(p(x, A) = 0\), then there is \(y \in \text{cl}_{\tau_0} \{x\}\) such that \(A(y) = 1\).

Definition 2.11[4] We say that a fuzzy mapping \(F\) on a quasi-metric space \((X, d)\) has a fixed point if there exists \(a \in X\) such that \((F(a))(a) = 1\).

3. Common fixed point theorem for fuzzy mappings.

We consider the set \(G\) of all continuous functions \(g : [0, \infty)^5 \to [0, \infty)\) with the following properties:

(i) \(g\) is non-decreasing in the 2nd, 3rd, 4th, 5th variable,

(ii) If \(u, v \in [0, \infty)\) are such that \(u \leq g(v, v, u, 0, 0)\) or \(u \leq g(v, u, v, 0, 0)\) then \(u \leq hv\), where \(0 < h < 1\) is a given constant,

Theorem 3.1 Let \((X, d)\) be a Smyth-complete quasi-metric space and let \(F_1, F_2\) be fuzzy mappings on \(X\) into \(\mathcal{A}(X)\). If there exists a \(g \in G\) such that for all \(x, y \in X\)

\[
D(F_1(x), F_2(y)) \leq g(d(x, y), p(x, F_1(x)), p(y, F_2(y)), p(x, F_2(y)), p(y, F_1(x))) \quad (3.1)
\]

then \(F_1, F_2\) have common fixed point.

\textbf{Proof.} Let \(x_0\) be an arbitrary point in \(X\). By lemma 2.4, there exists \(x_1 \in X\) such that \((F_1(x_0))(x_1) = 1\). By lemmas 2.4 and 2.5, there exists \(x_2 \in X\) such that \((F_2(x_1))(x_2) = 1\) and \(d(x_1, x_2) \leq D_\alpha(F_1(x_0), F_2(x_1))\).

Then we obtain
Common fixed point theorems

\[ d(x_1, x_2) \leq D_1(F(x_0), F(x_1)) \leq D(F(x_0), F(x_1)) \]
\[ \leq g(d(x_0, x_1), p(x_0, F(x_0)), p(x_1, F(x_1)), p(x_0, F(x_0), p(x_1, F(x_0))) \]

By lemma 2.7 \( p(x_0, F(x_0)) \leq d(x_0, x_1) \), \( p(x_1, F(x_1)) \leq d(x_1, x_2) \), \( p(x_0, F(x_0)) \leq d(x_0, x_1) \leq d(x_0, x_1) \) and \( p(x_1, F(x_0)) = 0 \).

Thus we have
\[ d(x_1, x_2) \leq g(d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2), 0) \]

By (ii) condition we have
\[ d(x_1, x_2) \leq h d(x_0, x_1), \quad 0 < h < 1. \]

Again
\[ d(x_2, x_3) \leq h d(x_1, x_2) \leq h^2 d(x_0, x_1). \]

By induction, we produce a sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \) such that for \( k \geq 0 \),
\[ (F_1(x_{2k}))(x_{2k+1}) = 1, \quad (F_2(x_{2k+1}))(x_{2k+2}) = 1 \]
and
\[ d(x_n, x_{n+1}) \leq h^n d(x_0, x_1). \]

Furthermore, for \( m > n \),
\[ d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \]
\[ \leq h^n + h^{n+1} + \ldots + h^{m-1} d(x_0, x_1) \]
\[ \leq \frac{h^n}{1-h} d(x_0, x_1) \]

It follows that \( (x_n)_{n \in \mathbb{N}} \) is a left \( K \)-Cauchy sequence in the Smyth-complete quasi-metric space \( (X, d) \) and so there exists a \( z \in X \) such that \( d^2(z, x_n) \to 0 \).

Now by lemma 2.8, we have
\[ p_1(z, F_2(z)) \leq d(z, x_{2n+1}) + p_1(x_{2n+1}, F_2(z)) \]
for all \( n \in \mathbb{N} \).

So by lemmas 2.6, 2.9 and inequality (3) we have
By lemmas 2.7, 2.8, we have

\[ p(x_{2n}, F_1(x_{2n})) \leq d(x_{2n}, x_{2n+1}) \]
\[ p(z, F_1(x_{2n})) \leq d(z, x_{2n+1}) \]
\[ p(x_{2n}, F_2(z)) \leq d(x_{2n}, z) + p(z, F_2(z)) \]

It follows that

\[ p(z, F_2(z)) \leq d(z, x_{2n+1}) + g(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), p(z, F_2(z)), d(x_{2n}, z) + p(z, F_2(z)), d(z, x_{2n+1})) \]

As \( n \to \infty \), we have

\[ p(z, F_2(z)) \leq g(0, 0, p(z, F_2(z)), p(z, F_2(z)), 0). \]

By (ii) condition we have \( p(z, F_2(z)) = 0 \). Similarly, we have \( p(z, F_1(z)) = 0 \). So by lemma 2.10 there exists \( z^* \in cl_{r(d^{-1})}\{z\} \) such that \( (F_2(z))(z^*) = 1 \). Since \( z^* \in cl_{r(d^{-1})}\{z\} \) we have \( d(z, z^*) = 0 \).

We will prove now that \( z^* \) is fixed point of \( F_2 \).

By \( d(x_n, z^*) \leq d(x_n, z) + d(z, z^*) \) we have \( d(x_n, z^*) \to 0 \) and \( x_n \to z^* \) as \( n \to \infty \).

By Lemmas 2.8 and 2.9 we have

\[ p(z^*, F_2(z^*)) \leq d(z^*, x_{2n+1}) + p(x_{2n}, F_2(z^*)) \]
\[ \leq d(z^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(z^*)) \]

Using the inequality (3.1), we have

\[ D(F_1(x_{2n}), F_2(z^*)) \leq g(d(x_{2n}, z^*), p(x_{2n}, F_1(x_{2n})), p(z^*, F_2(z^*)), p(x_{2n}, F_2(z^*)), p(z^*, F_1(x_{2n}))) \]

By Lemmas 2.7 and 2.9 we have
\[ p(x_{2n}, F_1(x_{2n})) \leq d(x_{2n}, x_{2n+1}), \]
\[ p(x_{2n}, F_2(z^*)) \leq d(x_{2n}, z^*) + p(z^*, F_2(z^*)), \quad p(z^*, F_1(x_{2n})) \leq d(z^*, x_{2n+1}) \]

So
\[ p(z^*, F_2(z^*)) \leq d(z^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(z^*)) \]
\[ \leq d(z^*, x_{2n+1}) + g(d(x_{2n}, z^*), d(x_{2n}, x_{2n+1}), p(z^*, F_2(z^*)), \quad (3.4) \]
\[ d(x_{2n}, z^*) + p(z^*, F_2(z^*)), d(z^*, x_{2n+1})) \]

As \( n \to \infty \) we have
\[ p(z^*, F_2(z^*)) \leq g(0, 0, p(z^*, F_2(z^*)), p(z^*, F_2(z^*)), 0) \]

So, by (ii) condition
\[ p(z^*, F_2(z^*)) \leq hp(z^*, F_2(z^*)) \quad \text{with} \quad 0 < h < 1, \]

and so
\[ p(z^*, F_2(z^*)) = 0 \quad (3.5) \]

By Definition 2.3 we see that \((F_2(z^*), z^*) = 1\) and so \(z^*\) is fixed point of \(F_2\).

Similarly, by Lemma 2.10 there exists \(z_1^* \in cl_{\epsilon(d, \gamma)}(z)\) such that \((F_1(z))(z_1^*) = 1\)
and \(z_1^*\) is fixed point of \(F_1\).

By the triangle inequality \(d(z^*, z_1^*) \leq d(z^*, x_n) + d(x_n, z_1^*)\) and
\(d(z_1^*, z^*) \leq d(z_1^*, x_n) + d(x_n, z^*)\).

So as \( n \to \infty, d(z^*, z_1^*) = d(z_1^*, z^*) = 0 \) and \(F_1, F_2\) have common fixed point.

If in theorem 3.1 \(F_1 = F_2 = F\), we get the following fixed point theorem.

**Theorem 3.2** Let \((X, d)\) be a Smyth-complete quasi-metric space and let \(F\) be fuzzy mapping on \(X\) into \(\mathcal{F}(X)\). If there exists a \(g \in G\) such that for all \(x, y \in X\)
\[ D(F(x), F(y)) \leq g(d(x, y), p(x, F(x)), p(y, F(y)), p(x, F(y), p(y, F(x)))) \]
then \(F\) have a fixed point.

**Corollary 3.3** [4;Theorem 1] Let \((X, d)\) be a Smyth-complete quasi-metric space and let \(F\) be fuzzy mapping from \(X\) into \(\mathcal{F}(X)\). If there exists a constant \(h, 0 \leq h < 1\), such that for each \(x, y \in X\)
Then, \( F \) has a fixed point.

**Proof.** We consider the function \( g : [0, \infty)^5 \to [0, \infty) \) defined by

\[
g(x_1, x_2, x_3, x_4, x_5) = h \max \{ x_1, x_2, x_3, 2 \cdot \frac{x_4 + x_5}{2} \}
\]

Since \( g \in G \) we can apply Theorem 3.2 and obtain Corollary 3.3.

**Corollary 3.4** [13; Theorem 3.4] Let \((X, d)\) be a Smyth-complete quasi-metric space and let \( F \) be fuzzy mapping from \( X \) into \( \mathcal{F}(X) \). If there exists a constant \( h, \ 0 \leq h < 1 \), such that for each \( x, y \in X \)

\[
D(F(x), F(y)) \leq h \max \{d(x, y), p(x, F(x)), p(y, F(y)), \frac{p(x, F(y)) + p(y, F(x))}{2}, \frac{r[p(x, F(y)) \cdot p(y, F(x))]^{1/2}}{2} \}
\]

Where \( rh < 2 \), then \( F \) has a fixed point.

**Proof.** We consider the function \( g : [0, \infty)^5 \to [0, \infty) \) defined by

\[
g(x_1, x_2, x_3, x_4, x_5) = h \max \{ x_1, x_2, x_3, 2 \cdot \frac{x_4 + x_5}{2} \}
\]

Since \( g \in G \) we can apply Theorem 3.2 and obtain Corollary 3.4.

**Remark 3.5** Theorems 3.1 and 3.2 generalize the results obtained in the [4, 9, 13, et al] in the settings of Smyth-complete quasi-metric spaces.

Note that Smyth-completeness cannot be relaxed to left \( K \)-completeness in Theorem 3.1 (see [4, Example 5]) hence the result of the Theorem 3.1 do not remain valid in the settings of left \( K \)-complete quasi-metric spaces.

### 4. Conclusions

In this paper we proved common fixed point theorems for a pair of the classes \( G \) of fuzzy mappings in the settings of Smyth-complete quasi-metric spaces. Well known results are special case of our results. The Smyth completeness cannot be relaxed to left \( K \)-completeness, therefore the issue of finding a more general class of fuzzy mappings such as the conclusions of our theorem stand, remains open.
References


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