BIFURCATION OF PEAKONS AND CUSPONS OF THE INTEGRABLE NOVIKOV EQUATION

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By applying the bifurcation theory of dynamical systems to the Novikov equation, a new feature of non-smooth traveling wave solutions, two peakons or two cuspons that coexist for the same wave speed, is put forward. It is shown that \( g = 0 \) is the peakon bifurcation value in the process of obtaining the bifurcation of phase portraits, where \( g \) is a certain integration constant. In particular, we obtain both stationary and periodic cusp solutions of the Novikov equation.

Key words: Novikov equation, peakon, cuspon, bifurcation.

1. INTRODUCTION

In recent years there has been a growing interest in certain integrable systems with non-smooth solitons such as peakons and cuspons since the study of the remarkable Camassa-Holm (CH) equation with peakon solutions [1–9]. In contrast to the integrable modified Korteweg-de Vries (KdV) equation with a cubic nonlinearity, the nonlinearity in the CH equation is quadratic. One may ask the question: are there integrable modified CH equations with cubic nonlinearity? The answer is positive. In 1996, an integrable CH-type equation with cubic nonlinearity

\[
(u - u_{xx})_t + [(u^2 - u_x^2)(u - u_{xx})]_x = 0,
\]

was introduced by Olver and Rosenau [10] by applying the general method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified KdV equation. Later, it was obtained again by Qiao [11] from the two-dimensional Euler equation. It was shown in [11] that Eq. (1) admits Lax pair and bi-Hamiltonian structures and possesses new cusped solitons, alias cuspons, instead of regular peakons \( ce^{-|x-ct|} \) with speed \( c \).

Very recently, Novikov [12] found a new integrable equation (we call it the Novikov equation in this paper)

\[
u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx},
\]

which was discovered in a symmetry classification of nonlocal partial differential equations (PDEs) with cubic nonlinearity. By using the perturbative symmetry approach, Novikov was able to isolate Novikov equation and find a few symmetries, and he subsequently found a scalar Lax pair for it, then proved that the equation is integrable. Hone and Wang [13] gave a matrix Lax pair for the Novikov equation, and showed how it was related by a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy. An infinite sequence of conserved quantities are found in [13], as well as a bi-Hamiltonian structure. Also, the Cauchy problem, well-posedness and blow up phenomena to the Novikov equation have been studied extensively [14–16].

It is well known that traveling wave solution is an important type of solution for PDEs. One interesting property of the Novikov equation (2) is that it admits peakon solutions [13], i.e. solutions of the form

\[
u(x, t) = \frac{1}{4} \left( u_{max} - u_{min} \right) \operatorname{sech}^2 \left( \frac{x - ct}{\sqrt{2}} \right).
\]
\[ u(x, t) = \pm \sqrt{c} e^{-|x-ct|}, \quad c > 0. \]  

Our goal in the present paper is to seek new non-smooth traveling wave solutions by using the qualitative analysis methods of dynamical systems.

More precisely, we look for the traveling wave solutions of (2) through a generic setting 
\[ u(x, t) = \varphi(x - ct), \quad \text{where } c \text{ is the wave speed.} \]

Let \( \xi = x - ct \), then \( u(x, t) = \varphi(\xi) \). Substituting it into Eq. (2) yields

\[-c\varphi' + c\varphi'' + 4\varphi^2\varphi' = 3\varphi\varphi'\varphi'' + \varphi^2\varphi'''.\]  

Multiplying both sides of (4) by \( \varphi \) and integrating the equation once we get

\[-c\varphi^2/2 + \varphi^4 + c\varphi\varphi'' - c(\varphi')^2/2 = \varphi^3\varphi'' - g,\]  

where \( g \) is an integration constant. Let \( y = \varphi' \), then we get a planar integrable system

\[ \frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{2\varphi^4 - c\varphi^2 - cy^2 + g}{2\varphi(\varphi^2 - c)} \]  

with the following first integral:

\[ H(\varphi, y) = \begin{cases} -\sqrt{|\varphi^2 - c|}(\varphi^2 - y^2 + g/c), & \text{when } c > 0, \\ -\varphi^2 + y^2 + \frac{g}{2\varphi^2}, & \text{when } c = 0. \end{cases} \]  

We notice that the right-hand side of the second equation in (6) is discontinuous when \( \varphi = 0 \) or \( \varphi = \pm \sqrt{c} \quad (c > 0) \). In other words, the function \( \varphi'^2 \) is not well defined on such straight lines of the phase plane \((\varphi, y)\). It implies that a smooth traveling wave equation may have non-smooth traveling wave solutions.

During the past decades, many works have been focused on discussing the reasons causing the appearance of non-smooth traveling wave solutions under the condition that there exists only one singular straight line in the corresponding phase plane [2–4, 8]. Here, we are interested in a study of the non-smooth traveling wave solutions when there exist two or more singular straight lines in the ordinary differential equation (6). Hence, we assume \( c \geq 0 \) throughout this work. We point out that:

(i) For \( c > 0 \) and \( g = 0 \), two peakon solutions coexist in the Novikov equation (2).

(ii) For \( -c^2 < g < 0 \), two periodic cuspon solutions coexist in the Novikov equation (2).

(iii) For \( c = 0 \), two stationary cuspon solutions coexist in the Novikov equation (2).

2. BIFURCATION CONDITIONS AND PHASE PORTRAITS

In this section, we will study the bifurcation of phase portraits of system (6) in its parameter space.

**Case 1**: \( c > 0 \). In this case, system (6) has three singular straight lines \( \varphi = 0 \), \( \varphi = \sqrt{c} \), and \( \varphi = -\sqrt{c} \). To avoid these lines temporarily, let \( d\xi = 2\varphi(\varphi^2 - c)d\tau \), then (6) is changed to a regular system

\[ \frac{d\varphi}{d\tau} = 2\varphi y(\varphi^2 - c), \quad \frac{dy}{d\tau} = 2\varphi^4 - c\varphi^2 - cy^2 + g. \]  

Since the first integral of system (6) is the same as system (8), system (6) should have the same topological phase portraits as system (8) except for the above three straight lines. Therefore, we should be able to obtain the topological phase portraits of (6) from those of (8).
Fig. 1 – The phase portrait bifurcation of system (8) when c > 0:
(a) $c^2 + g < 0$;  
(b) $c^2 + g = 0$; 
(c) $0 < -g < c^2$; 
(d) $g = 0$; 
(e) $0 < 8g < c^2$; 
(f) $c^2 - 8g = 0$; 
(g) $c^2 - 8g < 0$.
Now we consider the equilibrium points of system (8). We write that

\[ E_{1,2}(\pm \sqrt{c+\sqrt{c^2-8g}/2}, 0), \quad E_{3,4}(\pm \sqrt{c-\sqrt{c^2-8g}/2}, 0), \]

\[ B_{1,2}(\sqrt{c}, \pm \sqrt{c^2+g}/c), \quad B_{3,4}(-\sqrt{c}, \pm \sqrt{c^2+g}/c), \]

\[ A_{1,2}(0, \pm \sqrt{g/c}). \]

It is easy to see that the system (8) has two equilibrium points \( E_{1,2} \) on the \( \phi \)-axis when \( g < 0 \). For \( 0 < g < c^2/8 \), there exist four equilibrium points \( E_{1,2,3,4} \) on the \( \phi \)-axis. For \( g > 0 \), the system (8) has two equilibrium points \( A_{1,2} \) on the straight line \( \phi = 0 \). For \( g > -c^2 \), the system (8) has four equilibrium points \( B_{1,2} \) and \( B_{3,4} \) on the straight lines \( \phi = \sqrt{c} \) and \( \phi = -\sqrt{c} \), respectively. For the Hamiltonian \( H \) defined by (7), we define \( h_1 = H(E_{1,2}) \).

We see from the above discussion that in the \((g, c)\)-parameter half-plane, there exist three bifurcation curves:

\[ L_1: c^2-8g=0; \quad L_2: g=0; \quad L_3: c^2+g=0, \]

which partition the \((g, c)\)-parameter half-plane into four regions. According to the qualitative theory of dynamical systems, we have seven different phase portraits of (8) that are shown in Figs. 1(a–g).

**Case II:** \( c = 0 \). In this case, the system (6) has the same phase orbits as the following system:
\[ \frac{d\phi}{d\tau} = 2\phi^3 y, \quad \frac{dy}{d\tau} = 2\phi^4 + g, \]  
\hspace{1cm} (10) 

except for the straight line \( \phi = 0 \), where \( d\xi = 2\phi^3 d\tau \) for \( \phi \neq 0 \). It is obvious that the system (10) has only one equilibrium point \( O(0, 0) \) when \( g = 0 \) and two equilibrium points \( E_{1,2}(\pm\sqrt{-g/2}, 0) \) when \( g < 0 \).

By the qualitative analysis, we draw the bifurcations of phase portraits of (10) as shown in Fig. 2.

3. MAIN RESULTS AND THE THEORETICAL DERIVATIONS OF MAIN RESULTS

In this section, we will state our main results in Theorem 1.

**Theorem 1.** (i) When \( c > 0 \) and \( g = 0 \), for the same wave speed, two peakons coexist in Eq. (2). The peakons are expressed as

\[ u_1(x,t) = \sqrt{c}e^{-|x-ct|} \]  
\hspace{1cm} (11) 

and

\[ u_2(x,t) = -\sqrt{c}e^{-|x-ct|}. \]  
\hspace{1cm} (12) 

(ii) When \( c > 0 \) and \( 0 < -g < c^2 \), Eq. (2) has two periodic cuspons

\[ u_3(x,t) = \left(\alpha e^{-|x-ct-2nT|} - \beta e^{|x-ct-2nT|}\right) \]  
\hspace{1cm} (13) 

and

\[ u_4(x,t) = -\left(\alpha e^{-|x-ct-2nT|} - \beta e^{|x-ct-2nT|}\right), \]  
\hspace{1cm} (14) 

where \( n = 0, \pm 1, \pm 2, \ldots, x-ct \in [(2n-1)T,(2n+1)T], \) and

\[ \alpha = \frac{c + \sqrt{c^2 + g}}{2\sqrt{c}}, \]  
\hspace{1cm} (15) 

\[ \beta = \frac{g}{2\sqrt{c}(c + \sqrt{c^2 + g})}, \]  
\hspace{1cm} (16) 

\[ T = \ln \left(\frac{c + \sqrt{c^2 + g}}{\sqrt{-g}}\right). \]  
\hspace{1cm} (17) 

Further, the periodic cuspons \( u_3(x,t) \) and \( u_4(x,t) \) converge to the peakons (11) and (12), respectively, when \( g \) tends to the bifurcation parameter value 0.

(iii) When \( c = 0 \) and \( g < 0 \), the Novikov equation (2) has two stationary cuspons

\[ u_5(x,t) = \gamma \sqrt{1 - e^{-2|x-ct|}} \]  
\hspace{1cm} (18) 

and

\[ u_6(x,t) = -\gamma \sqrt{1 - e^{-2|x-ct|}}, \]  
\hspace{1cm} (19) 

where \( \gamma = \sqrt[4]{-g/2} \).
Fig. 3 – The wave profiles: a) peakon, $c = 2$, $g = 0$; b) valleyon, $c = 2$, $g = 0$; c) periodic peak cuspon, $c = 2$, $g = -\frac{1}{3}$; d) periodic valley cuspon, $c = 2$, $g = -\frac{1}{3}$; e) stationary valley cuspon, $c = 0$, $g = -4$; f) stationary peak cuspon, $c = 0$, $g = -4$. 
Proof. (i) When $c > 0$ and $g = 0$, from Fig. 1e, we see that there are two triangle orbits. By Eq. (7), we get the expression of the triangle orbits as follows:

$$y = \pm \varphi \quad \text{for} \quad 0 \leq |\varphi| < \sqrt{c}, \quad (20)$$

and

$$\varphi = \pm \sqrt{c} \quad \text{for} \quad |y| \leq \sqrt{c^2 + g/c}. \quad (21)$$

Substituting (20) into the first equation of (6) and integrating along the triangle orbits, we have

$$\int_{\xi}^{\varphi} \frac{d\varphi}{\varphi} = -\int_{\xi}^{0} \text{sgn}(\xi) d\xi \quad \text{for} \quad 0 \leq \varphi \leq \sqrt{c}, \quad (22)$$

and

$$\int_{\xi}^{\varphi} \frac{d\varphi}{\varphi} = -\int_{0}^{\varphi} \text{sgn}(\xi) d\xi \quad \text{for} \quad -\sqrt{c} \leq \varphi \leq 0. \quad (23)$$

From Eqs. (22) and (23) we immediately obtain the peakon solutions (11) and (12) (Figs. 3a–b).

(ii) From Fig. 1d it is seen that given $c > 0$, $0 < -g < c^2$ and $h \in (h_1, 0)$, the system (8) has two families of periodic orbits. Note that when $h$ tends to 0, the periodic orbits lose their smoothness and become non-smooth periodic cusped orbits. Let $\Gamma_{1,2}$ be the two limiting curves of the periodic orbits as $h \to 0$. By Eq. (7) $\Gamma_{1,2}$ consist of

$$y = \pm \sqrt{\varphi^2 + \frac{g}{c}} \quad \text{for} \quad \sqrt{-\frac{g}{c}} \leq |\varphi| < \sqrt{c}, \quad (24)$$

and

$$\varphi = \pm \sqrt{c} \quad \text{for} \quad |y| \leq \sqrt{\frac{g + c^2}{c}}. \quad (25)$$

Substituting (24) into the first equation of (6) and integrating $\Gamma_1$ and $\Gamma_2$, respectively, we have

$$\int_{\xi}^{\varphi} \frac{d\varphi}{\sqrt{\varphi^2 + \frac{g}{c}}} = -\int_{\xi}^{0} \text{sgn}(\xi) d\xi \quad \text{for} \quad \varphi \in \left[\sqrt{-\frac{g}{c}}, \sqrt{c}\right], \quad (26)$$

and

$$\int_{\xi}^{\varphi} \frac{d\varphi}{\sqrt{\varphi^2 + \frac{g}{c}}} = \int_{\xi}^{\varphi} \text{sgn}(\xi) d\xi \quad \text{for} \quad \varphi \in \left[-\sqrt{c}, -\sqrt{-\frac{g}{c}}\right], \quad (27)$$

which immediately yield the periodic cuspons (13) and (14) (Figs. 3e–d).

(iii) When $c = 0$ and $g < 0$, corresponding to the stable and unstable manifolds of (8), to the saddle points $E_{1,2}(\pm \gamma, 0)$, and approaching the straight line $\varphi = 0$ defined by $H(\varphi, y) = h_1$ (Fig. 2b), there exist two stationary cuspon solutions for Eq. (2). We see from (7) that

$$y^2 = \frac{2\varphi^4 + 2h_1\varphi^2 - g}{2\varphi^2} - \frac{(\varphi^2 - \gamma^2)^2}{\varphi^2} \quad \text{for} \quad 0 \leq |\varphi| < \gamma. \quad (28)$$

Thus, by using the first equation of (6) to do the integration, we get the explicit representations of two stationary cuspon solutions that are given in (18) and (19) (Figs. 3e–f).
4. CONCLUSIONS

In this paper we have studied in detail the bifurcation of peakon and cuspons of the integrable Novikov equation, by applying the bifurcation theory of dynamical systems to this nonlinear partial differential equation. We have found that either two peakons or two cuspons coexist for the same value of the wave speed. Both stationary and periodic cuspon solutions of the integrable Novikov equation are also reported.

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