Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

$\mathcal{H}_\infty$ Bounded Resilient state-feedback design for linear continuous-time systems - A robust control approach

C. Briat * J.J. Martinez **

* *ACCESS Linnaeus Centre, Division of Optimization and Systems Theory, Royal Institute of Technology (KTH), 100 44 Stockholm, Sweden. (e-mail: cbriat@math.kth.se, corentin@briat.info; url: http://www.briat.info).*
** *GIPSA-Lab, Domaine Universitaire - BP46, 38402 Saint Martin d'Hères - Cedex, FRANCE. (e-mail: john-jairo.martinez-molina@gipsa-lab.grenoble-inp.fr; url: http://www.gipsa-lab.inpg.fr/page_pro.php?vid=483).*

Abstract: The quadratic stabilization of LTI systems by bounded resilient state-feedback controllers is addressed. The controllers are guaranteed to be non-fragile with respect to either uniform or nonuniform maximal implementation errors. The design of resilient controller relies on an LMI problem while boundedness of the coefficients (a NP-Hard problem) can be guaranteed by solving a nonconvex problem involving a BMI. The latter problem is solved using proposed locally convergent iterative LMI algorithms. Finally, an example is considered in order to illustrate the effectiveness of the approach.

Keywords: Fragility, Bounded Controllers, Linear Systems, Robust Control, $\mathcal{H}_\infty$ control, Relaxation

1. INTRODUCTION

One of the main objectives of control theory is to provide efficient tools for the design of controllers satisfying some specifications, e.g. disturbance rejection, output tracking, robustness and performance. Modern control techniques gave rise to new specifications in terms of norms, such as $\mathcal{H}_\infty$-norm [Zames, 1966] which can be characterized through convex optimization problems involving Linear Matrix Inequality (LMI) constraints. This class of optimization problems belongs to the particular class of convex optimization problems, one of the most studied and fruitful field in optimization. These problems are known to be solvable in polynomial time using specific algorithms, such as interior points algorithms [Nesterov and Nemirovskii, 1994] as implemented in SeDuMi [Sturm, 2001].

In the past recent years, control devices moved from analog to digital. Although digital devices are more flexible than analog ones, new problems arose: e.g. implementation of controllers, stability of numerical algorithms, signals quantization and sampling. The problem of non-fragility of controllers (also referred to as 'resilience' in the literature), first emphasized in the seminal paper [Keel and Bhattacharyya, 1997], has been addressed quite late compared to its importance. It addresses the analysis and design of controllers which are unsensitive to small variations of their coefficients, in the sense that a small variation of their coefficients does not destabilize the closed-loop system. In [Keel and Bhattacharyya, 1997], it is shown that, in some cases, a relative variation of the coefficients of the order of 10^{-7} is sufficient to destabilize the closed-loop system. Even if the analysis is very often done in continuous time, the same results also holds for discrete time systems for which it is important to note that 10^{-7} is generally far smaller than the precision of processors used in controllers. Since the work by Keel and Bhattacharyya [1997], several papers have been devoted to the design of resilient controllers; see for instance [Dorato, 1998, Haddad and Corrado, 1998, Yang and Wang, 2001, Peaucelle et al., 2004, Zheng and Wu, 1998] and the references therein. In this paper, we will consider the problem of designing non-fragile state-feedback controllers and express the problem as an optimization problem involving LMIs. Two types of maximal implementation errors will be considered. The first one, which we refer to 'uniform maximal error', is constant and independent of the amplitude of the coefficient of the controller. The second one, referred to as 'nonuniform maximal error' is not and depends on the amplitude of the coefficients. The latter being possibly highly non linear, an affine outer approximation will be considered instead. This will be detailed in Section 2.1.

A final constraint on controllers, which is seldom taken into account in the design, is the boundedness of the controller coefficients within prescribed bounds [1], in the design procedure. The main difficulty lies in the fact that the problem of designing such bounded controllers is a NP-hard problem [Blondel and Tsitsiklis, 1997] and hence, cannot be expressed exactly as a convex optimization problem, e.g. involving LMIs. It will be shown that the problem can be exactly expressed as an optimization

[1] imposed by the controller processor precision and by the stability of the implemented algorithms.

problem involving a Bilinear Matrix Inequality (BMI). However, the particular structure of this BMI prevents the use of iterative LMI (iLMI) algorithms. An equivalent formulation based on liftings in then proposed and allows for the use of iLMI algorithms to solve the problem.

The objective of the paper is the design of resilient bounded state-feedback controllers of the form:

$$u(t) = Kx(t) \tag{1}$$

for systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ew(t) \\ z(t) &= Cx(t) + Du(t) + Fw(t) \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^q$, $z \in \mathbb{R}^r$ and $K \in \mathbb{R}^{m\times n}$ are the system state, the control input, the exogenous input, the controlled output and the controller gain respectively.

The paper is structured as follows. Section 2 provides necessary background consisting of definitions and preliminary results. In Section 3, the main results of the paper are detailed, namely the design of resilient controllers and norm-bounded controllers as well as algorithms solving the problem. Finally, in Section 4, the efficiency of the approach is illustrated through an example.

The notation is standard. For a matrix A, $||A||_2$ is the induced 2-norm, $\star$ denotes symmetric elements in symmetric matrices; $\otimes$ is the Kronecker product; $\mathbb{1}_{n_1\times n_2}$ denotes a $n_1 \times n_2$ matrix with entries equal to 1; for a square matrix A, $A^{\mathcal{S}}$ stands for the sum $A + A^T$; for a square integrable signal $v \in \mathcal{L}_2$ over $[0, +\infty)$, its $\mathcal{L}_2$-norm is defined as $||v||_{\mathcal{L}_2} = (\int_0^{+\infty} v(s)^*v(s)ds)^{1/2}$.

2. DEFINITIONS AND PRELIMINARY RESULTS

The concepts of non-fragility and boundedness of controllers are recalled here. A result from [Briat et al., 2008], [Briat, 2008, Section 3.3] is also restated.

2.1 Non-fragility

The resilience can be rigorously defined using the discrete set $\mathcal{I} \subset \mathbb{R}$ containing all the implementable values for the controller coefficients. It is, for the moment, assumed that $\mathcal{I}$ is unbounded. The bounded case will be considered in Section 2.2. Numerical tools, such as optimization problem solvers, used in control generally return real solutions (up to the machine/solver precision), so we will adopt the technique consisting of rounding the controller solution to the nearest element in $\mathcal{I}^{m\times n}$.

Two types of implementation errors will be considered: the first one considers a uniform maximal implementation errors while the second one considers a nonuniform maximal error depending on the controller gain.

2.1.1. Uniform maximal error:
In such a case, the controller can be decomposed as follows:

$$K_i = K_c + \delta K \tag{3}$$

where K_i is the implemented controller with coefficients in $\mathcal{I}$, K_c is the computed controller gain and δK is the implementation error. Let us introduce the matrices U and V such that

$$\delta K = U\Delta V \tag{4}$$

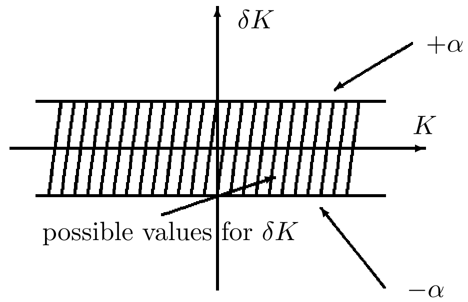

Fig. 1. Error Domain with respect to K - the constant worst case error.

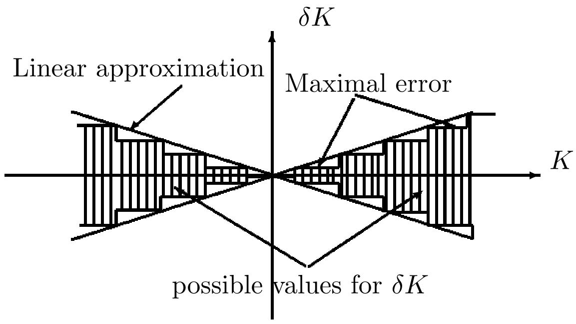

Fig. 2. Error Domain with respect to K - the linear worst-case error with $\Gamma = 0$.

with $\delta K = [\delta K_{k,\ell}]_{k,\ell}$, $\Delta = \text{diag}_{k=1}^m\{\text{diag}_{\ell=1}^n[\delta K_{k,\ell}]\}$ and

$$U = \operatorname*{diag}_{k=1}^{m} \mathbb{1}_{1\times n}, \quad V = \mathbb{1}_{m\times 1} \otimes I_n. \tag{5}$$

We have the following proposition:

Proposition 1. The following equality holds:

$$||\Delta||_2 \leq \alpha \tag{6}$$

where $\alpha > 0$ is the maximal implementation error (i.e. the maximal distance between two consecutive elements of $\mathcal{I}$).

2.1.2. Nonuniform maximal error
Another important coding scheme is the one where the distance between two consecutive points in $\mathcal{I}$ depends on their value in order to maintain a relative error small for small implemented values. This nonuniform coding strategy is typically nonlinear (e.g. logarithmic, stairs-like) since it depends on the computed value for the controller. However, it is possible to provide a 'conic' covering (see Fig. 2) of all the possible errors as:

$$\delta K(M) = \theta M + \Gamma, \qquad \Gamma = U\tilde{\Delta}V, \quad ||\tilde{\Delta}||_2 \leq \tilde{\alpha} \tag{7}$$

where M is the argument of the function $\delta K(\cdot)$, $\theta \in [-\theta_0, \theta_0]$ is the multiplicative scalar uncertainty with maximal value $\theta_0 > 0$, $\tilde{\Delta}$ is a diagonal uncertain matrix with coefficients in $[-\tilde{\alpha}, \tilde{\alpha}]$. The matrices U and V are defined in (5). Note that such a description can characterize a quantizer-like error function as depicted in Figure 2.

In such a case, the implemented controller takes the form:

$$K_i = (1 + \theta)K_c + U\tilde{\Delta}V \tag{8}$$

with $\theta \in [-\theta_0, \theta_0]$ and $||\tilde{\Delta}||_2 \leq \tilde{\alpha}$.

2.2 Bounded coefficients

Now, the idea is to add a compactness constraint on the set $\mathcal{I}$ so that the controller has bounded and controlled coefficients. The boundedness of the controllers will be characterized by their 2-norm. We refine first the implemented controller expression by adding a term K_0:

$$K_i = K_0 + K_c + \delta K \tag{9}$$

where $K_i \in \mathcal{I}^{m\times n}$ is the implemented controller, K_0 is a translation gain shifting the values of the computed controller gain K_c and δK is the implementation error. Since the computed controller gain K_c will be restricted to lie within a ball centered around 0 (since the 2-norm is the considered measure of boundedness), it is thus not possible to design a controller with asymmetric bounds. The term K_0 is here to allow for the computation of such controllers.

In what follows, we will consider that the matrix gain $K_c + \delta K$ belongs to the set $\mathcal{K}_\beta := \mathcal{I}^{m\times n} \cap [-\beta,\beta]^{m\times n}$ for a given $\beta > 0$. In such a case, we have the following proposition:

Proposition 2. If $K_c + \delta K \in \mathcal{K}_\beta$, then

$$||K_c + \delta K||_2 \leq \beta\sqrt{mn}. \tag{10}$$

Proof: The proof is omitted for brevity. ■

It is important to note that $K_c+\delta K \in \mathcal{K}_\beta$ implies (10) but the converse is not true in general. Therefore, in order to provide an LMI condition that guarantees $K_c + \delta K \in \mathcal{K}_\beta$, we shall consider the more general inequality:

$$||K_c + \delta K||_2 \leq s\beta\sqrt{mn} \tag{11}$$

where $s \in [(mn)^{-1/2}, 1]$ is a tuning parameter. If $s = (mn)^{-1/2}$ then the feasibility of (11) implies $K_c+\delta K \in \mathcal{K}_\beta$, in the sense that the nm-dimensional ball that includes the controller coefficients is contained in the hypercube $[-\beta,\beta]^{nm}$. On the other hand, when $s = 1$, the ball is the smallest ball that contains the hypercube $[-\beta,\beta]^{nm}$, and thus values outside $[-\beta,\beta]$ may be obtained. This motivates the introduction of the scaling term s allowing for a tradeoff between a 'strong' and a 'weak' boundedness obtained for $s = (mn)^{-1/2}$ and $s = 1$ respectively.

2.3 Simplifying nonconvex rational matrix inequalities

This section aims at providing a brief description of a theorem allowing to transform rational matrix inequalities with concave nonlinearities of the form:

$$\mathcal{M}(x) - \mathcal{P}(x)^T\mathcal{Q}(x)^{-1}\mathcal{P}(x) \prec 0 \tag{12}$$

with $\mathcal{M}(x) = \mathcal{M}(x)^T$, $\mathcal{Q}(x) = \mathcal{Q}(x)^T \succ 0$ and $\mathcal{P}(x)$ of appropriate dimensions depending on a real variable $x \in \mathcal{X} \subset \mathbb{R}^k$, $k \in \mathbb{N}$. In [Briat et al., 2008] and [Briat, 2008, Section 3.3], the following lemma is proved:

Lemma 3. The following statements are equivalent:

(1) There exists $x \in \mathcal{X}$ such that the matrix inequality:
$$\mathcal{M}(x) - \mathcal{P}(x)^T\mathcal{Q}(x)^{-1}\mathcal{P}(x) \prec 0 \tag{13}$$
holds.

(2) There exist $x \in \mathcal{X}$ and a matrix Λ of appropriate dimensions such that the matrix inequality:
$$\begin{bmatrix} \mathcal{M}(x) + [\Lambda^T\mathcal{P}(x)]^{\mathcal{S}} & \Lambda^T\mathcal{Q}(x) \\ \star & -\mathcal{Q}(x) \end{bmatrix} \prec 0 \tag{14}$$
holds.

Moreover, we have equality between the matrix expressions (modulo a Schur complement) for $\Lambda^* := -\mathcal{Q}(x)^{-1}\mathcal{P}(x)$.

Assuming the matrices $\mathcal{M}(x), \mathcal{P}(x)$ and $\mathcal{Q}(x)$ depend affinely on x, this result reformulates a rational nonlinear feasibility problem as a bilinear feasibility problem through the introduction of a lifting (or 'slack-variable'). Although the problem remains non-convex, the latter inequality is far simpler to solve since it allows for the use of iterative LMI algorithms, solving for x and for Λ alternatively. Such algorithms will be proposed in Section 3.3. An example of application of such a method can be found in [Briat et al., 2008] and [Briat, 2008, Section 5.1.3.2]. More complicated algorithms can also be used, such as techniques for polynomial optimization using, for instance, moments theory [Lasserre, 2001].

3. MAIN RESULTS

This section is devoted to the design of non-fragile state-feedback controllers with bounded coefficients. It is assumed that fundamentals on robust stability and stabilization are known; see [Scherer and Weiland, 2005] or [Briat, 2008, Appendix D.].

3.1 Design of unconstrained resilient state-feedback

The objective of this section is the design of a resilient state-feedback controller $K_i = K_c + K_0 + \delta K$, with no boundedness constraint such that the following closed-loop system:

$$\begin{aligned} \dot{x}(t) &= (A + BK_i)x(t) + Ew(t) \\ z(t) &= (C + DK_i)x(t) + Fw(t) \\ K_i &= K_0 + K_c + \delta K. \end{aligned} \tag{15}$$

is asymptotically stable and has minimal $\mathcal{L}_2$-gain of the channel $w \to z$, i.e. $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$, where $\gamma > 0$ is minimal.

To this aim, a robust control approach is considered and a controller will be designed such that for any implemented controller gain K_i inside the set

$$\{K + \delta K \in \mathbb{R}^{m\times n} : \ ||\delta K||_2 \leq \alpha, \ K = K_0 + K_c\}$$

the closed-loop system is asymptotically stable. That is, after implementation, the stability is ensured since the implemented controller will lie within the above set.

First, the case of uniform maximal implementation error is considered, then the nonuniform case will be addressed. The following theorem is obtained in which K_0 is a chosen and known matrix [2] :

Theorem 4. There exists an unconstrained resilient state-feedback of the form (3) with $\delta K = U\Delta V$ as defined in (4) and (5) which quadratically stabilizes system (2) if there exist a matrix $X = X^T \succ 0$, a diagonal matrix $Q \succ 0$, a matrix $Y \in \mathbb{R}^{m\times n}$ and a scalar $\gamma > 0$ such that the LMI

$$\begin{bmatrix} \mathcal{M}_{11} & E & XV^T & \mathcal{M}_{14} \\ \star & -\gamma I_q & 0 & F^T \\ \star & \star & -Q & 0 \\ \star & \star & \star & \mathcal{M}_{44} \end{bmatrix} \prec 0 \tag{16}$$

holds with

$$\begin{aligned} \mathcal{M}_{11} &= [AX + BK_0X + BY]^{\mathcal{S}} + \alpha^2 BUQU^TB^T \\ \mathcal{M}_{14} &= [CX + DK_0X + DY + \alpha^2 DUQU^TB^T]^T \\ \mathcal{M}_{44} &= -\gamma I_r + \alpha^2 DUQU^TD^T. \end{aligned}$$

In such a case, K_c is given by $K_c = YX^{-1}$ and the closed-loop system (15) satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$.

[2] can be set to 0 w.l.o.g. in the unconstrained case

Proof: Omitted for brevity. ■

Remark 3.1. It is also important to mention that when $\alpha \to 0$ then the result tends to the classical bounded real lemma. Indeed, in such a case, the LMI (16) becomes (with $K_0 = 0$ w.l.o.g.):

$$\begin{bmatrix} [AX+BY]^{\mathcal{S}} & E & XV^T & [CX+CY]^T \\ \star & -\gamma I & 0 & F^T \\ \star & \star & -Q & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} \prec 0$$

Hence, for a sufficiently large $Q = Q^T \succ 0$, this LMI is equivalent to the classical bounded real lemma.

The following theorem addresses the case when the implementation error is of the form (7):

Theorem 5. There exists an unconstrained resilient state-feedback of the form (3) with $\delta K = \theta(K_0 + K_c) + U\tilde{\Delta}V$ as defined in (7) which quadratically stabilizes system (2) if and only if there exist matrices $X = X^T, R = R^T \succ 0$, a diagonal matrix $S \succ 0$, a matrix $Y \in \mathbb{R}^{m\times n}$ and a scalar $\gamma > 0$ such that the LMI

$$\begin{bmatrix} \mathcal{M}_{11} & E & XV^T & \mathcal{M}_{14} & \mathcal{M}_{15} \\ \star & -\gamma I_q & 0 & F^T & 0 \\ \star & \star & -S & 0 & \mathcal{M}_{35} \\ \star & \star & \star & \mathcal{M}_{44} & 0 \\ \star & \star & \star & \star & -\theta_0^{-2}R \end{bmatrix} \prec 0 \qquad (17)$$

holds with

$$\begin{aligned} \mathcal{M}_{11} &= [AX + B(K_0X+Y)]^{\mathcal{S}} + R + \alpha^2 BUSU^TB^T \\ \mathcal{M}_{14} &= [CX + D(K_0X+Y)]^T + \alpha^2 BUSU^TD^T \\ \mathcal{M}_{15} &= B(K_0X+Y) \\ \mathcal{M}_{35} &= D(K_0X+Y) \\ \mathcal{M}_{44} &= -\gamma I_r + \alpha^2 DUSU^TD^T. \end{aligned}$$

In this case, K_c is given by $K_c = YX^{-1}$ and the closed-loop system (15) satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$.

Proof: Omitted for brevity. ■

Two results considering both types of maximal implementation errors have been obtained in terms of LMIs. The respectively obtained controllers are thus guaranteed to be resilient to uniform and nonuniform implementation errors.

3.2 Design of Non-fragile Bounded Controllers

This section aims at providing sufficient conditions to the existence of bounded controllers. As stated in Section 2.2, the synthesis of bounded controllers is an NP-Hard problem. It is thus not expected to find a nonconservative convex solution to this problem. Indeed, the use of the matrix 2-norm leads to the determination of a point (the coefficients of the controller) inside a nm-dimensional ball centered around 0. The primal formulation takes the form of BMIs which are difficult to solve due to their structure. It will be shown that using Lemma 3, it is possible to derive simpler BMIs, solvable using locally convergent iterative LMI algorithms.

The goal of the section is the development of additional conditions that can be 'plugged' to any other LMI-based design tools, like those of Section 3.1. This makes the results of this section very flexible and widely usable.

The first result concerns the design of bounded resilient controllers in the uniform maximal implementation error case.

Theorem 6. There exists a bounded resilient feedback controller as defined in (4) and (5) which quadratically stabilizes system (2) if there exist a matrix $X = X^T \succ 0$, diagonal matrices $Q = Q^T \succ 0, H = H^T \succ 0$, matrices $Y \in \mathbb{R}^{m\times n}, N \in \mathbb{R}^{n\times n}$ and a scalar $\gamma > 0$ such that the matrix inequalities

$$\begin{bmatrix} \Pi_{11} & Y & 0 & 0 \\ \star & [N^TX]^{\mathcal{S}} & V^T & N^T \\ \star & \star & -H & 0 \\ \star & \star & \star & -I_n \end{bmatrix} \preceq 0 \qquad (18)$$

and

$$\begin{bmatrix} \mathcal{M}_{11} & E & XV^T & \mathcal{M}_{14} \\ \star & -\gamma I_q & 0 & F^T \\ \star & \star & -Q & 0 \\ \star & \star & \star & \mathcal{M}_{44} \end{bmatrix} \prec 0 \qquad (19)$$

hold where

$$\begin{aligned} \Pi_{11} &= -s^2 mn\beta^2 I_m + \alpha^2 UHU^T \\ \mathcal{M}_{11} &= [AX + BK_0X + BY]^{\mathcal{S}} + \alpha^2 BUQU^TB^T \\ \mathcal{M}_{14}^T &= CX + DY + DK_0X + \alpha^2 DUQU^TB^T \\ \mathcal{M}_{44} &= -\gamma I_r + \alpha^2 DUQU^TD^T. \end{aligned}$$

In this case, K_c is obtained using $K_c = YX^{-1}$ and the closed-loop system (15) satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$.

Proof: Omitted for brevity. ■

The second part of this section is about the design of resilient bounded controllers with respect to a nonuniform maximal implementation error. We have the following theorem:

Theorem 7. There exists a bounded resilient feedback controller as defined in (7) which quadratically stabilizes system (2) if there exist matrices $X = X^T, R = R^T \succ 0, T = T^T \succ 0$, diagonal matrices $L, S \succ 0$ and full matrices Y, N of appropriate dimensions such that the following matrix inequalities

$$\begin{bmatrix} \mathcal{M}_{11} & E & XV^T & \mathcal{M}_{14} & \mathcal{M}_{15} \\ \star & -\gamma I_q & 0 & F^T & 0 \\ \star & \star & -S & 0 & \mathcal{M}_{35} \\ \star & \star & \star & \mathcal{M}_{44} & 0 \\ \star & \star & \star & \star & -\theta_0^{-2}R \end{bmatrix} \prec 0 \qquad (20)$$

$$\begin{bmatrix} \Pi_{11} & Y & K_0X+Y & 0 & 0 \\ \star & \Pi_{22} & 0 & XV^T & N^T \\ \star & \star & -\theta_0^{-2}T & 0 & 0 \\ \star & \star & \star & -L & 0 \\ \star & \star & \star & \star & -I_n \end{bmatrix} \preceq 0 \qquad (21)$$

hold with

$$\begin{aligned} \mathcal{M}_{11} &= [AX + B(K_0X+Y)]^{\mathcal{S}} + R + \alpha^2 BUSU^TB^T \\ \mathcal{M}_{14} &= [CX + D(K_0X+Y)]^T + \alpha^2 BUSU^TD^T \\ \mathcal{M}_{15} &= B(K_0X+Y) \\ \mathcal{M}_{35} &= D(K_0X+Y) \\ \mathcal{M}_{44} &= -\gamma I_r + \alpha^2 DUSU^TD^T \\ \Pi_{11} &= -s^2 mn\beta^2 I_m + \alpha^2 ULU^T \\ \Pi_{22} &= T + N^TX + XN^T. \end{aligned}$$

In this case, K_c is obtained using $K_c = YX^{-1}$ and the closed-loop system satisfies $||z||_{\mathcal{L}_2} \leq \gamma||w||_{\mathcal{L}_2}$.

Proof: The proof is similar as the one of Theorem 6. ■

3.3 Algorithms

This section is devoted to the presentation of two algorithms which can be used to solve the conditions of Theorems 6 and 7. Indeed, since the conditions are bilinear, they are nonconvex, and hence 'hard' to solve. There exist many algorithms for solving such problems, we propose here simple algorithms based on the iteration of LMI problems. Indeed, if N is kept fixed then the problem is linear in X and vice-versa. Thus iterative algorithms can be employed. It is important to mention that such algorithms are not applicable to the initial problem obtained directly from the controller boundedness condition since the same matrix (i.e. X) appears both affinely and quadratically (X^2). By using Lemma 3, a more convenient equivalent formulation involving an additional slack variable (lifting) is provided. It is also important to say that the relaxed BMI problem is still easier to solve than the original BMI stabilization problem in P and K. Indeed, the eigenvalues of the matrix inequality are very sensitive to the chosen values for P and K since they are fundamental matrices in the control and stabilization problems. This is not the case with the slack variable N which has a lower impact of the feasibility of the problem and can be solved separately, allowing then P and K to be determined (by a convex optimization problem) in the same step of the algorithms.

The following 2-step algorithm based on two LMI problems is first proposed:

Algorithm 1. (1) Initialize $k = 0$ and $N_0 \in \mathbb{R}^{n\times n}$
(2) Minimize $\gamma > 0$ with respect to LMIs of Theorem 6 or 7 with fixed $N = N_k$ and store value $X_k \leftarrow X$.
(3) Minimize $\gamma > 0$ with respect to LMIs of Theorem 6 or 7 with fixed $X = X_k$ and store value $N_{k+1} \leftarrow N$.
(4) If stopping criterion is satisfied then STOP else $k \leftarrow k+1$

Proof: Omitted for brevity. ∎

Another algorithm can also be obtained by using the optimal value of the slack-variable provided in Lemma 3. In such a case, the above algorithm can be modified to the following algorithm implementing only one LMI problem:

Algorithm 2. (1) Initialize $k = 0$, set $N_k \leftarrow N_0$
(2) Minimize objective with respect to LMIs of Theorem 6 or 7 with fixed N_k and store value $X_k \leftarrow X$.
(3) $N_{k+1} \leftarrow -X_k$.
(4) If stopping criterion is satisfied then STOP else $k \leftarrow k+1$

Proof: Omitted for brevity. ∎

The latter algorithm is less complex from a computational point of view while the speed of convergence seems, in practice, to be identical. It is also important to mention that the Cone Complementary Algorithm [El-Ghaoui et al., 1997] can also be used to deal with the nonlinear term $-X^2$ following the idea of [Gao and Wang, 2003] in a somehow different context. However, the above algorithms apply easily when the matrices are matrix functions (i.e. when considering LPV/LTV systems). Such cases cannot be addressed using the Cone Complementary Algorithm.

A difficulty of these algorithms lies in the choice of the value of N_0, the initial point. An obvious fact is the negative definiteness of $N_0^T X + X^T N_0 + N_0^T N_0 \prec 0$ and since $X = X^T \succ 0$, then choosing N_0 to be Hurwitz seems to be a good choice. Moreover, it is also interesting to keep the positive quadratic term $N_0^T N_0$ small. It turns out that $N_0 = -\varepsilon I$, $\varepsilon \in (0, 1]$ seems to be a correct initialization point. Indeed, such initial points are Hurwitz and maintain the positive quadratic term small.

4. EXAMPLES

Some examples are provided here in order to illustrate the effectiveness of the approach. Let us consider the unstable 4th order system of the form (2) given by the matrices

$$A = \begin{bmatrix} 2 & 1 & 7 & 1 \\ 1 & 7 & 1 & 2 \\ 7 & 1 & 0 & 1 \\ 2 & 1 & 6 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

First, we aim at stabilizing the system using an unconstrained resilient controller with uniform maximal implementation error. For brevity, only the case of uniform maximal uncertainty will be considered. Assume that controllers with integer coefficients are sought, then the maximal uncertainty is set to $\alpha = 1/2$ which is the maximal distance between a real number and the nearest integer. When an exact controller is considered, the minimal $\mathcal{H}_\infty$ norm for the closed-loop system is $\gamma_{opt} = 3.4155$ while for the implemented resilient controller the $\mathcal{H}_\infty$ norm is $\gamma_{res} = 3.4368$. This corresponds to a performance reduction of 0.62% which is very few. After rounding, the resilient controller is given by:

$$K = \begin{bmatrix} -4148 & -65751 & -10023 & -20577 \\ -22161 & -352006 & -53613 & -110142 \end{bmatrix}.$$

It is clear that the controller cannot be implemented since the coefficients are too large. In Figure 3, the worst-case gain in plotted with respect to α where, as expected, the performance deteriorates when α grows.

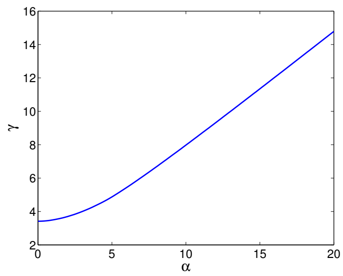

Fig. 3. Evolution of the worst-case $\mathcal{H}_\infty$-norm with respect to α.

Now, we aim at computing a bounded controller with integer coefficients belonging to $\mathcal{I} = [-\sigma, \sigma - 1] \cap \mathbb{Z}$. Let us consider, for instance, the case of signed integers coded over 8 bits, that is, we set $\sigma = 128$. To design such a controller, let $K_0 = -1/2 \cdot \mathbb{1}_{m\times n}$ (the center of the interval $\mathcal{I}$) and $\beta = \sigma - 1/2$ (the radius of the interval $\mathcal{I}$). We also choose $s = 1/\sqrt{mn}$ to ensure a strong boundedness of the coefficients. Using Algorithm 2 with $\varepsilon = -I_n$ and 10 iterations, the following controller is found [3]:

$$K_i = \begin{bmatrix} -20 & 19 & -34 & -35 \\ -14 & -95 & -23 & -39 \end{bmatrix}$$

ensuring a worst-case attenuation level $\gamma = 3.6671$. In such a case, the performance deterioration is about 7.37% which is quite few compared to the strong restriction of the controller coefficients. This illustrates that the proposed method can be useful to control the choice of the controller gain by the LMI solver.

When $\sigma = 32$, the following controller is obtained (still using Algorithm 2 with the same parameters):

$$K_i = \begin{bmatrix} -9 & 5 & -12 & -9 \\ -3 & -25 & -5 & -10 \end{bmatrix}$$

ensuring a performance index of $\gamma = 4.991$. Finally, when $\sigma = 16$, we obtain

$$K_i = \begin{bmatrix} -7 & 2 & -9 & -4 \\ -3 & -13 & 0 & -4 \end{bmatrix}$$

ensuring a performance index of $\gamma = 6.3144$. When $\sigma = 12$, the obtained controller is:

$$K_i = \begin{bmatrix} -7 & 1 & -7 & -4 \\ -2 & -10 & 0 & -3 \end{bmatrix}$$

and ensures a performance index of $\gamma = 7.7824$. However, the approach fails to find a controller for $\sigma < 12$.

In Figure 4, the $\mathcal{H}_\infty$ performance index γ is plotted with respect to the value of σ. The worst case performance is obtained by solving the iterative LMI problem, that is the maximal performance index for all controllers in the ball of radius 1/2 around K_c. Since one of them is K_i, then it is expected to obtain a better performance for the implemented controller, as seen in the Figure 4. We can also notice that the performance index is smaller when we tolerate larger coefficients for the controller.

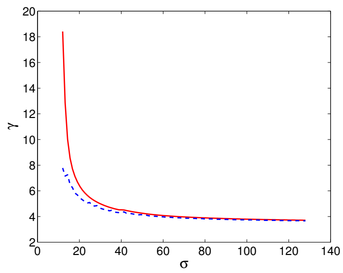

Fig. 4. Evolution of the $\mathcal{H}_\infty$-norm with respect to σ - Plain: worst case; Dashed: for the implemented controller.

ACKNOWLEDGMENTS

The authors thank the anonymous reviewers who helped to improve this work.

REFERENCES

V. Blondel and J.N. Tsitsiklis. NP-hardness of some linear control problems. *SIAM J. Control and Optimization*, 35(6):2118–2127, 1997.

[3] in 15.45 seconds on an Intel U7300 1.3Ghz processor with 4GB RAM

C. Briat. *Control and Observation of LPV Time-Delay Systems*. PhD thesis, Grenoble-INP, 2008. URL `http://www.briat.info/thesis/PhDThesis.pdf`.

C. Briat, O. Sename, and J-F. Lafay. Parameter dependent state-feedback control of LPV time delay systems with time varying delays using a projection approach. In *17th IFAC World Congress, Korea, Seoul*, 2008.

P. Dorato. Non-fragile controller design: an overview. In *American Control Conference*, pages 2829–2831, Philadelphia, Pennsylvania, USA, 1998.

L. El-Ghaoui, F. Oustry, and M. Ait Rami. A Cone Complementary linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control*, 42:1171–1176, 1997.

H. Gao and C. Wang. Comments and further results on 'a descriptor system approach to $\mathcal{H}_\infty$ control of linear time-delay systems'. *IEEE Transactions on Automatic Control*, 48(3):520–525, 2003.

W. M. Haddad and J.R. Corrado. Robust resilient dynamic controlers for systems with parametric uncertainty and controller gain variations. In *American Control Conference*, pages 2837–2841, Philadelphia, Pennsylvania, USA, 1998.

L. H. Keel and S. P. Bhattacharyya. Robust, fragile or optimal ? *IEEE Transactions on Automatic Control*, 42(8):1098–1105, 1997.

J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.*, 11(3):796–817, 2001.

Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithm in Convex Programming*. PA, SIAM, Philadelphia, 1994.

D. Peaucelle, D. Arzelier, and C. Farges. LMI results for resilient state-feedback with $\mathcal{H}_\infty$ performance. In *43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas*, 2004.

C. W. Scherer and S. Weiland. *Linear Matrix Inequalities in Control*. 2005.

J. F. Sturm. Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(12):625–653, 2001.

G.-H. Yang and J. L. Wang. Non-fragile $\mathcal{H}_\infty$ control for linear systems with multiplicative controller gain variations. *IEEE Transactions on Automatic Control*, 37:727–737, 2001.

G. Zames. On the input output stability of time-varying nonlinear feecback systems, part i : Conditions using concepts of loop gain, conicity and positivity. *IEEE Trans. on Aut. Control*, 11(2):228–238, 1966.

Q. Zheng and F. Wu. Resilient control design for polynomial nonlinear systems with controller gain variations. In *American Control Conference*, pages 2837–2841, Philadelphia, Pennsylvania, USA, 1998.