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Research Article

A Defect-Correction Mixed Finite Element Method for Stationary Conduction-Convection Problems

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A defect-correction mixed finite element method (MFEM) for solving the stationary conduction-convection problems in two-dimension is given. In this method, we solve the nonlinear equations with an added artificial viscosity term on a grid and correct this solution on the same grid using a linearized defect-correction technique. The stability is given and the error analysis in L^2 and H^1-norm of u, T and the L^2-norm of p are derived. The theory analysis shows that our method is stable and has a good precision. Some numerical results are also given, which show that the defect-correction MFEM is highly efficient for the stationary conduction-convection problems.

1. Introduction

In this paper, we consider the stationary conduction-convection problems in two dimension whose coupled equations governing viscous incompressible flow and heat transfer for the incompressible fluid are Boussinesq approximations to the stationary Navier-Stokes equations.

($\mathcal{P}$) Find $(u, p, T) \in X \times M \times W$ such that

$$\begin{gathered} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = \lambda j T, \quad x \in \Omega, \\ \operatorname{div} u = 0, \quad x \in \Omega, \\ -\Delta T + \lambda u \cdot \nabla T = 0, \quad x \in \Omega, \\ u = 0, \quad T = T_0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$. $u = (u_1(x), u_2(x))^T$ represents the velocity vector, $p(x)$ the pressure, $T(x)$ the temperature, $\lambda > 0$ the Grashoff number, $j = (0,1)^T$ the two-dimensional vector, and $\nu > 0$ the viscosity.

As we know the conduction-convection problem contains the velocity vector field, the pressure field and the temperature field, so finding the numerical solution of conduction-convection problems is very difficult. The conduction-convection problems is an important system of equations in atmospheric dynamics and dissipative nonlinear system of equations, so lots of works are devoted to this problem [1–6]. There are also some works devoted to the nonstationary conduction-convection problems [7–10]. In [8], Luo et al. gave an optimizing reduced PLSMFE for the nonstationary conduction-convection problems. They combined PLSMEF method with POD to deal with the problems. In [11], an analysis of conduction natural convection conjugate heat transfer in the gap between concentric cylinders under solar irradiation was studied. In [12], a Newton iterative mixed finite element method for the stationary conduction-convection problems was shown by Si et al. In [13], Si and He gave a coupled Newton iterative mixed finite element method for the stationary conduction-convection problems.

The defect-correction method is an iterative improvement technique for increasing the accuracy of a numerical solution without applying a grid refinement. Due to its good efficiency, there are many works devoted to this method, for example, [14–28]. In [18], a method making it possible to apply the idea of iterated defect correction to finite element methods was given. A method for solving the time-dependent Navier-Stokes equations, aiming at higher Reynolds' number, was presented in [23]. In [27], an accurate approximations for self-adjoint elliptic eigenvalues was presented. In [28], Stetter exposed the common structural principle of all these techniques and exhibit the principal modes of its implementation in a discretization context.

In this paper we present a defect-correction MFEM for the stationary conduction convection problems. In this method, we solve the nonlinear equations with an added artificial viscosity term on a finite element grid and correct this solution on the same grid using a linearized defect-correction technique. Actually, the defect-correction MFEM incorporates the artificial viscosity term as a stabilizing factor, making both the nonlinear system easier to resolve and the linearized system easier to precondition. The stability and error analysis of the coupled the defect-correction MFEM show that this method is stable and has a good precision. Some numerical experiments show that our analysis is proper and our method is effective. And it can be used for solving the convection-conduction problems with much small viscosity.

This paper is organized as follows. In Section 2, the functional settings and some assumptions are given. Section 3 is devoted to the defect-correction MFEM. Section 4 gives the stability analysis. Section 5 presents the error analysis. In Section 6, some numerical results and the numerical analysis to validate the effectiveness of the method are laid out.

2. Functional Setting for the Conduction Convection Problems

In this section, we aim to describe some of the notations and results which will be frequently used in this paper. The Sobolev spaces used in this context are standard [29]. For the mathematical setting of the conduction-convection problems and MFEM of conduction-convection problems (1.1), we introduce the Hilbert spaces

$$\begin{gathered} X = H_0^1(\Omega)^2, \qquad W = H^1(\Omega), \\ M = L_0^2(\Omega) \doteq \left\{ \varphi \in L^2(\Omega); \int_\Omega \varphi \, dx = 0 \right\}. \end{gathered} \tag{2.1}$$

$\Im_h$ is the uniformly regular family of triangulation of $\overline{\Omega}$, indexed by a parameter $h = \max_{K\in\Im_h}\{h_K; h_K = \operatorname{diam}(K)\}$. We introduce the finite element subspace $X_h \subset X$, $M_h \subset M$, $W_h \subset W$ as follows

$$\begin{aligned} X_h &= \left\{ v_h \in X \cap C^0\left(\overline{\Omega}\right)^2; v_h|_K \in P_\ell(K)^2, \forall K \in \Im_h \right\}, \\ M_h &= \left\{ q_h \in M \cap C^0\left(\overline{\Omega}\right); q_h|_K \in P_k(K), \forall K \in \Im_h \right\}, \\ W_h &= \left\{ \phi_h \in W \cap C^0\left(\overline{\Omega}\right); \phi_h|_K \in P_l(K), \forall K \in \Im_h \right\}, \end{aligned} \tag{2.2}$$

where $P_\ell(K)$ is the space of piecewise polynomials of degree ℓ on K, and $\ell \geqslant 1$, $k \geqslant 1$, $l \geqslant 1$ are three integers. $W_{0h} = W_h \cap H_0^1(\Omega)$, and (X_h, M_h) satisfies the discrete LBB condition

$$\sup_{v_h\in X_h} \frac{d(\varphi_h, v_h)}{\|\nabla v_h\|_0} \geqslant \beta \|\varphi_h\|_0, \quad \forall \varphi_h \in M_h, \tag{2.3}$$

where $d(\varphi, v) = (\varphi, \operatorname{div} v)$.

With the above notations, the Galerkin mixed variation and the mixed FEM problem for the conduction-convection problems $(\mathcal{P})$ are defined, respectively, as follows.

$(\mathcal{P}')$ Find $(u, p, T) \in X \times M \times W$ such that

$$\begin{gathered} \nu a(u, v) - d(p, v) + d(\varphi, u) + b(u, u, v) = \lambda(jT, v), \quad \forall v \in X,\ \varphi \in M, \\ \overline{a}(T, \psi) + \lambda \overline{b}(u, T, \psi) = 0, \quad \forall \psi \in W_0. \end{gathered} \tag{2.4}$$

$(\mathcal{P}'')$ Find $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$ such that

$$\begin{gathered} \nu a(u_h, v_h) - d(p_h, v_h) + d(\varphi_h, u_h) + b(u_h, u_h, v_h) = \lambda(jT_h, v_h), \quad \forall v_h \in X_h,\ \varphi_h \in M_h, \\ \overline{a}(T_h, \psi_h) + \lambda \overline{b}(u_h, T_h, \psi_h) = 0, \quad \forall \psi_h \in W_{0h}, \end{gathered} \tag{2.5}$$

where $a(u, v) = (\nabla u, \nabla v)$, $d(\varphi, v) = (\varphi, \operatorname{div} v)$, $\overline{a}(T, \psi) = (\nabla T, \nabla \psi)$, and

$$\begin{aligned} b(u, v, w) &= \frac{1}{2}\left[\int_\Omega \sum_{i,k=1}^{2} u_i \frac{\partial v_k}{\partial x_i} w_k dx - \sum_{i,k=1}^{2} u_i \frac{\partial w_k}{\partial x_i} v_k dx\right], \quad \forall u, v, w \in X, \\ \overline{b}(u, T, \psi) &= \frac{1}{2}\left[\int_\Omega \sum_{i=1}^{2} u_i \frac{\partial T}{\partial x_i} \psi\, dx - \sum_{i}^{2} u_i \frac{\partial \psi}{\partial x_i} T\, dx\right], \quad \forall u \in X,\ T, \psi \in W. \end{aligned} \tag{2.6}$$

The following assumptions and results are recalled (see [7, 29–31]).

(A_1) There exists a constant C_0 which only depends on Ω, such that

(i) $\|u\|_0 \le C_0\|\nabla u\|_0$, $\|u\|_{0,4} \le C_0\|\nabla u\|_0$, for all $u \in H_0^1(\Omega)^2$ (or $H_0^1(\Omega)$),

(ii) $\|u\|_{0,4} \le C_0\|u\|_1$, for all $u \in H^1(\Omega)^2$,

(iii) $\|u\|_{0,4} \le \sqrt{2}\|\nabla u\|_0^{1/2}\|u\|_0^{1/2}$, for all $u \in H_0^1(\Omega)^2$ (or $H_0^1(\Omega)$).

(A_2) Assuming $\partial\Omega \in C^{k,\alpha}$ ($k \geqslant 0$, $\alpha > 0$), then, for $T_0 \in C^{k,\alpha}(\partial\Omega)$, there exists an extension in $C_0^{k,\alpha}$ ($\mathbb{R}^2$) (denote T_0 also), such that

$$\|T_0\|_{k,q} \le \varepsilon, \qquad k \geqslant 0,\ 1 \le q \le \infty, \tag{2.7}$$

where ε is an arbitrary positive constant.

(A_3) $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$ have the following properties.

(i) For all $u \in X$, $v, w \in X$ (or $T, \varphi \in H_0^1(\Omega)$), there holds that

$$b(u,v,v) = 0, \qquad b(u,v,w) = -b(u,w,v), \tag{2.8}$$

$$\overline{b}(u,T,T) = 0, \qquad \overline{b}(u,T,\varphi) = -\overline{b}(u,\varphi,T). \tag{2.9}$$

(ii) For all $u \in X$, $v \in H^1(\Omega)^2$ (or $T \in H^1(\Omega)$), for all $w \in X$ (or $\varphi \in H_0^1(\Omega)$), there holds that

$$|b(u,v,w)| \le N\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0, \tag{2.10}$$

$$\left|\overline{b}(u,T,\varphi)\right| \le \overline{N}\|\nabla u\|_0\|\nabla T\|_0\|\nabla\varphi\|_0, \tag{2.11}$$

where

$$N = \frac{\sup_{u,v,w}|b(u,v,w)|}{(\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0)}, \qquad \overline{N} = \frac{\sup_{u,T,\varphi}\left|\overline{b}(u,T,\varphi)\right|}{(\|\nabla u\|_0\|\nabla T\|_0\|\nabla\varphi\|_0)}. \tag{2.12}$$

We recall the following existence, uniqueness and regularity result of ($\mathcal{P}'$) (see [7, Chapter 4]).

Theorem 2.1 (see [7]). *Under the assumption of (A_1)~(A_3), letting $A \equiv 2\nu^{-1}\lambda(3C_0+1)\|T_0\|_1$, $B \equiv 2\|\nabla T_0\|_0 + 2(C_0^2\lambda)^{-1}A$, there exist $0 < \delta_1, \delta_2 \le 1$ such that*

$$\nu^{-1}NA \le 1-\delta_1, \qquad \delta_1^{-1}\nu^{-1}C_0^2\lambda^2B\overline{N} \le 1-\delta_2. \tag{2.13}$$

Then, there exists a unique solution $(u,p,T) \in X \times M \times W$ for ($\mathcal{P}'$), and

$$\|\nabla u\|_0 \le A, \qquad \|\nabla T\|_0 \le B. \tag{2.14}$$

Some estimates of the trilinear form b are given in the following lemma and the proof can be found in [30, 32–34].

Lemma 2.2. *The trilinear form b satisfies the following estimate:*

$$|b(u_h, v_h, w)| + |b(v_h, u_h, w)| + |b(w, u_h, v_h)| \le C_0 \left|\log h\right|^{1/2} \|\nabla v_h\|_0 \|\nabla u_h\|_0 \|w\|_0, \tag{2.15}$$

for all $u_h, v_h \in V_h$, $w \in L^2(\Omega)^2$.

Lemma 2.3. *Suppose that (A_1)~(A_3) are valid and ε is a positive constant, such that*

$$\frac{32C_0^2\lambda^2\overline{N}\varepsilon}{3\nu} < 1, \qquad \|\nabla T_0\|_0 \le \frac{\varepsilon}{4}, \qquad \|T_0\|_0 \le \frac{C_0\varepsilon}{4}, \tag{2.16}$$

then $(\mathcal{P}'')$ has a unique solution $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$, such that $T|_{\partial\Omega} = T_0$ and

$$\|\nabla u_h\|_0 \le \frac{5C_0^2\lambda\varepsilon}{3\nu}, \qquad \|\nabla T_h\|_0 \le \varepsilon. \tag{2.17}$$

Proof. The proof of the existence and the uniqueness of the solution has been given by Luo [7]. Let $T_h = \omega_h + T_0$, $\varphi_h = \omega_h$ in (2.5), we can get

$$\overline{a}(\omega_h, \omega_h) = -\lambda\overline{b}(u_h, T_0, \omega_h) - \overline{a}(T_0, \omega_h). \tag{2.18}$$

Using (2.11) and (2.16), we deduce

$$\|\nabla \omega_h\|_0 \le \|\nabla T_0\|_0 + \lambda\overline{N}\varepsilon\|\nabla u_h\|_0. \tag{2.19}$$

Letting $v_h = u_h$, $\varphi_h = p_h$ in the first equation of (2.5), we get

$$\nu\|\nabla u_h\|_0^2 = \left|\lambda\left(jT_h, u_h\right)\right| \le \lambda C_0\|T_h\|_0\|\nabla u_h\|_0. \tag{2.20}$$

By (2.16), we can obtian

$$\begin{aligned}
\|\nabla u_h\|_0 &\le \nu^{-1}\lambda C_0\|T_h\|_0 \\
&\le \nu^{-1}\lambda C_0(\|\omega_h\|_0 + \|T_0\|_0) \\
&\le \nu^{-1}\lambda C_0^2\|\nabla\omega_h\|_0 + \nu^{-1}\lambda C_0\|T_0\|_0 \\
&\le \nu^{-1}\lambda C_0\|T_0\|_0 + \nu^{-1}\lambda C_0^2\|\nabla T_0\|_0 + \nu^{-1}\lambda^2 C_0^2\overline{N}\varepsilon\|\nabla u_h\|_0.
\end{aligned} \tag{2.21}$$

Using (2.16) again, we get

$$\|\nabla u_h\|_0 \le \frac{5C_0^2\lambda\varepsilon}{3\nu}. \tag{2.22}$$

By (2.19), we deduce

$$\begin{aligned}\|\nabla T_h\|_0 &\le \|\nabla \omega_h\|_0 + \|\nabla T_0\|_0 \\ &\le 2\|\nabla T_0\|_0 + \lambda\overline{N}\varepsilon\|\nabla u_h\|_0 \\ &\le 2\|\nabla T_0\|_0 + \frac{5C_0^2\lambda^2\overline{N}\varepsilon^2}{3\nu} \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned} \tag{2.23}$$

□

We introduce the Laplace operator

$$\mathcal{A}u = -\Delta u, \quad \forall u \in D(\mathcal{A}) = H^2(\Omega)^2 \cap X. \tag{2.24}$$

Lemma 2.4 (see [35, 36]). *For all $u, w \in X$, $v \in D(A)$ there holds that*

$$|b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)| \le C\|\mathcal{A}v\|_0\|w\|_0\|\nabla u\|_0. \tag{2.25}$$

3. The Defect-Correction Method

The aim of this section is to give a method for solving the nonlinear system (2.5) on a coarser mesh than one uses when employing the standard FEM; the coarse-mesh solution is corrected using the same grid in our method. The defect-correction method in which we consider incorporates an artificial viscosity parameter σh as a stabilizing factor in the solution algorithm. For a fixed grid parameter h the method requires the solution of one nonlinear system and a few linear correction steps. It is described in the following paragraphs. We consider the following problems which is identical to (2.5) except for an artificial viscosity term.

($\mathcal{P}^*$) Find $(u_h^0, p_h^0, T_h^0) \in X_h \times M_h \times W_h$ such that

$$\begin{aligned}&(\nu + \sigma h)a\left(u_h^0, v_h\right) - d\left(p_h^0, v_h\right) + d\left(\varphi_h, u_h^0\right) + b\left(u_h^0, u_h^0, v_h\right) = \lambda\left(jT_h^0, v_h\right), \\ &\qquad\qquad \forall v_h \in X_h,\ \varphi_h \in M_h, \\ &(1+\sigma h)\overline{a}\left(T_h^0, \psi_h\right) + \lambda\overline{b}\left(u_h^0, T_h^0, \psi_h\right) = 0, \quad \forall \psi_h \in W_{0h}.\end{aligned} \tag{3.1}$$

We define the residual or named defect $R(u_h^0, p_h^0, T_h^0)$, $Q(u_h^0, p_h^0, T_h^0)$ for the momentum systems as follows:

$$\begin{aligned}\left(R\left(u_h^0, p_h^0, T_h^0\right), v_h\right) &= \lambda\left(jT_h^0, v_h\right) - \nu a\left(u_h^0, v_h\right) + d\left(p_h^0, v_h\right) \\ &\quad - d\left(\varphi_h, u_h^0\right) - b\left(u_h^0, u_h^0, v_h\right), \\ \left(Q\left(u_h^0, p_h^0, T_h^0\right), \psi_h\right) &= -\overline{a}\left(T_h^0, \psi_h\right) - \lambda\overline{b}\left(u_h^0, T_h^0, \psi_h\right).\end{aligned} \tag{3.2}$$

Define the correction $(\varepsilon_h^0, \varrho_h^0, \tau_h^0)$ satisfying the following linear problem:

$$\begin{aligned}
&(\nu+\sigma h)a\left(\varepsilon_h^0, v_h\right) - d\left(\varrho_h^0, v_h\right) + d\left(\varphi_h, \varepsilon_h^0\right) + b\left(\varepsilon_h^0, u_h^0, v_h\right) + b\left(u_h^0, \varepsilon_h^0, v_h\right) \\
&\quad = \left(R\left(u_h^0, p_h^0, T_h^0\right), v_h\right), \quad \forall v_h \in X_h,\ \varphi_h \in M_h, \\
&(1+\sigma h)\overline{a}\left(\tau_h^0, \psi_h\right) + \lambda\overline{b}\left(u_h^0, \tau_h^0, \psi_h\right) + \lambda\overline{b}\left(\varepsilon_h^0, T_h^0, \psi_h\right) \\
&\quad = \left(Q\left(u_h^0, p_h^0, T_h^0\right), \psi_h\right), \quad \forall \psi_h \in W_{0h}.
\end{aligned} \tag{3.3}$$

Define $u_h^1 = u_h^0 + \varepsilon^0$, $p_h^1 = p_h^0 + \varrho_h^0$, $T_h^1 = T_h^0 + \tau_h^0$, which are hoped to be better solutions of the problems. In order to obtain the equations for (u_h^1, p_h^1, T_h^1), we use the residual equation (3.2) to rewrite the linear problems (3.3); we obtain

$$\left(p^\dagger\right) \quad \begin{cases}
(\nu+\sigma h)a(u_h^1, v_h) - d(p_h^1, v_h) + d(\varphi_h, u_h^1) + b(u_h^0, u_h^1, v_h) + b(u_h^1, u_h^0, v_h) \\
\qquad = \lambda(jT_h^1, v_h) + \sigma h a(u_h^0, v_h) + b(u_h^0, u_h^0, v_h), \quad \forall v_h \in X_h,\ \varphi_h \in M_h, \\
(1+\sigma h)\overline{a}(T_h^1, \psi_h) + \lambda\overline{b}(u_h^1, T_h^0, \psi_h) + \lambda\overline{b}(u_h^0, T_h^1, \psi_h) \\
\qquad = \sigma h\overline{a}(T_h^0, \psi_h) + \lambda\overline{b}(u_h^0, T_h^0, \psi_h), \quad \forall \psi_h \in W_{0h}.
\end{cases} \tag{3.4}$$

In general, this method can be described as follows.

Step 1. Solve the nonlinear systems (3.1) for (u_h^0, p_h^0, T_h^0).

Step 2. For $i = 1, 2, \ldots, m$, solve the linear equations

$$\left(p^\ddagger\right) \quad \begin{cases}
(\nu+\sigma h)a(u_h^i, v_h) - d(p_h^i, v_h) + d(\varphi_h, u_h^i) + b(u_h^{i-1}, u_h^i, v_h) + b(u_h^i, u_h^{i-1}, v_h) \\
\qquad = (T_h^i, v_h) + \sigma h a(u_h^{i-1}, v_h) + b(u_h^{i-1}, u_h^{i-1}, v_h), \quad \forall v_h \in X_h,\ \varphi_h \in M_h, \\
(1+\sigma h)\overline{a}(T_h^i, \psi_h) + \lambda\overline{b}(u_h^i, T_h^{i-1}, \psi_h) + \lambda\overline{b}(u_h^{i-1}, T_h^i, \psi_h) \\
\qquad = \sigma h\overline{a}(T_h^{i-1}, \psi_h) + \lambda\overline{b}(u_h^{i-1}, T_h^{i-1}, \psi_h), \quad \forall \psi_h \in W_{0h}.
\end{cases} \tag{3.5}$$

For each i the residual is given by

$$\begin{aligned}
\left(R\left(u_h^i, p_h^i, T_h^i\right), v_h\right) &= \lambda\left(jT_h^i, v_h\right) - \nu a\left(u_h^i, v_h\right) + d\left(p_h^i, v_h\right) \\
&\quad - d\left(\varphi_h, u_h^i\right) - b\left(u_h^i, u_h^i, v_h\right), \\
\left(Q\left(u_h^i, p_h^i, T_h^i\right), \psi_h\right) &= -\overline{a}\left(T_h^i, \psi_h\right) - \lambda\overline{b}\left(u_h^i, T_h^i, \psi_h\right).
\end{aligned} \tag{3.6}$$

The correction $(\varepsilon_h^i, \varrho_h^i, \tau_h^i)$ is given by

$$\begin{aligned}
&(\nu+\sigma h)a\left(\varepsilon_h^i, v_h\right) - d\left(\varrho_h^i, v_h\right) + d\left(\varphi_h, \varepsilon_h^i\right) + b\left(\varepsilon_h^i, u_h^i, v_h\right) + b\left(u_h^i, \varepsilon_h^i, v_h\right) \\
&\quad = \left(R\left(u_h^i, p_h^i, T_h^i\right), v_h\right), \quad \forall v_h \in X_h,\ \varphi_h \in M_h, \\
&(1+\sigma h)\overline{a}\left(\tau_h^i, \psi_h\right) + \lambda\overline{b}\left(u_h^i, \tau^i, \psi_h\right) + \lambda\overline{b}\left(\varepsilon_h^i, T_h^i, \psi_h\right) \\
&\quad = \left(Q\left(u_h^i, p_h^i, T_h^i\right), \psi_h\right), \quad \forall \psi_h \in W_{0h}.
\end{aligned} \tag{3.7}$$

Remark 3.1. From the numerical experiments, we see that one or two correction steps is adequate. And this is as same as [24].

4. Stability Analysis

In this section, we give the stability analysis. It is given by the following theorems.

Theorem 4.1. *Under the assumptions of Lemma 2.3, then* (u_h^0, T_h^0) *defined by* $(\mathcal{P}^*)$ *satisfies*

$$\left\|\nabla u_h^0\right\|_0 \le \frac{5C_0^2\lambda\varepsilon}{3(\nu+\sigma h)}, \qquad \left\|\nabla T_h^0\right\|_0 \le \varepsilon. \tag{4.1}$$

Moreover, if

$$\frac{25C_0^2 N\lambda\varepsilon}{3(\nu+\sigma h)^2} < 1, \tag{4.2}$$

$(\mathcal{P}^*)$ *admits a unique solution.*

Proof. We define the set

$$\mathcal{B}_M = \left\{\tilde{v}_h \in X_h; \|\nabla\tilde{v}_h\|_0 \le \frac{5C_0^2\lambda\varepsilon}{3(\nu+\sigma h)}\right\}. \tag{4.3}$$

Let $\tilde{u}_h$ be in $\mathcal{B}_M$. Then

$$(1+\sigma h)\overline{a}\left(T_h^0, \psi_h\right) + \lambda\overline{b}\left(\tilde{u}_h, T_h^0, \psi_h\right) = 0, \quad \forall \psi_h \in W_{0h} \tag{4.4}$$

has a unique solution $T_h^0 \in W_h$ such that $T_h|_{\partial\Omega} = T_0$. For a given T_h^0, we consider the following problem:

$$\begin{aligned}
(\nu+\sigma h)a\left(u_h^{0*}, v_h\right) - d\left(p_h^{0*}, v_h\right) + d\left(\varphi_h, u_h^{0*}\right) + b\left(u_h^{0*}, u_h^{0*}, v_h\right) = \lambda\left(jT_h^0, v_h\right), \\
\forall v_h \in X_h,\ \varphi_h \in M_h.
\end{aligned} \tag{4.5}$$

By the theory of the Navier-Stokes equations, we get (4.5) has a unique solution $(u_h^{0*}, p_h^{0*}) \in X_h \times M_h$ (see [31]). It means that (4.4) and (4.5) give a unique $u_h^{0*} \in X_h$ for a given $\tilde{u}_h \in X_h$, we denote

$$u_h^{0*} = \ell_h \tilde{u}_h. \tag{4.6}$$

Setting $T_h^0 = \omega_h^0 + T_0$, $\psi_h = \omega_h^0$ in (4.4) and using (2.9), we can obtain

$$(1+\sigma h)\overline{a}\left(\omega_h^0, \omega_h^0\right) = -\lambda \overline{b}\left(\tilde{u}_h, T_0, \omega_h^0\right) - (1+\sigma h)\overline{a}\left(T_0, \omega_h^0\right). \tag{4.7}$$

Using (2.7), (2.11), and (2.16), we can get

$$\begin{aligned} (1+\sigma h)\left\|\nabla \omega_h^0\right\|_0 &\le \lambda \overline{N}\|\nabla \tilde{u}_h\|_0 \|\nabla T_0\|_0 + (1+\sigma h)\|\nabla T_0\|_0, \\ \left\|\nabla \omega_h^0\right\|_0 &\le \frac{\lambda \overline{N}\varepsilon}{4}\|\nabla \tilde{u}_h\|_0 + \|\nabla T_0\|_0. \end{aligned} \tag{4.8}$$

Using the triangle inequality, we have

$$\begin{aligned} \left\|\nabla T_h^0\right\|_0 &\le \|\nabla T_0\|_0 + \left\|\nabla \omega_h^0\right\|_0 \\ &\le \frac{\lambda \overline{N}\varepsilon}{4}\|\nabla \tilde{u}_h\|_0 + 2\|\nabla T_0\|_0 \\ &\le \frac{5C_0^2 \overline{N}\lambda\varepsilon^2}{12(\nu+\sigma h)} + \frac{\varepsilon}{2} \le \varepsilon. \end{aligned} \tag{4.9}$$

Letting $v_h = u_h^{0*}$, $\varphi_h = p_h^0$ in (4.5) and using (2.8), we get

$$(\nu+\sigma h)a\left(u_h^{0*}, u_h^{0*}\right) = \lambda\left(jT_h^0, u_h^{0*}\right). \tag{4.10}$$

Letting $T_h^0 = \omega_h^0 + T_0$ and using (2.9), we have

$$\begin{aligned} (\nu+\sigma h)\left\|\nabla u_h^{0*}\right\|_0 &\le C_0^2\lambda\left\|\nabla \omega_h^0\right\|_0 + C_0\lambda\|T_0\|_0 \\ &\le C_0^2\lambda^2\overline{N}\|\nabla \tilde{u}_h\|_0\|\nabla T_0\|_0 + C_0\lambda(1+C_0)\|\nabla T_0\|_0 \\ &\le C_0^2\lambda\varepsilon. \end{aligned} \tag{4.11}$$

Namely,

$$\left\|\nabla u_h^{0*}\right\|_0 \le \frac{5C_0^2\lambda\varepsilon}{3(\nu+\sigma h)}. \tag{4.12}$$

Hence, we proved that ℓ_h maps $\mathcal{B}_M$ to $\mathcal{B}_M$. It follows from Brouwer's fixed-point theorem that there exits a solution to system $(\mathcal{P}^*)$.

To prove the uniqueness, assume that $(u_h^{01}, p_h^{01}, T_h^{01}), (u_h^{02}, p_h^{02}, T_h^{02}) \in X_h \times M_h \times W_h$, and $T_h^{01}|_{\partial\Omega} = T_h^{02}|_{\partial\Omega} = T_0$ are two solutions of $(\mathcal{P}^*)$. Then, we obtain that

$$\begin{aligned}
&(\nu + \sigma h) a\left(u_h^{01} - u_h^{02}, v_h\right) - d\left(p_h^{01} - p_h^{02}, v_h\right) + d\left(\varphi_h, u_h^{01} - u_h^{02}\right) + b\left(u_h^{01} - u_h^{02}, u_h^{01}, v_h\right) \\
&\quad + b\left(u_h^{02}, u_h^{01} - u_h^{02}, v_h\right) = \lambda\left(j\left(T_h^{01} - T_h^{02}\right), v_h\right), \quad \forall v_h \in X_h,\ \varphi_h \in M_h, \\
&(1 + \sigma h)\overline{a}\left(T_h^{01} - T_h^{02}, \varphi_h\right) + \lambda\overline{b}\left(u_h^{02}, T_h^{01} - T_h^{02}, \varphi_h\right) + \lambda\overline{b}\left(u_h^{01} - u_h^{02}, T_h^{01}, \varphi_h\right) = 0, \quad \forall \varphi_h \in W_{0h}.
\end{aligned} \tag{4.13}$$

Let $v_h = u_h^{01} - u_h^{02}$, $\varphi_h = p_h^{01} - p_h^{02}$ in the first equation of (4.13), we can get

$$(\nu + \sigma h)\left\|\nabla\left(u_h^{01} - u_h^{02}\right)\right\|_0 \le N\left\|\nabla\left(u_h^{01} - u_h^{02}\right)\right\|_0\left\|\nabla u_h^{01}\right\|_0 + C_0^2\lambda\left\|\nabla\left(T_h^{01} - T_h^{02}\right)\right\|_0. \tag{4.14}$$

Setting $\varphi_h = T_h^{01} - T_h^{02}$ in the second equation of (4.13), we obtain

$$(1 + \sigma h)\left\|\nabla\left(T_h^{01} - T_h^{02}\right)\right\|_0 \le \lambda\overline{N}\left\|\nabla\left(u_h^{01} - u_h^{02}\right)\right\|_0\left\|\nabla T_h^{01}\right\|_0. \tag{4.15}$$

By (4.14) and (4.15), we deduce

$$\begin{aligned}
&(\nu + \sigma h)\left\|\nabla\left(u_h^{01} - u_h^{02}\right)\right\|_0 \\
&\quad \le N\left\|\nabla u_h^{01}\right\|_0\left\|\nabla(u_h^{01} - u_h^{02})\right\|_0 + C_0^2\lambda^2\overline{N}\left\|\nabla T_h^{01}\right\|_0\left\|\nabla(u_h^{01} - u_h^{02})\right\|_0 \\
&\quad \le \frac{5C_0^2 N\lambda\varepsilon}{3(\nu + \sigma h)}\left\|\nabla(u_h^{01} - u_h^{02})\right\|_0 + \frac{C_0^2\lambda\overline{N}\varepsilon}{4\nu}\left\|\nabla(u_h^{01} - u_h^{02})\right\|_0.
\end{aligned} \tag{4.16}$$

Using (4.2), we obtain

$$\left\|\nabla\left(u_h^{01} - u_h^{02}\right)\right\|_0 \le \frac{2}{5}\left\|\nabla\left(u_h^{01} - u_h^{02}\right)\right\|_0. \tag{4.17}$$

Namely,

$$\left\|\nabla\left(u_h^{01} - u_h^{02}\right)\right\|_0 = 0. \tag{4.18}$$

By (4.15), we see that $\|\nabla(T_h^{01} - T_h^{02})\|_0 = 0$. Therefore, it follows that $(\mathcal{P}^*)$ has a unique solution.

Then, we give the prove of (4.1) without using (4.2). Letting $v_h = u_h^0$, $\varphi_h = p_h^0$ in the first equation of (3.1) and using (2.8), we get

$$(\nu + \sigma h) a\left(u_h^0, u_h^0\right) = \lambda\left(jT_h^0, u_h^0\right). \tag{4.19}$$

Letting $T_h^0 = \omega_h^0 + T_0$, we have

$$(\nu + \sigma h)\left\|\nabla u_h^0\right\|_0 \leq C_0^2\lambda\left\|\nabla \omega_h^0\right\|_0 + C_0\lambda\|T_0\|_0. \tag{4.20}$$

Letting $T_h^0 = \omega_h^0 + T_0$, $\varphi_h = \omega_h^0$ in the second equation of (3.1) and using (2.9), we can obtain

$$(1+\sigma h)\overline{a}\left(\omega_h^0, \omega_h^0\right) = -\lambda\overline{b}\left(u_h^0, T_0, \omega_h^0\right) - (1+\sigma h)\overline{a}\left(T_0, \omega_h^0\right). \tag{4.21}$$

Using (2.7), (2.11), and (2.16), we can get

$$\begin{gathered}(1+\sigma h)\left\|\nabla \omega_h^0\right\|_0 \leq \lambda\overline{N}\left\|\nabla u_h^0\right\|_0\|\nabla T_0\|_0 + (1+\sigma h)\|\nabla T_0\|_0, \\ \left\|\nabla \omega_h^0\right\|_0 \leq \frac{\lambda\overline{N}\varepsilon}{4}\left\|\nabla u_h^0\right\|_0 + \|\nabla T_0\|_0.\end{gathered} \tag{4.22}$$

By (4.20) and (4.22), we can deduce

$$\begin{aligned}\left\|\nabla u_h^0\right\|_0 &\leq (\nu+\sigma h)^{-1}\lambda C_0^2\left\|\nabla \omega_h^0\right\|_0 + (\nu+\sigma h)^{-1}C_0\lambda\|T_0\|_0 \\ &\leq (\nu+\sigma h)^{-1}\left(\lambda C_0\|T_0\|_0 + C_0^2\lambda\|\nabla T_0\|_0 + \frac{C_0^2\lambda^2\overline{N}\varepsilon}{4}\left\|\nabla u_h^0\right\|_0\right).\end{aligned} \tag{4.23}$$

Using (2.16), we get

$$\left\|\nabla u_h^0\right\|_0 \leq \frac{5C_0^2\lambda\varepsilon}{3(\nu+\sigma h)}. \tag{4.24}$$

Using (2.7), (2.11), (2.16), and (4.20), we can get

$$\begin{aligned}\left\|\nabla \omega_h^0\right\|_0 &\leq \lambda\overline{N}\left\|\nabla u_h^0\right\|_0\|\nabla T_0\|_0 + \|\nabla T_0\|_0 \\ &\leq \frac{\lambda\overline{N}\varepsilon}{4}\left\|\nabla u_h^0\right\|_0 + \|\nabla T_0\|_0 \\ &\leq \frac{3\varepsilon}{4}, \\ \left\|\nabla T_h^0\right\|_0 &\leq \left\|\nabla \omega_h^0\right\|_0 + \|\nabla T_0\|_0 \\ &\leq \varepsilon.\end{aligned} \tag{4.25}$$

Therefore, we finish the proof. □

Theorem 4.2. *Under the assumptions of Lemma 2.3, and*

$$\frac{25C_0^2 N\lambda\varepsilon}{3(\nu+\sigma h)^2} < 1, \tag{4.26}$$

(u_h^1, T_h^1) *defined by* (3.4) *satisfies*

$$\left\|\nabla u_h^1\right\|_0 \le \delta, \qquad \left\|\nabla T_h^1\right\|_0 \le \frac{5\varepsilon}{6} + \lambda\overline{N}\delta\varepsilon + \sigma h\varepsilon, \tag{4.27}$$

where $\delta \doteq (103C_0^2\lambda\varepsilon/48 + \sigma h(5C_0^2\lambda\varepsilon/3\nu))/(7/10)(\nu+\sigma h)$.

Proof. Letting $v_h = u_h^1$, $\varphi_h = p_h^1$ in the first equation of (3.4) and using (2.8), we get

$$(\nu+\sigma h)a\left(u_h^1, u_h^1\right) + b\left(u_h^1, u_h^0, u_h^1\right) = b\left(u_h^0, u_h^0, u_h^1\right) + \sigma h a\left(u_h^0, u_h^1\right) + \lambda\left(jT_h^0, u_h^1\right). \tag{4.28}$$

Letting $T_h^0 = \omega_h^0 + T_0$ and using (2.10), we have

$$\begin{aligned}(\nu+\sigma h)\left\|\nabla u_h^1\right\|_0 &\le N\left\|\nabla u_h^1\right\|_0\left\|\nabla u_h^0\right\|_0 + \sigma h\left\|\nabla u_h^0\right\|_0 + N\left\|\nabla u_h^0\right\|_0^2 \\ &\quad + C_0^2\lambda\left\|\nabla\omega_h^1\right\|_0 + C_0\lambda\|T_0\|_0.\end{aligned} \tag{4.29}$$

Let $T_h^1 = \omega_h^1 + T_0$, $\psi_h = \omega_h^1$ in the second equation of (3.4), we can obtain

$$\begin{aligned}(1+\sigma h)\overline{a}\left(\omega_h^1, \omega_h^1\right) &= -\lambda\overline{b}\left(u_h^0, T_0, \omega_h^1\right) - \lambda\overline{b}\left(u_h^1, T_h^0, \omega_h^1\right) + \lambda\overline{b}\left(u_h^0, T_h^0, \omega_h^1\right) \\ &\quad + \sigma h\overline{a}\left(T_h^0, \omega_h^1\right) - (1+\sigma h)\overline{a}\left(T_0, \omega_h^1\right).\end{aligned} \tag{4.30}$$

Using (2.11) and (2.16), we can get

$$\begin{aligned}(1+\sigma h)\left\|\nabla\omega_h^1\right\|_0 &\le \lambda\overline{N}\left\|\nabla u_h^0\right\|_0\|\nabla T_0\|_0 + \lambda\overline{N}\left\|\nabla u_h^1\right\|_0\left\|\nabla T_h^0\right\|_0 \\ &\quad + \lambda\overline{N}\left\|\nabla u_h^0\right\|_0\left\|\nabla T_h^0\right\|_0 + \sigma h\left\|\nabla T_h^0\right\|_0 + (1+\sigma h)\|\nabla T_0\|_0,\end{aligned} \tag{4.31}$$

$$\begin{aligned}\left\|\nabla\omega_h^1\right\|_0 &\le \lambda\overline{N}\left\|\nabla u_h^0\right\|_0\|\nabla T_0\|_0 + \lambda\overline{N}\left\|\nabla u_h^1\right\|_0\left\|\nabla T_h^0\right\|_0 \\ &\quad + \lambda\overline{N}\left\|\nabla u_h^0\right\|_0\left\|\nabla T_h^0\right\|_0 + \sigma h\left\|\nabla T_h^0\right\|_0 + \|\nabla T_0\|_0.\end{aligned} \tag{4.32}$$

Using (4.29), we get

$$
\begin{aligned}
&(\nu+\sigma h)\left\|\nabla u_h^1\right\|_0 \\
&\quad\le N\left\|\nabla u_h^1\right\|_0\left\|\nabla u_h^0\right\|_0+\sigma h\left\|\nabla u_h^0\right\|_0+N\left\|\nabla u_h^0\right\|_0^2 \\
&\qquad+C_0^2\lambda\Big(\lambda\overline{N}\left\|\nabla u_h^0\right\|_0\|\nabla T_0\|_0+\lambda\overline{N}\left\|\nabla u_h^1\right\|_0\left\|\nabla T_h^0\right\|_0+\lambda\overline{N}\left\|\nabla u_h^0\right\|_0\left\|\nabla T_h^0\right\|_0 \\
&\qquad\qquad+\left\|\nabla T_h^0\right\|_0+\|\nabla T_0\|_0\Big)+C_0\lambda\|T_0\|_0. \\
&\Big(\nu+\sigma h-N\left\|\nabla u_h^0\right\|_0-C_0^2\lambda^2\overline{N}\left\|\nabla T_h^0\right\|_0\Big)\left\|\nabla u_h^1\right\|_0 \\
&\quad\le N\left\|\nabla u_h^0\right\|_0^2+\sigma h\left\|\nabla u_h^0\right\|_0+C_0^2\lambda^2\overline{N}\left\|\nabla u_h^0\right\|_0\|\nabla T_0\|_0+C_0^2\lambda^2\overline{N}\left\|\nabla u_h^0\right\|_0\left\|\nabla T_h^0\right\|_0 \\
&\qquad+C_0^2\lambda\left\|\nabla T_h^0\right\|_0+C_0^2\lambda\|\nabla T_0\|_0+C_0\lambda\|T_0\|_0.
\end{aligned}
\tag{4.33}
$$

Using (2.16), (4.26), and Theorem 4.2, we can obtain

$$
\begin{aligned}
\frac{7}{10}(\nu+\sigma h)\left\|\nabla u_h^1\right\|_0 &\le \frac{25NC_0^4\lambda^2\varepsilon^2}{9(\nu+\sigma h)^2}+\sigma h\frac{5C_0^2\lambda\varepsilon}{3(\nu+\sigma h)}+\frac{10\overline{N}C_0^4\lambda^3\varepsilon^2}{3(\nu+\sigma h)}+\frac{3C_0^2\lambda^2\varepsilon^2}{2} \\
&\le \frac{103C_0^2\lambda\varepsilon}{48}+\sigma h\frac{5C_0^2\lambda\varepsilon}{3(\nu+\sigma h)}.
\end{aligned}
\tag{4.34}
$$

Namely,

$$
\left\|\nabla u_h^1\right\|_0 \le \frac{103C_0^2\lambda\varepsilon/48+\sigma h\left(5C_0^2\lambda\varepsilon/3\nu\right)}{(7/10)(\nu+\sigma h)} \doteq \delta. \tag{4.35}
$$

Using (2.16), (4.31), and (4.35), we can get

$$
\begin{aligned}
\left\|\nabla\omega_h^1\right\|_0 &\le \frac{10\overline{N}C_0^2\lambda^2\varepsilon^2}{3(\nu+\sigma h)}+\lambda\overline{N}\delta\varepsilon+\sigma h\varepsilon+\frac{\varepsilon}{4} \\
&\le \frac{\varepsilon}{3}+\frac{\varepsilon}{4}+\lambda\overline{N}\delta\varepsilon+\sigma h\varepsilon \\
&= \frac{7\varepsilon}{12}+\lambda\overline{N}\delta\varepsilon+\sigma h\varepsilon.
\end{aligned}
\tag{4.36}
$$

Using the triangle inequality, we can get

$$
\left\|\nabla T_h^1\right\|_0 \le \left\|\nabla\omega_h^1\right\|_0+\|\nabla T_0\|_0 \le \frac{5\varepsilon}{6}+\lambda\overline{N}\delta\varepsilon+\sigma h\varepsilon. \tag{4.37}
$$

Therefore, we finish the proof. □

5. Error Analysis

In this section, we establish the H^1 and L^2-bounds of the error $u - u_h^i$, $T - T_h^i$, $i = 0,1$ and L^2-bound of the error $p - p_h^i$, $i = 0,1$. In order to obtain the error estimates, we define the Galerkin projection $(R_h, Q_h) = (R_h(u,p), Q_h(u,p)) : (X, M) \to (X_h, M_h)$, such that

$$a(R_h - u, v_h) - d(Q_h - p, v_h) + d(q_h, R_h - u) = 0, \quad \forall (u,p) \in (X, M), \ (v_h, q_h) \in (X_h, M_h). \tag{5.1}$$

Lemma 5.1 (see [37, 38]). *The Galerkin projection (R_h, Q_h) satisfies*

$$\|R_h - u\|_0 + h(\|\nabla(R_h - u)\|_0 + \|Q_h - p\|_0) \le Ch^{r+1}(\nu\|u\|_{r+1} + \|p\|_r), \quad r = 1,2. \tag{5.2}$$

Lemma 5.2 (see [7]). *There exits $\tilde{r}_h : W \to W_h$ for all $\varphi \in W$ holds that*

$$(\nabla(\varphi - \tilde{r}_h\varphi), \nabla\varphi_h) = 0, \quad \forall \varphi_h \in W_h, \tag{5.3}$$

$$\int_\Omega (\varphi - \tilde{r}_h\varphi)dx = 0, \qquad \|\nabla\tilde{r}_h\varphi\|_0 \le \|\nabla\varphi\|_0. \tag{5.4}$$

When $\varphi \in W^{k,q}(\Omega)$ $(1 \le q \le \infty)$, there holds

$$\|\varphi - \tilde{r}_h\varphi\|_{-s,q} \le Ch^{k+s}|\varphi|_{k,q}, \quad -1 \le s \le m, \ 0 \le k \le r+1. \tag{5.5}$$

There exits $\overline{r}_h : W_0 \to W_{0h}$ for all $\varphi \in W_0$ holds that

$$(\nabla(\varphi - \overline{r}_h\varphi), \nabla\varphi_h) = 0, \quad \forall \varphi_h \in W_{0h}, \quad \|\nabla\overline{r}_h\varphi\|_0 \le \|\nabla\varphi\|_0. \tag{5.6}$$

When $\varphi \in W^{r,q}(\Omega)$ $(1 \le q \le \infty)$, there holds

$$\|\varphi - \overline{r}_h\varphi\|_{-s,q} \le Ch^{k+s}|\varphi|_{r,q}, \quad -1 \le s \le r, \ 0 \le k \le r+1. \tag{5.7}$$

Lemma 5.3 (see [7]). *If (A_1)~(A_3) hold and $(u,p,T) \in H^{r+1}(\Omega) \times H^r(\Omega) \times H^{r+1}(\Omega)$ and (u_h, P_h, T_h) are the solution of problem $(\mathcal{P}')$ and $(\mathcal{P}'')$, respectively, then there holds that*

$$\|\nabla(u - u_h)\|_0 + \|p - p_h\|_0 + \|\nabla(T - T_h)\|_0 \le Ch^r(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}). \tag{5.8}$$

Lemma 5.4. *Under the assumptions of Lemma 2.3,* (u_h, p_h, T_h) *is the solution of* (3.1), (u_h^0, p_h^0, T_h^0) *defined by* (3.4), *then there hold*

$$\begin{gathered}\left\|\nabla(u_h - u_h^0)\right\|_0 \le \frac{50\sigma C_0^2\lambda h}{21\nu} + \frac{10C_0^2\lambda\sigma\varepsilon h}{7\nu},\\ \left\|\nabla(T_h - T_h^0)\right\|_0 \le 2\sigma h\varepsilon + \frac{\sigma h\varepsilon}{14\nu},\\ \beta\left\|p_h - p_h^0\right\|_0 \le \frac{95\sigma h C_0^2\lambda\varepsilon}{21\nu} + \frac{19\sigma h C_0^2\lambda\varepsilon}{7} + \sigma h C_0^2\lambda + \frac{\sigma h C_0^2\lambda}{14\nu}.\end{gathered} \tag{5.9}$$

Proof. Subtracting (3.1) from (2.5) we get the error equations, namely $(u_h - u_h^0, p_h - p_h^0, T_h - T_h^0)$ satisfy

$$\begin{aligned}&\nu a\left(u_h - u_h^0, v_h\right) - \sigma h a\left(u_h^0, v_h\right) + d\left(\varphi_h, u_h - u_h^0\right) - d\left(p_h - p_h^0, v_h\right) + b\left(u_h^0, u_h - u_h^0, v_h\right)\\ &\quad + b\left(u_h - u_h^0, u_h, v_h\right) = \lambda\left(j\left(T_h - T_h^0\right), v_h\right), \quad \forall v_h \in X_h,\ \varphi_h \in M_h,\\ &\overline{a}\left(T_h - T_h^0, \varphi_h\right) - \sigma h\overline{a}\left(T_h^0, \varphi_h\right) + \lambda\overline{b}\left(u_h - u_h^0, T_h, \varphi_h\right) + \overline{b}\left(u_h^0, T_h - T_h^0, \varphi_h\right) = 0, \quad \forall \varphi_h \in W_{0h}.\end{aligned} \tag{5.10}$$

Letting $v_h = u_h - u_h^0, \varphi_h = p_h - P_h^0$ in the first equation of (5.10) and using (2.11), (2.8), and (A_1), we can get

$$\nu\left\|\nabla\left(u_h - u_h^0\right)\right\|_0 \le \sigma h\left\|\nabla u_h^0\right\|_0 + N\left\|\nabla\left(u_h - u_h^0\right)\right\|_0\|\nabla u_h\|_0 + C_0^2\lambda\left\|\nabla\left(T_h - T_h^0\right)\right\|_0. \tag{5.11}$$

Hence, we deduce

$$(\nu - N\|\nabla u_h\|_0)\left\|\nabla\left(u_h - u_h^0\right)\right\|_0 \le \sigma h\left\|\nabla u_h^0\right\|_0 + C_0^2\lambda\left\|\nabla\left(T_h - T_h^0\right)\right\|_0. \tag{5.12}$$

Letting $\varphi_h = T_h - T_h^0$ in the second equation of (5.10) and using (2.9), we obtain

$$\overline{a}\left(T_h - T_h^0, T_h - T_h^0\right) + \sigma h\overline{a}\left(T_h^0, T_h - T_h^0\right) + \lambda\overline{b}\left(u_h - u_h^0, T_h, T_h - T_h^0\right) = 0. \tag{5.13}$$

Using (2.11), we can get

$$\left\|\nabla(T_h - T_h^0)\right\|_0 \le \sigma h\left\|\nabla T_h^0\right\|_0 + \lambda\overline{N}\left\|\nabla(u_h - u_h^0)\right\|_0\|\nabla T_h\|_0. \tag{5.14}$$

By (5.12), we deduce

$$(\nu - N\|\nabla u_h\|_0)\left\|\nabla\left(u_h - u_h^0\right)\right\|_0 \le \sigma h\left\|\nabla u_h^0\right\|_0 + C_0^2\lambda\sigma h\left\|\nabla T_h^0\right\|_0 + C_0^2\lambda^2\overline{N}\left\|\nabla(u_h - u_h^0)\right\|_0\|\nabla T_h\|_0. \tag{5.15}$$

Using (4.1), we can obtain

$$\begin{aligned}\left(\nu - N\|\nabla u_h\|_0 - \frac{3\nu}{32}\right)\left\|\nabla\left(u_h - u_h^0\right)\right\|_0 &\le \sigma h\left\|\nabla u_h^0\right\|_0 + C_0^2\lambda\sigma h\left\|\nabla T_h^0\right\|_0 \\ &\le \frac{5\sigma h C_0^2\lambda\varepsilon}{3(\nu+\sigma h)} + C_0^2\lambda\sigma h\varepsilon.\end{aligned} \tag{5.16}$$

By using (2.16) and (2.17), there holds

$$\nu - N\|\nabla u_h\|_0 - \frac{3\nu}{32} \ge \frac{7\nu}{10}. \tag{5.17}$$

Therefore, we can deduce

$$\left\|\nabla(u_h - u_h^0)\right\|_0 \le \frac{50\sigma h C_0^2\lambda\varepsilon}{21\nu(\nu+\sigma h)} + \frac{10C_0^2\lambda\sigma h\varepsilon}{7\nu}. \tag{5.18}$$

By (5.14) and (5.18), we can have

$$\begin{aligned}\left\|\nabla(T_h - T_h^0)\right\|_0 &\le \sigma h\varepsilon + \lambda\overline{N}\varepsilon\left(\frac{50\sigma h C_0^2\lambda\varepsilon}{21\nu(\nu+\sigma h)} + \frac{10C_0^2\lambda\sigma h\varepsilon}{7\nu}\right) \\ &\le 2\sigma h\varepsilon + \frac{\sigma h\varepsilon}{14\nu}.\end{aligned} \tag{5.19}$$

Letting $\varphi_h = 0$, $v_h = u_h - u_h^0$ in the first equation of (5.10) and using (2.3), we have

$$\begin{aligned}\beta\left\|p_h - p_h^0\right\|_0 &\le \nu\left\|\nabla(u_h - u_h^0)\right\|_0 + \sigma h\left\|\nabla u_h^0\right\|_0 + N\left\|\nabla(u_h - u_h^0)\right\|_0\|\nabla u_h\|_0 + C_0^2\lambda\left\|\nabla(T_h - T_h^0)\right\|_0 \\ &\le \frac{50\sigma h C_0^2\lambda\varepsilon}{21(\nu+\sigma h)} + \frac{10\sigma h C_0^2\lambda\varepsilon}{7} + \frac{5\sigma h C_0^2\lambda\varepsilon}{3\nu} + \frac{10\sigma h C_0^2\lambda\varepsilon}{21\nu} + \frac{2\sigma h C_0^2\lambda\varepsilon}{7} \\ &\quad + \sigma h C_0^2\lambda\varepsilon + \sigma h C_0^2\lambda + \frac{\sigma h C_0^2\lambda}{14\nu} \\ &\le \frac{95\sigma h C_0^2\lambda\varepsilon}{21(\nu+\sigma h)} + \frac{19\sigma h C_0^2\lambda\varepsilon}{7} + \sigma h C_0^2\lambda + \frac{\sigma h C_0^2\lambda}{14\nu}.\end{aligned} \tag{5.20}$$

Hence, we finish the proof. □

Theorem 5.5. *Under the assumptions of Lemmas 2.3 and 5.3, the following inequality*

$$\left\|\nabla(u-u_h^0)\right\|_0+\left\|p-p_h^0\right\|_0+\left\|\nabla(T-T_h^0)\right\|_0\le Ch^r\left(\|u\|_{r+1}+\|p\|_r+\|T\|_{r+1}\right)+Ch, \tag{5.21}$$

holds, where C is a positive constant numbers.

Proof. By Lemmas 5.3, 5.4, and the triangle inequality this theorem is obviously true. □

Lemma 5.6. *For all $u\in H^2(\Omega)\cap X$, $\omega\in W_0$, $\psi\in H^2(\Omega)\cap W_0$, there hold that*

$$\left|\overline{b}(u-R_h,\omega,\psi)\right|\le C\|u-R_h\|_0\|\mathcal{A}\omega\|_0\|\nabla\psi\|_0, \tag{5.22}$$

$$\left|\overline{b}(u,T-\tilde{r}_hT,\psi)\right|\le C\|\mathcal{A}u\|_0\|T-\tilde{r}_hT\|_0\|\nabla\psi\|_0. \tag{5.23}$$

Proof. Letting $\overline{\omega}=(\omega,0)^T$, we have

$$\overline{b}(u-R_h,\omega,\psi)=b(u-R_h,\overline{\omega},\psi). \tag{5.24}$$

Using (2.25), we can deduce (5.22). Because $T-\tilde{r}_hT\in W_0$, (5.23) holds. □

Theorem 5.7. *Under the assumptions of Lemmas 2.3 and 5.3, the following inequality:*

$$\left\|u-u_h^0\right\|_0+\left\|T-T_h^0\right\|_0\le Ch^{r+1}\left(\|u\|_{r+1}+\|p\|_r+\|T\|_{r+1}\right)+Ch, \tag{5.25}$$

holds, where C is a positive constant.

Proof. Subtracting (3.1) from (2.4) we get the error equations, namely,

$$\begin{aligned}
&\nu a\left(u-u_h^0,v_h\right)-\sigma ha\left(u_h^0,v_h\right)+b\left(u-u_h^0,u_h^0,v_h\right)+b\left(u,u-u_h^0,v_h\right)-d\left(p-p_h^0,v_h\right)\\
&\qquad+d\left(\varphi_h,u-u_h^0\right)=\lambda\left(j\left(T-T_h^0\right),v_h\right),\quad \forall v_h\in X_h,\ \varphi_h\in M_h,\\
&\overline{a}\left(T-T_h^0,\psi_h\right)-\sigma h\overline{a}\left(T_h^0,\psi_h\right)+\lambda\overline{b}\left(u-u_h^0,T,\psi_h\right)+\lambda\overline{b}\left(u_h^0,T-T_h^0,\psi_h\right)=0,\quad \forall\psi_h\in W_{0h}.
\end{aligned} \tag{5.26}$$

Letting $e_h^0=R_h-u_h^0$, $\eta_h^0=Q_h-p_h^0$, $\xi_h^0=\tilde{r}_hT-T_h^0$ and using (5.1) and (5.3), we can get

$$\begin{aligned}
&\nu a\left(e_h^0,v_h\right)-\sigma ha\left(u_h^0,v_h\right)+b\left(u-u_h^0,u_h^0,v_h\right)+b\left(u,u-u_h^0,v_h\right)-d\left(\eta_h^0,v_h\right)+d\left(\varphi_h,e_h^0\right)\\
&\qquad=\lambda\left(j\left(T-T_h^0\right),v_h\right),\quad \forall v_h\in X_h,\ \varphi_h\in M_h,\\
&\overline{a}(\xi_h,\psi_h)-\sigma h\overline{a}\left(T_h^0,\psi_h\right)+\lambda\overline{b}\left(u-u_h^0,T,\psi_h\right)+\lambda\overline{b}\left(u_h^0,T-T_h^0,\psi_h\right)=0,\quad \forall\psi_h\in W_{0h}.
\end{aligned} \tag{5.27}$$

Taking $v_h = e_h^0$, $\varphi_h = \eta_h^0$ in the first equation of (5.27), we obtain

$$\begin{aligned} &\nu a\left(e_h^0, e_h^0\right) - \sigma h a\left(u_h^0, e_h^0\right) + b\left(e_h^0, u_h^0, e_h^0\right) + b\left(u - R_h, u_h^0, e_h^0\right) + b\left(u, u - R_h, e_h^0\right) \\ &\quad = \lambda\left(j\left(T - T_h^0\right), e_h^0\right), \quad \forall v_h \in X_h,\ \varphi_h \in M_h. \end{aligned} \tag{5.28}$$

Using (2.10) and (A_1), we deduce

$$\begin{aligned} \left(\nu - N\left\|\nabla u_h^0\right\|_0\right)\left\|\nabla e_h^0\right\|_0^2 &\le \left|b\left(u - R_h, u, e_h^0\right)\right| + \left|b\left(u_h^0, u - R_h, e_h^0\right)\right| \\ &\quad + \left|\lambda\left(j\left(T - T_h^0\right), e_h^0\right)\right| + \left|\sigma h a\left(u_h^0, e_h^0\right)\right| \\ &\le N\left(\|\nabla u\|_0 + \left\|\nabla u_h^0\right\|_0\right)\|\nabla(u - R_h)\|_0\left\|\nabla e_h^0\right\|_0 \\ &\quad + C_0^2\lambda\left\|\nabla(T - T_h^0)\right\|_0\left\|\nabla e_h^0\right\|_0 + \sigma h\left\|\nabla u_h^0\right\|_0\left\|\nabla e_h^0\right\|_0. \end{aligned} \tag{5.29}$$

Using Theorem 2.1, (2.16), (4.1), and (5.2), we can obtain

$$\left\|\nabla e_h^0\right\|_0 \le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch. \tag{5.30}$$

Taking $\psi_h = \xi_h^0$ in the second equation of (5.27) and using (2.9) we have

$$\overline{a}\left(\xi_h^0, \xi_h^0\right) - \sigma h\overline{a}\left(T_h^0, \xi_h^0\right) + \lambda\overline{b}\left(u_h^0, T - \tilde{r}_h T, \xi_h^0\right) + \lambda\overline{b}\left(u - R_h, T, \xi_h^0\right) + \lambda\overline{b}\left(e_h^0, T, \xi_h^0\right) = 0. \tag{5.31}$$

By (2.9), we have

$$\begin{aligned} &\overline{b}\left(u_h^0, T - \tilde{r}_h T, \xi_h^0\right) + \overline{b}\left(u - R_h, T, \xi_h^0\right) + \overline{b}\left(e_h^0, T, \xi_h^0\right) \\ &\quad = -\overline{b}\left(e_h^0, T - \tilde{r}_h T, \xi_h^0\right) - \overline{b}\left(u - R_h, T - \tilde{r}_h T, \xi_h^0\right) + \overline{b}\left(u, T - \tilde{r}_h T, \xi_h^0\right) \\ &\quad + \overline{b}\left(u - R_h, T, \xi_h^0\right) + \overline{b}\left(e_h^0, T - \tilde{r}_h T, \xi_h^0\right) + \overline{b}\left(e_h^0, \xi_h^0, \xi_h^0\right) + \overline{b}\left(e_h^0, T_h^0, \xi_h^0\right). \end{aligned} \tag{5.32}$$

Letting $T = \omega + T_0$, $\omega \in W_0$ and using Lemma 5.6, we can get

$$\begin{aligned}
&\left|\overline{b}\left(u_h^0, T - \tilde{r}_h T, \xi_h^0\right) + \overline{b}\left(u - R_h, T, \xi_h^0\right) + \overline{b}\left(e_h, T, \xi_h^0\right)\right| \\
&\quad \le \overline{N}\left\|\nabla e_h^0\right\|_0 \|\nabla(T - \tilde{r}_h T)\|_0 \left\|\nabla \xi_h^0\right\|_0 + \overline{N}\|\nabla(u - R_h)\|_0 \|\nabla(T - \tilde{r}_h T)\|_0 \left\|\nabla \xi_h^0\right\|_0 \\
&\qquad + \overline{N}\left\|\nabla e_h^0\right\|_0 \|\nabla(T - \tilde{r}_h T)\|_0 \left\|\nabla \xi_h^0\right\|_0 + \overline{N}\left\|\nabla e_h^0\right\|_0 \left\|\nabla T_h^0\right\|_0 \left\|\nabla \xi_h^0\right\|_0 \\
&\qquad + C\|\mathcal{A}u\|_0 \|T - \tilde{r}_h T\|_0 \left\|\nabla \xi_h^0\right\|_0 + C\|u - R_h\|_0 \|\mathcal{A}\omega\|_0 \left\|\nabla \xi_h^0\right\|_0 \\
&\qquad + C\|\nabla(u - R_h)\|_0 \|\nabla T_0\|_0 \left\|\nabla \xi_h^0\right\|_0.
\end{aligned} \tag{5.33}$$

By assumption (A_2), letting $\varepsilon < h$ and using Lemma 5.1 and (2.16), (4.26), and (5.33), we can deduce

$$\left\|\nabla \xi_h^0\right\|_0 \le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch. \tag{5.34}$$

Hence, we have

$$\begin{aligned}
\left\|T - T_h^0\right\|_0 &\le \|T - \tilde{r}_h T\|_0 + \left\|\xi_h^0\right\|_0 \\
&\le \|T - \tilde{r}_h T\|_0 + C_0\left\|\nabla \xi_h^0\right\|_0 \\
&\le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch.
\end{aligned} \tag{5.35}$$

By (2.10) and (2.25), we can deduce

$$\begin{aligned}
&\left|b\left(u - R_h, u, e_h^0\right)\right| + \left|b\left(u_h^0, u - R_h, e_h^0\right)\right| \\
&\quad \le \left|b\left(u - R_h, u, e_h^0\right)\right| + \left|b\left(u, u - R_h, e_h^0\right)\right| \\
&\qquad + \left|b\left(u - R_h, u - R_h, e_h^0\right)\right| + \left|b\left(e_h, u - R_h, e_h^0\right)\right| \\
&\quad \le C\|\mathcal{A}u\|_0 \|u - R_h\|_0 \left\|\nabla e_h^0\right\|_0 \\
&\qquad + N\left(\|\nabla(u - R_h)\|_0 + \left\|\nabla e_h^0\right\|_0\right)\|\nabla(u - R_h)\|_0 \left\|\nabla e_h^0\right\|_0 \\
&\quad \le Ch^2\left\|\nabla e_h^0\right\|_0.
\end{aligned} \tag{5.36}$$

Using (5.29), we can obtain

$$\begin{aligned}\left(\nu - N\left\|\nabla u_h^0\right\|_0\right)\left\|\nabla e_h^0\right\|_0^2 &\le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right)\left\|\nabla e_h^0\right\|_0 \\ &\quad + C_0\lambda\left\|T - T_h^0\right\|_0\left\|\nabla e_h^0\right\|_0 + \sigma h\left\|\nabla u_h^0\right\|_0\left\|\nabla e_h^0\right\|_0.\end{aligned} \tag{5.37}$$

By using (2.16) and (4.1), there holds

$$\nu - N\left\|\nabla u_h^0\right\|_0 \ge \frac{4\nu}{5}. \tag{5.38}$$

Hence, we can deduce from (5.37)

$$\left\|\nabla e_h^0\right\|_0 \le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch. \tag{5.39}$$

Therefore, we can deduce

$$\begin{aligned}\|u - u_h^0\|_0 &\le \|u - R_h\|_0 + \left\|e_h^0\right\|_0 \\ &\le \|u - R_h\|_0 + C_0\left\|\nabla e_h^0\right\|_0 \\ &\le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch.\end{aligned} \tag{5.40}$$

□

Theorem 5.8. *Under the assumptions of Lemmas 2.3 and 5.3, then there holds*

$$\begin{aligned}\left\|\nabla(u - u_h^1)\right\|_0 + \left\|\nabla(T - T_h^1)\right\|_0 &\le Ch^r\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch^2, \\ \left\|u - u_h^1\right\|_0 + \left\|T - T_h^1\right\|_0 + h\left\|p - p_h^1\right\|_0 &\le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch^2,\end{aligned} \tag{5.41}$$

where C is a positive constant.

Proof. Subtracting (3.4) from (2.4) we get the error equations, namely,

$$\begin{aligned}&\nu a\left(u - u_h^1, v_h\right) - \sigma h a\left(u_h^1, v_h\right) + b(u, u, v_h) - b\left(u_h^1, u_h^0, v_h\right) \\ &\quad - b\left(u_h^0, u_h^1, v_h\right) - d\left(p - p_h^1, v_h\right) + d\left(\varphi_h, u - u_h^1\right) \\ &\quad = \lambda\left(j\left(T - T_h^1\right), v_h\right) - b\left(u_h^0, u_h^0, v_h\right) - \sigma h a\left(u_h^0, v_h\right), \quad \forall v_h \in X_h,\ \varphi_h \in M_h, \\ &\overline{a}\left(T - T_h^1, \psi_h\right) - \sigma h\overline{a}\left(T_h^1, \psi_h\right) + \lambda\overline{b}(u, T, \psi_h) - \lambda\overline{b}\left(u_h^1, T_h^0, \psi_h\right) - \lambda\overline{b}\left(u_h^0, T_h^1, \psi_h\right) \\ &\quad = -\sigma h\overline{a}\left(T_h^0, \psi_h\right) - \lambda\overline{b}\left(u_h^0, T_h^0, \psi_h\right), \quad \forall \psi_h \in W_{0h}.\end{aligned} \tag{5.42}$$

Letting $e_h^1 = R_h - u_h^1$, $\eta_h^1 = Q_h - p_h^1$, $\xi_h^1 = \tilde{r}_h T - T_h^1$, using (5.1) and (5.3) and adding and subtracting appropriate terms in the above expression yields

$$\begin{aligned}
&(\nu + \sigma h) a\left(e_h^1, v_h\right) + b\left(u_h^0, u - u_h^1, v_h\right) + b\left(u - u_h^1, u_h^0, v_h\right) - d\left(\eta_h^1, v_h\right) + d\left(\varphi_h, e_h^1\right) \\
&\quad = \lambda\left(j\left(T - T_h^0\right), v_h\right) + \sigma h a\left(R_h - u_h^0, v_h\right) - b\left(u - u_h^0, u - u_h^0, v_h\right), \quad \forall v_h \in X_h,\ \varphi_h \in M_h, \\
&(1 + \sigma h)\overline{a}\left(\xi_h^1, \psi_h\right) + \lambda \overline{b}\left(u_h^0, T - T_h^1, \psi_h\right) - \lambda \overline{b}\left(u - u_h^1, T_h^0, \psi_h\right) \\
&\quad = \sigma h \overline{a}\left(\tilde{r}_h T - T_h^0, \psi_h\right) - \lambda \overline{b}\left(u - u_h^0, T - T_h^0, \psi_h\right), \quad \forall \psi_h \in W_{0h}.
\end{aligned} \tag{5.43}$$

Letting $v_h = e_h^1$, $\varphi_h = \eta_h^1$ in the first equation of (5.43), we can deduce

$$\begin{aligned}
&(\nu + \sigma h) a\left(e_h^1, e_h^1\right) + b\left(u_h^0, u - R_h, e_h^1\right) + b\left(u - R_h, u_h^0, e_h^1\right) + b\left(e_h^1, u_h^0, e_h^1\right) \\
&\quad = \lambda\left(j\left(T - T_h^0\right), e_h^1\right) + \sigma h a\left(R_h - u_h^0, e_h^1\right) - b\left(u - u_h^0, u - u_h^0, e_h^1\right)
\end{aligned} \tag{5.44}$$

By (2.10) and (2.25), we can deduce

$$\begin{aligned}
\left(\nu + \sigma h - N\left\|\nabla u_h^0\right\|_0\right)\left\|\nabla e_h^1\right\|_0 &\le C\left\|\mathcal{A} u_h^0\right\|_0 \|u - R_h\|_0 + \sigma h\left\|\nabla(R_h - u_h^0)\right\|_0 \\
&\quad + N\left\|\nabla(u - u_h^0)\right\|_0^2 + \lambda C_0\left\|T - T_h^0\right\|_0.
\end{aligned} \tag{5.45}$$

Using (2.16), (4.1), (4.26), (5.21), and (5.2), we can obtain

$$\left\|\nabla e_h^1\right\|_0 \le C h^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + C h^2. \tag{5.46}$$

Using (5.2) and triangle inequality, we can have

$$\begin{aligned}
\left\|\nabla(u - u_h^1)\right\|_0 &\le \|\nabla(u - R_h)\|_0 + \|\nabla e_1\|_0 \\
&\le C h^r\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + C h^2, \\
\left\|u - u_h^1\right\|_0 &\le \|u - R_h\|_0 + \|e_1\|_0 \\
&\le \|u - R_h\|_0 + C_0\|\nabla e_1\|_0 \\
&\le C h^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + C h^2.
\end{aligned} \tag{5.47}$$

Letting $\psi_h = \xi_h^1$ in the second equation of (5.43) and using (2.8), we can deduce

$$\begin{aligned}(1+\sigma h)\overline{a}\left(\xi_h^1,\xi_h^1\right) + \lambda\overline{b}\left(u_h^0, T-\widetilde{r}_h T, \xi_h^1\right) - \lambda\overline{b}\left(u-u_h^1, T_h^0, \xi_h^1\right)\\ = \sigma h\overline{a}\left(T-T_h^0,\xi_h^1\right) - \lambda\overline{b}\left(u-u_h^0, T-T_h^0, \xi_h^1\right).\end{aligned} \tag{5.48}$$

Letting $T_h^0 = \omega_h^0 + T_0$ and using (2.11), (5.22), and (5.23), we have

$$\begin{aligned}(1+\sigma h)\left\|\nabla\xi_h^1\right\|_0 \le{}& C\lambda\left\|\mathcal{A}u_h^0\right\|_0\|T-\widetilde{r}_hT\|_0 + C\lambda\left\|u-u_h^1\right\|_0\left\|\mathcal{A}\omega_h^0\right\|_0\\ &+\overline{N}\lambda\left\|\nabla(u-u_h^1)\right\|_0\|\nabla T_0\|_0 + \sigma h\left\|\nabla(T-T_h^0)\right\|_0\\ &+\lambda\overline{N}\left\|\nabla(u-u_h^0)\right\|_0\left\|\nabla(T-T_h^0)\right\|_0.\end{aligned} \tag{5.49}$$

Using (5.5), (5.21), (5.47), we can obtain

$$\left\|\nabla\xi_h^1\right\|_0 \le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch^2. \tag{5.50}$$

Using (5.2) and triangle inequality, we can have

$$\begin{aligned}\left\|\nabla(T-T_h^1)\right\|_0 &\le \|\nabla(T-\widetilde{r}_hT)\|_0 + \left\|\nabla\xi_h^1\right\|_0\\ &\le Ch^r\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch^2,\\ \left\|T-T_h^1\right\|_0 &\le \|T-\widetilde{r}_hT\|_0 + \left\|\xi_h^1\right\|_0 \le \|T-\widetilde{r}_hT\|_0 + C_0\left\|\nabla\xi_h^1\right\|_0\\ &\le Ch^{r+1}\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch^2.\end{aligned} \tag{5.51}$$

Taking $\varphi_h = 0$, $v_h = R_h - u_h^1$ in the first equation of (5.43) and using (2.3), we have

$$\begin{aligned}\beta\left\|\eta_h^1\right\|_0 \le{}& (\nu+\sigma h)\left\|\nabla e_h^1\right\|_0 + \sigma h\left\|\nabla(u-u_h^0)\right\|_0 + 2N\left\|\nabla u_h^0\right\|\left\|\nabla(u-u_h^1)\right\|_0\\ &+C_0\lambda\left\|T-T_h^0\right\|_0 + N\left\|\nabla(u-u_h^0)\right\|_0^2.\end{aligned} \tag{5.52}$$

By (4.1), (5.21), and (5.47), we can deduce

$$\left\|\eta_h^1\right\|_0 \le Ch^r\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch^2. \tag{5.53}$$

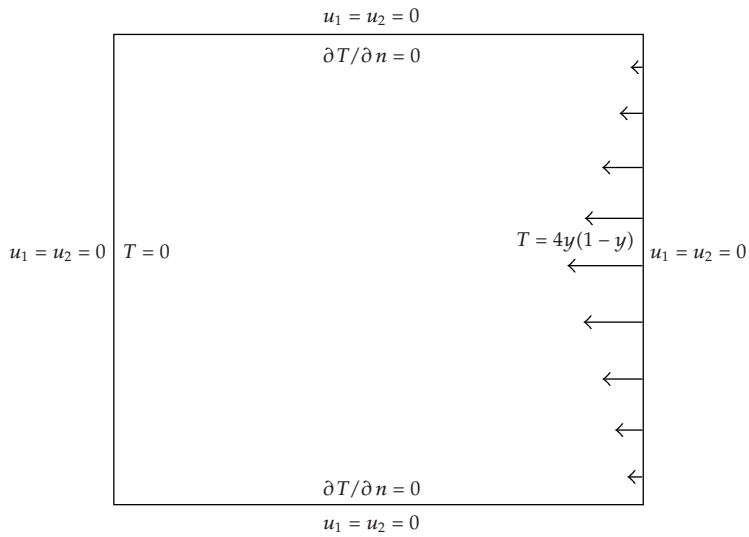

Figure 1: Physics model of the cavity flows.

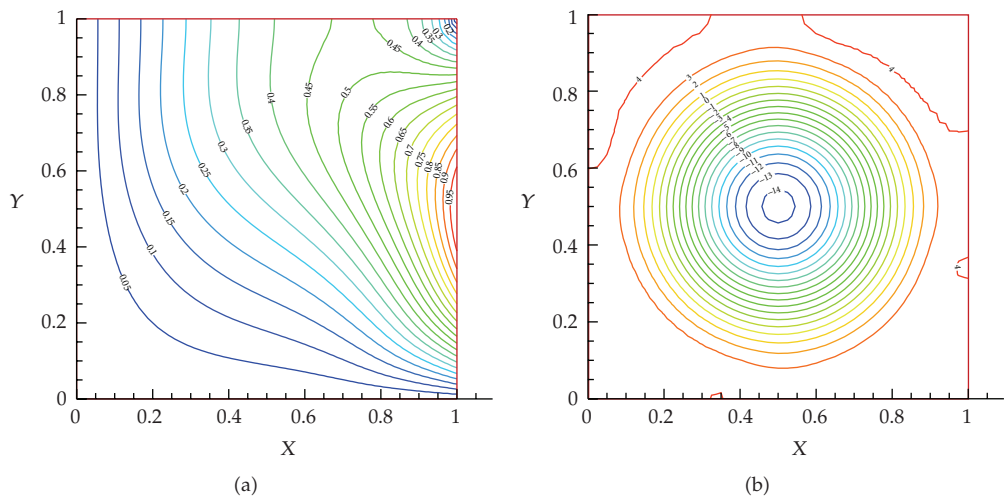

Figure 2: The numerical Isotherms (a) and the numerical Isobar (b) for $\nu = 1/2000$ by the defect-correction MFEM with $h = \sqrt{2}/40$, $\sigma = 0.4$.

Using (5.2) and triangle inequality, we can have

$$\left\|p - p_h^1\right\|_0 \le \|p - Q_h\|_0 + \left\|\eta_h^1\right\|_0 \le Ch^r\left(\|u\|_{r+1} + \|p\|_r + \|T\|_{r+1}\right) + Ch^2. \tag{5.54}$$

□

6. Numerical Experiments

In this section, we present some numerical examples with a physical model of square cavity stationary flow. We choose different ν for comparison. The side length of the square cavity and the boundary conditions are given by Figure 1. From Figure 1, we can see that the $T = 0$

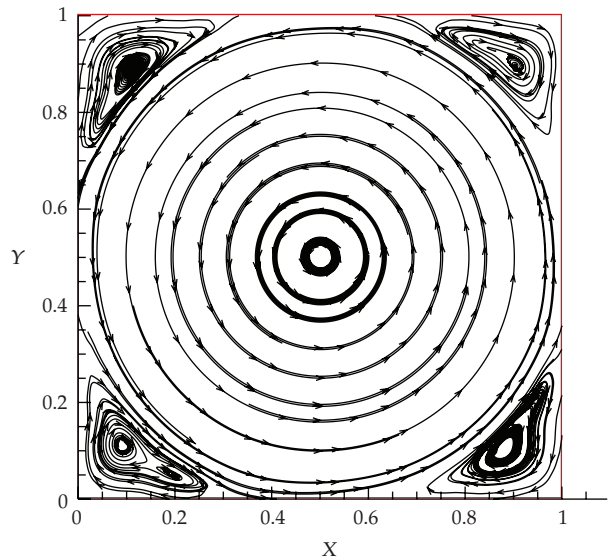

Figure 3: The numerical streamline for $\nu = 1/2000$ by the defect-correction MFEM with $h = \sqrt{2}/40$, $\sigma = 0.4$.

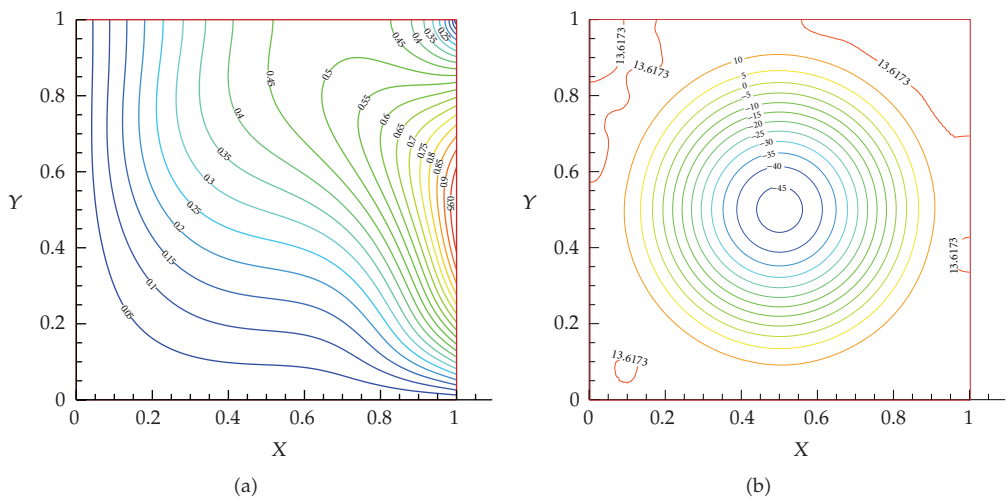

Figure 4: The numerical Isotherms (a) and the numerical Isobar (b) for $\nu = 1/5000$ by the defect-correction MFEM with $h = \sqrt{2}/100$, $\sigma = 0.4$.

on left and lower boundaries, $\partial T/\partial n = 0$ on upper boundary, and $T = 4y(1-y)$ on right boundary of the cavity. We use $P_2 - P_1 - P_2$ finite element here.

Firstly, we choose $\nu = 1/2000$, $\sigma = 0.4$ and divide the cavity into $M \times N = 40 \times 40$, that is, $h = \sqrt{2}/40$. Figure 2 gives the numerical isotherms (a) and the numerical isobar (b). Figure 3 gives the numerical streamline. From the numerical results, we can see that our method is stable and has a good precision.

Secondly, we choose $\nu = 1/5000$, $\sigma = 0.4$ to show our our method suiting for solving the conduction convection problems with small viscosity. It is well known that it is more and more difficult to solve the problem by numerical method as ν changing smaller and smaller.

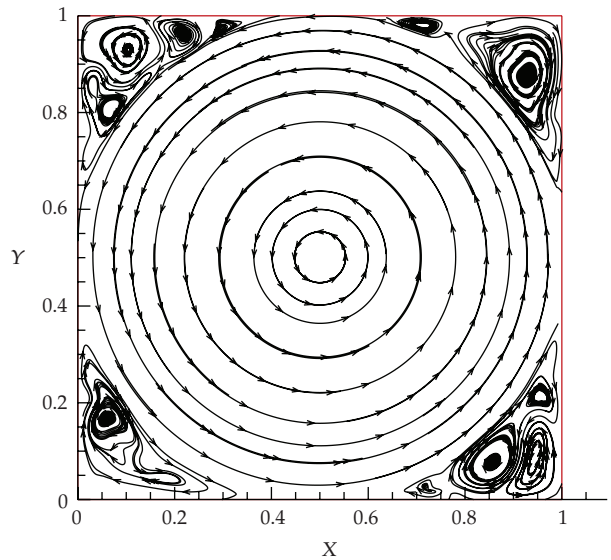

Figure 5: The numerical streamline for $\nu = 1/5000$ by the defect-correction MFEM with $h = \sqrt{2}/100$, $\sigma = 0.4$.

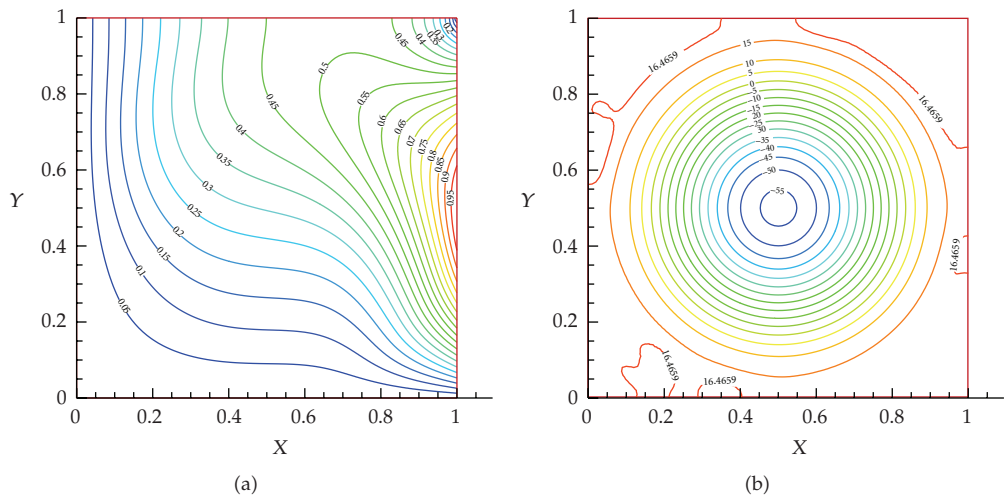

Figure 6: The numerical Isotherms (a) and the numerical Isobar (b) for $\nu = 1/6000$ by the defect-correction MFEM with $h = \sqrt{2}/100$, $\sigma = 0.4$.

Hence, we divide the cavity into $M \times N = 100 \times 100$, namely $h = \sqrt{2}/100$. Figure 4 gives the numerical isotherms (a) and the numerical isobar (b), and Figure 5 shows the numerical streamline. At last, we choose $\nu = 1/6000$, $\sigma = 0.4$. Figure 6 gives the numerical isotherms (a) and the numerical isobar (b), and Figure 7 shows the numerical streamline.

Just as Remark 3.1, we only use one correction step in our numerical experiments. From the numerical, we can see that when $\nu = 0.5 \times 10^{-3}$ the numerical streamline is very regular. The pressure is small near the wall. But the numerical streamline changes more and more immethodical with ν changing smaller and smaller. And the pressure changes bigger

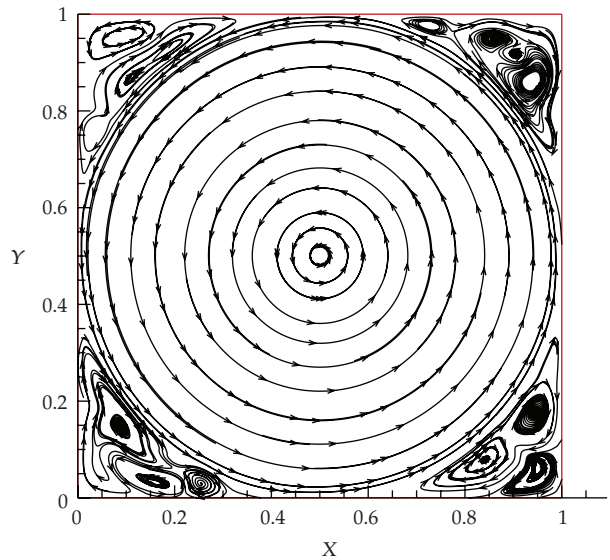

Figure 7: The numerical streamline for $\nu = 1/6000$ by the defect-correction MFEM with $h = \sqrt{2}/100$, $\sigma = 0.4$.

near the wall. In conclusion, the defect-correction MFEM is highly efficient for the stationary conduction-convection problems and it can be used for solving the convection-conduction problems with much small viscosity.

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