A note on the total domination number of a tree

1Mustapha Chellali and 2Teresa W. Haynes

1Department of Mathematics, University of Blida.
B.P. 270, Blida, Algeria.
E-mail: m_chellali@yahoo.com

2Department of Mathematics, East Tennessee State University
Johnson City, TN 37614 USA
E-mail: haynes@mail.etsu.edu

Abstract
A set \( S \) of vertices is a total dominating set of a graph \( G \) if every vertex of \( G \) is adjacent to some vertex in \( S \). The minimum cardinality of a total dominating set is the total domination number \( \gamma_t(G) \). We show that for a nontrivial tree \( T \) of order \( n \) and with \( \ell \) leaves, \( \gamma_t(T) \geq (n + 2 - \ell)/2 \), and we characterize the trees attaining this lower bound.

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1 Introduction
In a graph \( G = (V, E) \), the open neighborhood of a vertex \( v \in V \) is \( N(v) = \{u \in V \mid uv \in E\} \), and the closed neighborhood is \( N(v) \cup \{v\} \). The degree of a vertex \( v \) denoted by \( \text{deg}_G(v) \) is the cardinality of its open neighborhood. A leaf of a tree \( T \) is a vertex of degree one, while a support vertex of \( T \) is a vertex adjacent to a leaf.

A subset \( S \subseteq V \) is a dominating set of \( G \) if every vertex in \( V - S \) has a neighbor in \( S \) and is a total dominating set, abbreviated TDS, if every vertex in \( V \) has a neighbor in \( S \). The domination number \( \gamma(G) \) (respectively, total domination number \( \gamma_t(G) \)) is the minimum cardinality of a dominating set (respectively, total dominating set) of \( G \). A total dominating set of \( G \) with
minimum cardinality is called a \( \gamma_t(G) \)-set. Total domination was introduced by Cockayne, Dawes and Hedetniemi [2]. For a comprehensive survey of domination in graphs and its variations, see [3, 4].

Recently, the authors [1] showed that every tree \( T \) of order \( n \geq 3 \) and with \( s \) support vertices satisfies \( \gamma_t(T) \leq (n + s)/2 \). In this note we give a lower bound on the total domination number of a tree \( T \) in terms of the order \( n \) and the number of leaves \( \ell \), namely, \( \gamma_t(T) \geq (n + 2 - \ell)/2 \), and we characterize the extremal trees. Note that Lemańska [5] proved that \( \gamma(T) > (n + 2 - \ell)/3 \) for every tree \( T \) of order at least three.

## 2 Main results

Before presenting our main results, we make a couple of straightforward observations.

**Observation 1** If \( v \) is a support vertex of a graph \( G \), then \( v \) is in every \( \gamma_t(G) \)-set.

**Observation 2** For any connected graph \( G \) with diameter at least three, there exists a \( \gamma_t(G) \)-set that contains no leaves of \( G \).

In order to characterize extremal trees attaining our lower bound, we define the family \( T \) of trees to consist of all trees \( T \) that can be obtained from a sequence \( T_1, T_2, \ldots, T_k \) (\( k \geq 1 \)) of trees such that \( T_1 \) is the path \( P_4 \) with support vertices \( x \) and \( y \), \( T = T_k \), and, if \( k \geq 2 \), \( T_{i+1} \) can be obtained recursively from \( T_i \) by one of the following operations. Let \( A(T_1) = \{x, y\} \) and \( H \) be a path \( P_4 \) with support vertices \( u \) and \( v \).

- **Operation** \( O_1 \): Attach a vertex by adding an edge to any vertex of \( A(T_i) \). Let \( A(T_{i+1}) = A(T_i) \).
- **Operation** \( O_2 \): Attach a copy of \( H \) by adding an edge from a leaf of \( H \) to any leaf in \( T_i \). Let \( A(T_{i+1}) = A(T_i) \cup \{u, v\} \).
- **Operation** \( O_3 \): Attach a copy of \( H \) by adding a new vertex \( w \) and edges \( uw \) and \( wz \), where \( z \) is a leaf of \( T_i \). Let \( A(T_{i+1}) = A(T_i) \cup \{u, v\} \).

**Lemma 3** If \( T \in T \), then \( A(T) \) is a \( \gamma_t(T) \)-set of size \( (n + 2 - \ell)/2 \).

**Proof.** We use the terminology of the construction for the tree \( T = T_k \), the set \( A(T) \), and the graph \( H \) with support vertices \( u \) and \( v \). To show
that $A(T)$ is a $\gamma_t(T)$-set of cardinality $(n + 2 - \ell)/2$, we use induction on the number of operations $k$ performed to construct $T$. The property is true for $T_1 = P_4$. Suppose the property is true for all trees of $T$ constructed with $k - 1 \geq 0$ operations. Let $T = T_k$ with $k \geq 2$, $D$ be a $\gamma_t(T)$-set, and $T' = T_{k-1}$. Assume that $T'$ has order $n'$ and $\ell'$ leaves.

If $T$ was obtained from $T'$ by Operation $O_1$, then $\gamma_{pr}(T) = \gamma_{pr}(T')$, $n = n' + 1$, and $\ell = \ell' + 1$. By induction on $T'$, $A(T') = A(T)$ is a $\gamma_t(T)$-set of cardinality $(n + 2 - \ell)/2$.

Assume now that $T$ was obtained from $T'$ using Operation $O_2$ or $O_3$. Then we have $n = n' + 4$ and $\ell = \ell'$ or $n = n' + 5$ and $\ell = \ell' + 1$, respectively. Since $A(T) = A(T') \cup \{u, v\}$ is a TDS of $T$, $\gamma_t(T) \leq |A(T)| = \gamma_t(T') + 2$. Now by Observations 1 and 2, $D$ contains $u$ and $v$, and, without loss of generality, $D$ contains no neighbor of $u$ besides $v$, for otherwise it can be replaced by a vertex of $T'$. Thus, $D - \{u, v\}$ is a TDS of $T'$ and $\gamma_t(T') \leq \gamma_t(T) - 2$. It follows that $\gamma_t(T) = \gamma_t(T') + 2$ and $A(T)$ is a $\gamma_t(T)$-set. By induction on $T'$, it is routine matter to check that $|A(T)| = (n + 2 - \ell)/2$.

We now are ready to establish our main result.

**Theorem 4** If $T$ is a nontrivial tree of order $n$ and with $\ell$ leaves, then $\gamma_t(T) \geq (n + 2 - \ell)/2$ with equality if and only if $T \in T$.

**Proof.** If $T \in T$, then by Lemma 3, $\gamma_t(T) = (n + 2 - \ell)/2$. To prove that if $T$ is a tree of order $n \geq 2$, then $\gamma_t(T) \geq (n + 2 - \ell)/2$ with equality only if $T \in T$, we perform by induction on the order $n$. If $diam(T) \in \{1, 2\}$, then $\gamma_t(T) = 2 > (n + 2 - \ell)/2$. If $diam(T) = 3$, then $T$ is a double star where $T \in T$ and $\gamma_t(T) = (n + 2 - \ell)/2$. In this case if $T$ is different from $T_1 = P_4$, then it can be obtained from $T_1$ by using Operation $O_1$. This establishes the base cases.

Assume that every tree $T'$ of order $2 \leq n' < n$ and with $\ell'$ leaves satisfies $\gamma_t(T') \geq (n' + 2 - \ell'/2)$ with equality only if $T' \in T$. Let $T$ be a tree of order $n$ with $\ell$ leaves.

If any support vertex, say $x$, of $T$ is adjacent to two or more leaves, then let $T'$ be the tree obtained from $T$ by removing a leaf adjacent to $x$. Then $\gamma_t(T') = \gamma_t(T), n' = n - 1$, and $\ell' = \ell - 1$. Applying the inductive hypothesis to $T'$, we obtain the desired inequality. Further if $\gamma_t(T) = (n + 2 - \ell)/2$, then $\gamma_t(T') = (n + 2 - \ell)/2 = (n' + 2 - \ell')/2$, and $T' \in T$. Thus, $T \in T$ and is obtained from $T'$ by using Operation $O_1$. Henceforth, we can assume that every support vertex of $T$ is adjacent to exactly one leaf.

We now root $T$ at a vertex $r$ of maximum eccentricity $diam(T) \geq 4$. Let $v$ be a support vertex at maximum distance from $r$, $u$ be the parent of $v$,
and $w$ be the parent of $u$ in the rooted tree. Note that $\deg_T(w) \geq 2$. Let $S$ be a $\gamma_t(T)$-set that contains no leaves. Denote by $T_v$ the subtree induced by a vertex $v$ and its descendants in the rooted tree $T$. We distinguish between two cases.

**Case 1.** $\deg_T(u) \geq 3$. Then either $u$ has a child $b \neq v$ that is a support vertex or every child of $u$ except $v$ is a leaf.

Suppose first that $u$ has a child $b \neq v$ that is a support vertex. Let $T' = T - T_v$. Then $n' = n - 2 \geq 4$ and $\ell' = \ell - 1$. By Observation 1, $v$ and $b$ are in $S$. Observation 2 and our choice of $S$ imply that $S$ contains $u$. Therefore $S - \{v\}$ is a TDS of $T'$ and $\gamma_t(T') \leq \gamma_t(T) - 1$. Again by Observation 2, there is a $\gamma_t(T')$-set that contains $b$ and $u$, and such a set can be extended to a TDS of $T$ by adding $v$. Hence $\gamma_t(T) \leq \gamma_t(T') + 1$ implying that $\gamma_t(T') = \gamma_t(T) - 1$. By induction on $T'$, we have $\gamma_t(T) = \gamma_t(T') + 1 \geq (n' + 2 - \ell')/2 + 1 = (n + 2 - \ell + 1)/2$. Thus $\gamma_t(T) > (n + 2 - \ell)/2$.

Now assume that every child of $u$ except $v$ is a leaf. Since $u$ is adjacent to exactly one leaf, $\deg_T(u) = 3$. If $\deg_T(w) \geq 3$, then let $T' = T - T_u$. Then $n' = n - 4 \geq 3$, $\ell' = \ell - 2$, and $\gamma_t(T) \leq \gamma_t(T') + 2$ since any $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding the set $\{u, v\}$. Also since $\deg_T(w) \geq 3$, $w$ is a support vertex or $w$ has a descendant $x \neq u$ that is a support vertex. By our choice of $v$, the vertex $x$ is at distance at most two from $w$. In any case Observation 1 and our choice $S$ imply that $w$ is total dominated by $S - \{u, v\}$, and hence $\gamma_t(T') \leq \gamma_t(T) - 2$. It follows that $\gamma_t(T) = \gamma_t(T') - 2$. By induction on $T'$, we obtain $\gamma_t(T) = \gamma_t(T') + 2 \geq (n' + 2 - \ell')/2 + 2 = (n + 2 - \ell + 2)/2 > (n + 2 - \ell)/2$.

If $\deg_T(w) = 2$, then let $T' = T - T_w$. Then $n' = n - 5 \geq 1$. If $n' = 1$, then $T$ is a corona of $P_3$, where $\gamma_t(T) = 3 > (n + 2 - \ell)/2$. Thus we assume that $n' \geq 2$ and so $\ell - 1 \geq \ell'$. Then $S$ contains $v$ and $u$, and without loss of generality, $w \notin S$ (else substitute $w$ by a vertex from the closed neighborhood of the parent of $w$). Hence, $S - \{u, v\}$ is a TDS of $T'$ and $\gamma_t(T') \leq \gamma_t(T) - 2$. Also $\gamma_t(T) \leq \gamma_t(T') + 2$ since every $\gamma_t(T')$-set can be extended to a TDS of $T$ by adding $\{u, v\}$. It follows that $\gamma_t(T') = \gamma_t(T) - 2$. Now by induction on $T'$, we obtain $\gamma_t(T) = \gamma_t(T') + 2 \geq (n' + 2 - \ell')/2 + 2 = (n + 2 - \ell)/2$.

Further if $\gamma_t(T) = (n + 2 - \ell)/2$, then we have equality throughout this inequality chain. In particular, $\gamma_t(T') = (n' + 2 - \ell')/2$ and $\ell - 1 = \ell'$, that is, the parent of $w$ in $T$ is a leaf in $T'$. Thus by the inductive hypothesis on $T'$, $T' \in T$. Since $T$ is obtained from $T'$ by using Operation $O_3$, it follows that $T \in T$.

**Case 2.** $\deg_T(u) = 2$. If $\deg_T(w) \geq 3$, then let $T' = T - T_u$. Clearly, $n' = n - 3$ and $\ell' = \ell - 1$. Using an argument similar to one in Case 1, it is straightforward to show that $\gamma_t(T) = \gamma_t(T') + 2$. By induction on $T'$, we
have $\gamma_t(T) = \gamma_t(T') + 2 \geq (n' + 2 - \ell')/2 + 2 = (n + 2 - \ell + 2)/2 > (n + 2 - \ell)/2$

Assume now that $\deg_T(w) = 2$. Let $T' = T - T_w$. Then $n' = n - 4$ and $\ell' \leq \ell$. Further we assume that $n' \geq 2$ else $T$ is path $P_5$ where $\gamma_t(T) = 3 > (n + 2 - \ell)/2$. Also, as before it is straightforward to show that $\gamma_t(T) = \gamma_t(T') + 2$. Applying the inductive hypothesis to $T'$, it follows that $\gamma_t(T) = \gamma_t(T') + 2 \geq (n' + 2 - \ell')/2 + 2 = (n + 2 - \ell)/2$.

Further if $\gamma_t(T) = (n + 2 - \ell)/2$, then we must have equality throughout this inequality chain. In particular, $\gamma_t(T') = (n' + 2 - \ell')/2$ and $\ell = \ell'$, that is, the parent of $w$ is a leaf in $T'$. Thus by the inductive hypothesis, $T' \in T$. Since $T$ is obtained from $T'$ using Operation $O_2$, it follows that $T \in T$. □

References


