Remarks on “Distributed synchronization under uncertainty: A fuzzy approach ”

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Abstract

In a recent paper by G. Oliva, S. Panzieri, and R. Setola the synchronization problem of a network of identical systems with fuzzy initial conditions is introduced, as a convenient framework to obtain a shared estimation of a distributed system when the state is affected by ambiguity and/or vagueness. In these remarks the conditions under which the results of the cited paper hold are discussed. We show that these conditions are indeed very restrictive and prevent many practical applications among which the one in the field of critical infrastructure protection reported as a case study in their paper. A technical error in the application of the proposed method to this case is discussed.

1 Introduction: assumptions needed in the proposed approach

The synchronization problem concerns the convergence of an array of identical systems to a common trajectory. This issue has been widely investigated in literature under a variety of challenging conditions related to the network topology and the presence of noise and interference [1, 2, 4]. Paper [3] considers the synchronization problem in the context of fuzzy systems. In this proposal, a specific synchronization scheme is considered for ordinary discrete-time linear systems. Subsequently, discrete-time fuzzy systems (DFS) are introduced and the authors derive a general stability condition for these systems. Using this framework, it is shown that the extension of the output feedback scheme previously proposed for ordinary systems works also for the DFS case and solves the synchronization problem of an array of identical and linear DFS. This approach is used in a case study in the area of critical infrastructure protection.

The distributed system considered in [3] is composed by $p$ identical linear systems, where each system is in the form

$$
\begin{align*}
  z_i(k+1) &= Az_i(k) + u_i(k) \\
  y_i(k) &= Cz_i(k),
\end{align*}
$$

with $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$, where $m \leq n$, $z_i, u_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}^m$. The only information available for each system is given by

$$
e_i(k) = \sum_{j=1}^{p} \gamma_{ij} (y_j(k) - y_i(k)),
$$

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where the $\gamma_{ij}$ allow the $i$-th system to communicate only with its neighbors, according to the $p \times p$ interconnection matrix. The feedback law used to synchronize the distributed system has the form

$$u_i(k) = \Omega C \sum_{j=1}^{p} \gamma_{ij} (z_j(k) - z_i(k)),$$

(3)

where $\Omega$ is a $n \times m$ matrix, whose computation solves the synchronization problem.

The computation of $\Omega$ is in general a non trivial task. In [3], Theorem 2.1, the authors state that under some additional hypotheses the problem can be greatly simplified and $\Omega C$ can be made diagonal. The hypotheses needed are that (i) all the elements in $A$ are nonnegative; (ii) the sum $\sum_{j=1}^{n} a_{ij}$ of the rows of $A$ is $\leq 1$; (iii) $CTC$ is non-singular. Restrictive as they may appear, the first two hypotheses may be verified in some special cases. The third hypothesis is however more restrictive that it appears in the formulation of [3]. In fact, if $CTC$ is non-singular it is necessarily a square $n \times n$ matrix, that is, $m = n$. This means that the output information has the same size as the full state vector. Moreover, if a square matrix $C$ is singular, $CTC$ is also singular. This is a consequence of the fact that Ker$(CTC) = \text{Ker}(C)$, therefore, the non-singularity of $CTC$ implies the non-singularity of $C$, that in turn implies that $z_k$ can be immediately reconstructed as $z_i(k) = C^{-1}y_i(k)$.

### 2 A case study to which the approach of [3] cannot be applied

Obviously, it is licit to assume that $C$ is square and non-singular, however this hypothesis should have been made explicit. As it is formulated it may give the impression that the result is more general that it actually is. Apparently, the first ones to be confused by their misleading formulation were the same authors, who in their case study try to apply their method to a non-square matrix $C$. Moreover, in an attempt to prove that $\det(CTC) \neq 0$, they incur in an additional mistake. Actually, in Section 6.3 of [3] about the case of networked systems with total information sharing they correctly assume $C = I_{2n}$, which is square and non-singular. However, in Section 6.4 devoted to the case of incomplete information sharing (even if the title is the same as the previous section, a typographic error) they propose, see eq. (51), the following $(n + 1) \times 2n$ matrix $C$

$$C = \begin{bmatrix} -I_n & (I - A)^{-1}B \\ v^T & 0, \ldots, 0 \end{bmatrix},$$

(4)

where $v^T = [1/n, \ldots, 1/n]$. Thus $CTC$ is $2n \times 2n$ and it is obvious that it has rank at most $n + 1$ and $\det(CTC) = 0$. The authors attempt to prove $\det(CTC) \neq 0$, and to this aim they write explicitly $CTC$. However, the expression (52) in [3] is wrong, as the first block is not $P = (1 + 1/n^2)I_n$ but $P = I_n + vv^T$. Writing correctly the terms one gets

$$\det(CTC) = \det(P) \det(H)^2 \det(I_n - P^{-1}),$$

(5)

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where \( H = (I - A)^{-1}B \). It is easy to see that the last term is null,

\[
P^{-1}P = I_n \Rightarrow P^{-1}(I_n + vv^T) = I_n \Rightarrow I_n - P^{-1} = P^{-1}vv^T, \quad (6)
\]

and therefore \( \det(I_n - P^{-1}) = 0 \), which we already knew, because the product of two rectangular matrices along the shorter dimension is a singular matrix.

3 Conclusions

Since as they authors of [3] state “a complete information sharing approach, although effective, represents an unfeasible solution” we are forced to conclude that the approach described in the paper cannot be used for the synchronization in the Input-output Inoperability Model. Actually, as we have highlighted in these comments, the hypotheses of Theorem 2.1 are unfortunately so restrictive that the results reported in [3] are likely to be of limited relevance to deal with real applications.

References


