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# One-parameter Potential from Darboux Theorem

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**Abstract:** We consider the stationary one-dimensional Schrödinger equation with potential $u(x;i) = \sum_{j=-2}^{2} f_j(i)x^j$, where the coefficients $f_j(i)$ are functions of a discrete parameter $i$. We establish the most general form of the coefficients $f_j(i)$ and obtain the ladder operators for the solution of Schrödinger equation by a Darboux transform. Generally speaking, the Darboux transform is obtained through a so-called superpotential $W(x)$, which is derived from a Riccati equation. We first propose a convenient *ansatz* for the function $W'(x)$ and then yield a set of nine difference equations for the coefficients $f_j(i)$. This set of difference equations establishes the explicit form of the coefficients $f_j(i)$, in the potential $u(x;i)$. Our results are consistent with some well-known quantum potentials in special cases.


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## 1. Introduction

The classical Darboux theorem was formulated in the 19th century [1-2], and define basically a mapping between solutions of a pair of second-order differential equations of the same form. This mapping called Darboux transformation, is functionally parametrized by a pair of solutions of the differential equation and the transform vanishes if the solutions coincide. The Darboux transformations are very closely related to the SUSYQM, intertwining operators and inverse scattering techniques as well as other topics [3].

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In this work by applying the Darboux theorem to the Schrödinger equation with the potential $u(x;i) = \sum_{j=-2}^{2} f_j(i)x^j$ dependent of a discrete parameter $i$, different to principal quantum number $n$ since this potential is superposition of some important potentials in quantum mechanics [4], we obtain the explicit form of the coefficients $f_j(i)$ such that the ladder operators for the solution of Schrödinger equation are identified by a Darboux transform.

This paper is organized as follows. In Section 2 we briefly review the Darboux theorem. Section 3 is devoted to showing how from a convenient *ansatz* for the function involved in the Darboux transform to obtain a complicated coupling equation for the coefficients $f_j(i)$. A set of nine difference equations for coefficients $f_j(i)$ is presented. The ladder operators for corresponding solution are constructed by the Darboux transform technique. In Section 4, some particular $i$-dependent potentials and their applications in quantum mechanics are discussed. Finally, in Section 5 we give our conclusions.

## 2. Darboux Theorem

This theorem addresses that if the $W(x)$ is an arbitrary function, $\psi_n(x)$ a solution of eigenvalues equation

$$\frac{d^2}{dx^2}\psi_n(x) - 2u(x)\psi_n(x) = -2\lambda_n\psi_n(x), \tag{2.1}$$

then the function $\varphi_n(x)$ defined as

$$\varphi_n(x) \equiv B\psi_n(x), \quad B \equiv \left(\frac{d}{dx} - W(x)\right), \tag{2.2}$$

satisfies the following differential equation

$$\varphi_n''(x) - 2\left[u(x) - W'(x)\right]\varphi_n(x) = -2\lambda_n\varphi_n(x), \tag{2.3}$$

only if the expression

$$W^2(x) + W'(x) - 2u(x) \equiv k, \tag{2.4}$$

is independent of variable $x$. The prime denotes the derivative with respect to $x$. By taking into Eq.(2.4) account, Eq.(2.3) can be rewritten as

$$\varphi_n''(x) - \left[W^2(x) - W'(x)\right]\varphi_n(x) = -2\left(\lambda_n + \frac{k}{2}\right)\varphi_n(x). \tag{2.5}$$

As a result, it is shown that the inverse transformation of Eq.(2.2) becomes[1]

$$\left(\frac{d}{dx} + W(x)\right)\varphi_n(x) = -2\left(\lambda_n + \frac{k}{2}\right)\psi_n(x). \tag{2.6}$$

[1] It should be noted that the classical Darboux theorem can also be established if a solution $\psi_m(x)$ of Eq.(2.1) is known *a priory*. Actually, it is shown from the definition $W(x) = \psi_m'(x)/\psi_m(x)$ that $k = -2\lambda_m$ in Eq.(2.4). If taking $\psi_m(x) \equiv e^{v(x)}$, Eq.(2.1) is thus transformed to a Riccati equation $w^2(x) + w'(x) - 2u(x) = -2\lambda_n, w \equiv v'$. Nevertheless, throughout this paper the function $W(x)$ is not connected *a priory* with some eigenfunction of Eq.(2.1), as made in the classical Darboux transform.

## 3. Ladder Operators for One-parameter Potential

It is well known that in quantum mechanics the function $u(x)$ given in Eq.(2.1) may be dependent of a parameter $i$, but independent of the principal quantum number $n$. If so, the $u(x)$ and $\psi_n(x)$ shown above should be denoted as $u(x;i)$ and $\psi_n^{(i)}(x)$ for clearness.

Let us consider the following *ansatz* for $W'(x)$

$$W'(x) = U(x) + \Lambda, \tag{3.1}$$

where

$$U(x) \equiv u(x;i) - u(x;i+\beta), \qquad \Lambda \equiv \lambda_{n+\alpha} - \lambda_n, \quad \alpha, \beta \in \text{integer}. \tag{3.2}$$

If this *ansatz* satisfies Eq.(2.4), then substituting it into Eq.(2.5) leads to

$$\varphi_n''(x) - 2u(x;i+\beta)\varphi_n(x) = -2\lambda_{n+\alpha}\varphi_n(x). \tag{3.3}$$

Consequently, the function $\varphi_n(x)$ must be identified as eigenfunction $\psi_{n+\alpha}^{i+\beta}(x)$, and the operator $B$ given in Eq.(2.2) becomes a ladder operator acting on the parameters $i$ and $n$ of function $\psi_n^{(i)}(x)$, with steps $\beta$ and $\alpha$, respectively, *i.e.*

$$\psi_{n+\alpha}^{(i+\beta)}(x) = \left(\frac{d}{dx} - W(x)\right)\psi_n^{(i)}(x). \tag{3.4}$$

The different $u(x;i)$ corresponds to different ladder operators. For potential application in physics, we first consider a general potential with the form

$$u(x;i) = \sum_{j=-q}^{n} f_j(i)x^j, \tag{3.5}$$

where the coefficients $f_j(i)$ are to be determined. By integrating Eq.(3.1) we get

$$W\ (x) = \int U(x)dx + \int \Lambda dx = \sum_{\substack{j=-q \\ j\neq -1}}^{n} \frac{F_j(i)}{j+1}x^{j+1} + F_{-1}(i)\ln x + \Lambda x + C, \tag{3.6}$$

where

$$U(x) = \sum_{j=-q}^{n} F_j(i)x^j, \quad F_j(i) \equiv f_j(i) - f_j(i+\beta), \quad C = \text{constant}. \tag{3.7}$$

If $f_{-1}(i)$ taken as the numerical value $\mathfrak{f}_{-1}$, then the logarithmic term in Eq.(3.6) drops out due to Eq.(3.7). By using Eqs.(3.1), (3.2), (3.5) and (3.6), we are able to explicitly express Eq. (2.4) as follows:

$$\begin{aligned}&\left(\sum_{\substack{j=-q\\j\neq-1}}^{n}\frac{F_j(i)}{j+1}x^{j+1}\right)^2+\Lambda^2x^2+C^2\\&+2\left(\sum_{\substack{j=-q\\j\neq-1}}^{n}\frac{F_j(i)}{j+1}x^{j+1}\right)\Lambda x+2\Lambda Cx+2\left(\sum_{\substack{j=-q\\j\neq-1}}^{n}\frac{F_j(i)}{j+1}x^{j+1}\right)C\\&+\sum_{\substack{j=-q\\j\neq-1}}^{n}F_j(i)x^j+\Lambda-2\sum_{j=-q}^{n}f_j(i)x^j=k.\end{aligned} \tag{3.8}$$

This is a rather complicated coupling equation for the coefficients $f_j(i)$. For clearness, in this work we attempt to study the particular case

$$u(x;i)=\sum_{\substack{j=-2\\j\neq-1}}^{2}f_j(i)x^j+\mathfrak{f}_{-1}x^{-1}. \tag{3.9}$$

In this case, Eq.(3.8) yields immediately a set of nine difference equations for $f_j(i)$ by equating the coefficients of $x^j$

$$(F_{-2})^2+F_{-2}-2f_{-2}(i)=0, \tag{3.10}$$

$$-F_{-2}C-\mathfrak{f}_{-1}=0, \tag{3.11}$$

$$-2F_{-2}F_0+C^2-2F_{-2}\Lambda+F_0+\Lambda-2f_0(i)=k, \tag{3.12}$$

$$-F_{-2}F_1+2\Lambda C+2F_0C+F_1-2f_1(i)=0, \tag{3.13}$$

$$(F_0)^2-\frac{2}{3}F_{-2}F_2+\Lambda^2+2F_0\Lambda+F_1C+F_2-2f_2(i)=0, \tag{3.14}$$

$$F_0F_1+F_1\Lambda+\frac{2}{3}F_2C=0, \tag{3.15}$$

$$\frac{(F_1)^2}{4}+\frac{2}{3}F_0F_2+\frac{2}{3}F_2\Lambda=0, \tag{3.16}$$

$$\frac{1}{3}F_1F_2=0, \tag{3.17}$$

$$\frac{(F_2)^2}{9}=0. \tag{3.18}$$

From the last four equations we deduce $F_2=F_1=0$, or equivalently by Eq.(3.7) $f_2(i)$ and $f_1(i)$ are the constants $\mathfrak{f}_2$ and $\mathfrak{f}_1$, respectively. The equations (3.10)-(3.14) are reduced to

$$(F_{-2})^2+F_{-2}-2f_{-2}(i)=0, \tag{3.19}$$

$$F_{-2}C=-\mathfrak{f}_{-1}, \tag{3.20}$$

$$(-2F_{-2}+1)(F_0+\Lambda)+C^2-2f_0(i)=k, \tag{3.21}$$

$$(F_0 + \Lambda)\, C = \mathfrak{f}_1, \tag{3.22}$$

$$(F_0 + \Lambda)^2 = 2\mathfrak{f}_2. \tag{3.23}$$

It is shown from Eq.(3.19) that $f_{-2}(i)$ must be zero if it is any constant function and from Eq.(3.20) that the coefficient $\mathfrak{f}_{-1} = 0$. Thus, the term $f_{-2}(i)$ is in general zero or equal to

$$f_{-2}^{\pm}(i;\beta) = \frac{1}{2}\left(\frac{i}{\beta}\right)^2 + b_{\pm}\left(\frac{i}{\beta}\right) + c, \quad \beta \neq 0, \tag{3.24}$$

where

$$b_{\pm} = \pm\sqrt{\frac{1}{4} + 2c}; \quad c \geq -\frac{1}{8}, \tag{3.25}$$

since this is the solution of difference equation (3.19) [5]. Note that

$$f_{-2}^{+}(i;-\beta) = f_{-2}^{-}(i;\beta), \tag{3.26}$$

so if choosing

$$f_{-2}(i) \equiv f_{-2}^{+}(i;\beta), \tag{3.27}$$

then we must consider the case $-\beta$. To do this we write the difference $F_{-2}$ in Eq.(3.7) as

$$F_{-2}^{\pm} = f_{-2}(i) - f_{-2}(i \pm \beta) = \mp\frac{1}{2}\left(\frac{2i}{\beta} \pm 1\right) \mp b_{+}. \tag{3.28}$$

Additionally, equation (3.23) implies that $F_0$ is independent of the parameter $i$, because as indicated in Eq.(3.2), $\Lambda$ does not depend on $i$. For this reason, $f_0(i)$ has the form

$$f_0(i) = -\frac{d}{\beta} i + \mathfrak{f}_0; \quad F_0 \equiv d. \tag{3.29}$$

The solution of Eq.(3.23) is

$$\Lambda_{\pm} = -d \pm \sqrt{2\mathfrak{f}_2}; \quad \mathfrak{f}_2 = 0 \text{ or } \mathfrak{f}_2 > \frac{d^2}{2}. \tag{3.30}$$

Hence, the second difference equation (3.2) implies

$$\Lambda_{+} = \lambda_{n+\alpha} - \lambda_n, \quad \Lambda_{-} = \lambda_{n+\widetilde{\alpha}} - \lambda_n, \tag{3.31}$$

from which we deduce that

$$\lambda_n = \frac{\Lambda_{+}}{\alpha} n + \lambda_0, \tag{3.32}$$

and

$$\widetilde{\alpha} = \frac{\Lambda_{-}}{\Lambda_{+}}\alpha. \tag{3.33}$$

As shown above, we have considered Eq.(2.1) with given $u(x;i)$ in Eq.(3.9) and found that the operators defined in Eq.(3.4) are nothing but the ladder operators of the function $\psi_n^{(i)}(x)$ if and only if $u(x;i)$ has the following form

$$u(x;i) = f_0(i) + \mathfrak{f}_1 x + \mathfrak{f}_2 x^2, \tag{3.34}$$

or

$$u(x;i) = f_{-2}(i)x^{-2} + \mathfrak{f}_{-1}x^{-1} + f_0(i) + \mathfrak{f}_1 x + \mathfrak{f}_2 x^2, \tag{3.35}$$

where $\mathfrak{f}_{-1}$, $\mathfrak{f}_1$, $\mathfrak{f}_2$ are real numbers, $f_{-2}(i)$ and $f_0(i)$ are the functions of the parameter $i$, given in Eq.(3.27) and Eq.(3.29), such that they make Eqs.(3.19)-(3.23) consistent. Moreover, Eqs.(3.24) and (3.30) imply, in principle, that there are four ladder operators contained in Eq.(3.4). To clarify this, we consider specific examples below.

# 4. Discussions and Applications

## 4.1 Potential with $f_{-2}(i) = 0$

Substituting Eq.(3.30) into Eq.(3.22) yields

$$C_\pm = \frac{\mathfrak{f}_1}{\Lambda_\pm + d} = \frac{\mathfrak{f}_1}{\pm\sqrt{2\mathfrak{f}_2}}; \quad \mathfrak{f}_2 \neq 0. \tag{4.1}$$

Considering this and substituting Eq.(3.30) into Eq.(3.21) allow us to obtain

$$k_\pm = -2f_0(i) + \frac{(\mathfrak{f}_1)^2}{2\mathfrak{f}_2} \pm \sqrt{2\mathfrak{f}_2}. \tag{4.2}$$

As a result, if taking $u(x;i)$ as the form of Eq.(3.34), the function

$$\varphi_{n(\pm)}(x) = \left(\frac{d}{dx} - W_\pm(x)\right)\psi_n^{(i)}(x), \tag{4.3}$$

with (see Eq.(3.6))

$$W_\pm(x) = \pm\sqrt{2\mathfrak{f}_2}x + C_\pm, \tag{4.4}$$

satisfies Eq. (2.5)

$$\begin{aligned} &\varphi''_{n(\pm)}(x) - 2\left[\frac{(\mathfrak{f}_1)^2}{4\mathfrak{f}_2} + \mathfrak{f}_1 x + \mathfrak{f}_2 x^2 \mp \frac{\sqrt{2\mathfrak{f}_2}}{2}\right]\varphi_{n(\pm)}(x) = \\ &-2\left(\lambda_n - f_0(i) + \frac{(\mathfrak{f}_1)^2}{4\mathfrak{f}_2} \pm \frac{\sqrt{2\mathfrak{f}_2}}{2}\right)\varphi_{n(\pm)}(x), \quad \mathfrak{f}_2 \neq 0, \end{aligned} \tag{4.5}$$

which can be rearranged as

$$\varphi''_{n(\pm)}(x) - 2\left[f_0(i \pm \beta) + \mathfrak{f}_1 x + \mathfrak{f}_2 x^2\right]\varphi_{n(\pm)}(x) = -2(\lambda_n \pm \Lambda_+)\varphi_{n(\pm)}(x). \tag{4.6}$$

This can be directly identified as

$$\varphi_{n(\pm)}(x) = \psi^{(i)}_{\lambda_n \pm \sqrt{2f_2}}(x). \tag{4.7}$$

In particular, when $f_0(i) = 0$, $\mathfrak{f}_1 = 0$ (or 1), and $\mathfrak{f}_2 = 1/2$, equation (2.1) is the Schrödinger equation for a simple one-dimensional harmonic oscillator. The substitution of Eqs.(4.1) and (4.4) into Eq.(4.3) leads to well-known raising and lowering operators of this system.

## 4.2 Potential with $f_{-2}(i) \neq 0$ and $\mathfrak{f}_{-1} = 0$

Due to Eq. (3.28), we have for Eq. (3.20)

$$F_{-2}^{\pm} C = -\mathfrak{f}_{-1}. \tag{4.8}$$

By taking into Eq.(3.30) account, Eq.(3.22) becomes

$$\pm\sqrt{2\mathfrak{f}_2} C = \mathfrak{f}_1. \tag{4.9}$$

At first glance, these equations sound paradoxical, since Eq.(4.8) implies that $C$ is function of parameter $i$, while Eq.(4.9) shows that $C$ is independent of parameter $i$. In such a case, the only constraint imposed is the consistence of these equations. For example, if $\mathfrak{f}_{-1} = 0$ then $C = 0$ and $\mathfrak{f}_1 = 0$, Eq.(3.21) can be written as

$$k_{+\pm} = +\sqrt{2\mathfrak{f}_2}(1 - 2F_{-2}^{\pm}) - 2f_0(i), \tag{4.10}$$

$$k_{-\pm} = -\sqrt{2\mathfrak{f}_2}(1 - 2F_{-2}^{\pm}) - 2f_0(i). \tag{4.11}$$

Therefore, by taking

$$u(x;i) = f_{-2}(i)x^{-2} + f_0(i) + \mathfrak{f}_2 x^2, \tag{4.12}$$

in Eq.(2.1), the functions

$$\varphi_{n(+\pm)}^{(i)}(x) = \left(\frac{d}{dx} - W_{+\pm}(x)\right)\psi_n^{(i)}(x), \tag{4.13}$$

$$\varphi_{n(-\pm)}^{(i)}(x) = \left(\frac{d}{dx} - W_{-\pm}(x)\right)\psi_n^{(i)}(x), \tag{4.14}$$

with (see Eq.(3.6) )

$$W_{+\pm}(x) = -F_{-2}^{\pm}x^{-1} + \sqrt{2\mathfrak{f}_2}x, \tag{4.15}$$

$$W_{-\pm}(x) = -F_{-2}^{\pm}x^{-1} - \sqrt{2\mathfrak{f}_2}x, \tag{4.16}$$

satisfy (see Eq.(2.5))

$$\varphi_{n(+\pm)}^{(i)\prime\prime}(x) - \left[W_{+\pm}^2(x) - W_{+\pm}'(x)\right]\varphi_{n(+\pm)}^{(i)}(x) = -2\left(\lambda_n + \frac{k_{+\pm}}{2}\right)\varphi_{n(+\pm)}^{(i)}(x), \tag{4.17}$$

$$\varphi_{n(-\pm)}^{(i)\prime\prime}(x) - \left[W_{-\pm}^2(x) - W_{-\pm}'(x)\right]\varphi_{n(-\pm)}^{(i)}(x) = -2\left(\lambda_n + \frac{k_{-\pm}}{2}\right)\varphi_{n(-\pm)}^{(i)}(x), \tag{4.18}$$

or explicitly

$$\varphi_{n(++)}^{(i)\prime\prime}(x) - 2(\Delta_{+\beta})\varphi_{n(++)}^{(i)}(x) = -2\left(\lambda_n + \Lambda_+\right)\varphi_{n(++)}^{(i)}(x), \tag{4.19}$$

$$\varphi_{n(+-)}^{(i)\prime\prime}(x) - 2(\Delta_{-\beta})\varphi_{n(+-)}^{(i)}(x) = -2\left(\lambda_n - \Lambda_-\right)\varphi_{n(+-)}^{(i)}(x), \tag{4.20}$$

$$\varphi_{n(-+)}^{(i)\prime\prime}(x) - 2(\Delta_{+\beta})\varphi_{n(-+)}^{(i)}(x) = -2\left(\lambda_n + \Lambda_-\right)\varphi_{n(-+)}^{(i)}(x), \tag{4.21}$$

$$\varphi_{n(--)}^{(i)\prime\prime}(x) - 2(\Delta_{-\beta})\varphi_{n(--)}^{(i)}(x) = -2\left(\lambda_n - \Lambda_+\right)\varphi_{n(--)}^{(i)}(x), \tag{4.22}$$

where

$$F_{-2}^{\pm}(F_{-2}^{\pm}-1)=2f_{-2}(i\pm\beta),\qquad \Delta_{\pm\beta}=f_{-2}(i\pm\beta)x^{-2}+\mathfrak{f}_2x^2+f_0(i\pm\beta) \tag{4.23}$$

are used. We immediately identify that

$$\varphi_{n(+\pm)}^{(i)}(x)=\psi_{\lambda_n\pm\Lambda_\pm}^{(i\pm\beta)}(x), \tag{4.24}$$

$$\varphi_{n(-\pm)}^{(i)}(x)=\psi_{\lambda_n\pm\Lambda_\mp}^{(i\pm\beta)}(x). \tag{4.25}$$

It should be noted that there are two cases $\mathfrak{f}_2=0$ and $\mathfrak{f}_2\neq 0$ for the potential with $f_{-2}(i)\neq 0$ and $\mathfrak{f}_{-1}=0$. The results for these two special cases are given in Appendixes A and B, respectively.

## 4.3 Potential with $f_{-2}(i)\neq 0$ and $\mathfrak{f}_{-1}\neq 0$

It is shown from Eqs.(3.28) and (4.8) that $C$ is a general function of the parameter $i$. However, this and Eq.(4.9) imply that $\mathfrak{f}_1=\mathfrak{f}_2=0$. By using Eq.(4.8) and Eq.(3.30) in Eqs.(3.21) and (3.6), we arrive at

$$k_\pm=\left(\frac{\mathfrak{f}_{-1}}{F_{-2}^{\pm}}\right)^2-2f_0(i), \tag{4.26}$$

$$W_\pm=-F_{-2}^{\pm}x^{-1}-\frac{\mathfrak{f}_{-1}}{F_{-2}^{\pm}}. \tag{4.27}$$

respectively. These equations reduce, for instance to

$$k_\pm=\left(\pm i-\frac{1}{2}\mp\frac{1}{2}\right)^{-2}-2f_0(i), \tag{4.28}$$

$$W_\pm=-\left(\pm i-\frac{1}{2}\mp\frac{1}{2}\right)x^{-1}+\frac{1}{\left(\pm i-\frac{1}{2}\mp\frac{1}{2}\right)}, \tag{4.29}$$

when

$$u(x;i)=\frac{1}{2}i(i-1)x^{-2}-x^{-1}+f_0(i). \tag{4.30}$$

Generally speaking, from Eq.(4.29) we can obtain the radial ladder operators of N-dimensional hydrogen atom, which has the effective potential [8-10]

$$V_{eff}(r)=\frac{L(L-1)}{2r^2}-\frac{1}{r};\quad L\equiv l+\frac{1}{2}(N-1). \tag{4.31}$$

## Conclusions

In this work we have carried out the one-dimensional Schrödinger equation with potential function (3.9) and the coefficient $\mathfrak{f}_{-1}$ independent of the parameter $i$ by applying the Darboux theorem. By studying Eqs.(3.19)-(3.23) we have established the coefficients

$f_{-2}(i)$ and $f_0(i)$ through Eq.(3.27) and Eq.(3.29), respectively. The remained coefficients must be constants $\mathfrak{f}_1$ and $\mathfrak{f}_2$. However, the consistence of those difference equations allows us to consider only the possibilities for $u(x;i)$ expressed in Eq.(3.34), Eq.(4.12) and potential $u(x;i) = f_{-2}(i)x^{-2} + \mathfrak{f}_{-1}x^{-1} + f_0(i)$. For example, the coefficient $f_{-2}(i)$ allows the coefficient $\mathfrak{f}_{-1}$ to appear in $u(x;i)$, but this can not be compatible with the inclusion of linear and quadratic terms.

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**Appendix A: Potential with $f_{-2}(i) \neq 0$, $\mathfrak{f}_{-1} = 0$ but $\mathfrak{f}_2 = 0$**

In this appendix, we are going to show the special results for those obtained in subsection 4.2 in the case of $\mathfrak{f}_2 = 0$. On the other hand, we shall show the results for two typical examples in quantum mechanics.

When $\mathfrak{f}_2 = 0$, the obtained results (3.30), (4.10), (4.11), (4.15), (4.16), (4.13), (4.14) are now reduced to

$$\Lambda_\pm = -d, \quad k_{+\pm} = k_{-\pm} = -2f_0(i) \equiv k_\pm, \tag{A.1}$$

$$W_{+\pm}(x) = W_{-\pm}(x) = -F^\pm_{-2}x^{-1} \equiv W_\pm(x), \tag{A.2}$$

$$\varphi^{(i)}_{n(+\pm)}(x) = \varphi^{(i)}_{n(-\pm)}(x) \equiv \varphi^{(i)}_{n(\pm)}(x), \tag{A.3}$$

and those (4.19)-(4.22) are simplified as

$$\varphi^{(i)\prime\prime}_{n(+)}(x) - 2\left[f_{-2}(i+\beta)x^{-2} + f_0(i+\beta)\right]\varphi^{(i)}_{n(+)}(x) = -2\left(\lambda_n - d\right)\varphi^{(i)}_{n(+)}(x), \tag{A.4}$$

$$\varphi^{(i)\prime\prime}_{n(-)}(x) - 2\left[f_{-2}(i-\beta)x^{-2} + f_0(i-\beta)\right]\varphi^{(i)}_{n(-)}(x) = -2\left(\lambda_n + d\right)\varphi^{(i)}_{n(-)}(x), \tag{A.5}$$

The corresponding (4.24) and (4.25) now become

$$\varphi^{(i)}_{n(+)}(x) = \psi^{(i+\beta)}_{\lambda_n - d}(x), \tag{A.6}$$

$$\varphi^{(i)}_{n(-)}(x) = \psi^{(i-\beta)}_{\lambda_n + d}(x), \tag{A.7}$$

where $\psi^{(i)}_{\lambda_n}(x)$ satisfies

$$\frac{d^2}{dx^2}\psi^{(i)}_{\lambda_n}(x) - 2\left[f_{-2}(i)x^{-2} + f_0(i)\right]\psi^{(i)}_{\lambda_n}(x) = -2\lambda_n\psi^{(i)}_{\lambda_n}(x). \tag{A.8}$$

Let us illustrate two typical examples. First, we study

$$u(x;i) = \frac{1}{2}\left(i^2 - \frac{1}{4}\right)x^{-2} + f_0(i). \tag{A.9}$$

The coefficient $f_{-2}(i)$ has the form of Eq.(3.27). For $\beta = 1$ and $c = -1/8$, Eq.(3.28) is simplified as

$$F^\pm_{-2} = \mp\frac{1}{2}\left(2i \pm 1\right). \tag{A.10}$$

It should be noted from Eq.(A.1) $k_\pm = -2f_0(i)$ and $\Lambda_\pm = -d$. By substituting $F^\pm_{-2}$ into Eq.(A.2), we get

$$W_\pm(x) = \pm\frac{1}{2}\,(2i \pm 1)\,x^{-1}. \tag{A.11}$$

So, if the function $\psi^{(i)}_n(x)$ fulfills

$$\psi^{(i)\prime\prime}_n(x) - 2\left[\frac{1}{2}\left(i^2 - \frac{1}{4}\right)x^{-2} + f_0(i)\right]\psi^{(i)}_n(x) = -2\lambda_n\psi^{(i)}_n(x), \tag{A.12}$$

then

$$\psi^{(i\pm1)}_{\lambda_n\pm d}(x) = \left(\frac{d}{dx} \mp \frac{1}{2}\,(2i \pm 1)\,x^{-1}\right)\psi^{(i)}_n(x). \tag{A.13}$$

As a consequence, when $f_0(i) = 0$, we have $k_\pm = 0$ and $\Lambda_\pm = 0$. However, based on Eq.(3.32) we have $\lambda_n \equiv \lambda_0$. Equation (2.1) can be written as

$$\frac{d^2}{dx^2}\psi^{(i)}(x) - \left(i^2 - \frac{1}{4}\right)x^{-2}\psi^{(i)}(x) = -2\lambda\psi^{(i)}(x), \quad \lambda \equiv \lambda_0. \tag{A.14}$$

For special case $\lambda = 1/2$, its solution is given by

$$\psi^{(i)}(x) = x^{1/2}J_i(x), \tag{A.15}$$

where $J_i(x)$ is the Bessel function of the first kind [6]. Generally, we are able to obtain the ladder operators acting on the order of the Bessel function from the operators given in Eq.(A.13).

Second, let us consider another case of $i$-dependent potential

$$u(x; i) = \frac{i(i-1)}{2}x^{-2} + f_0(i). \tag{A.16}$$

In comparison with Eq.(3.27), we have $\beta = -1$, $c = 0$. Eq.(3.28) now becomes

$$F^\pm_{-2} = \mp\frac{1}{2}\,(-2i \pm 1) \mp \frac{1}{2} = \pm i - \frac{1}{2} \mp \frac{1}{2}. \tag{A.17}$$

Again, by Eq.(A.1) we have $k_\pm = -2f_0(i)$ and $\Lambda_\pm = -d$. From Eqs.(A.2), (A.6) and (A.7) we can say that

$$\psi^{(i\mp1)}_n(x) = \left[\frac{d}{dx} + \left(\pm i - \frac{1}{2} \mp \frac{1}{2}\right)x^{-1}\right]\psi^{(i)}_n(x), \tag{A.18}$$

where $\psi^{(i)}_n(x)$ is solution of the following differential equation

$$\psi^{(i)\prime\prime}_n(x) - 2\left[\frac{1}{2}i\,(i-1)\,x^{-2} + f_0(i)\right]\psi^{(i)}_n(x) = -2\lambda_n\psi^{(i)}_n(x). \tag{A.19}$$

A special case occurs for $f_0(i) = 0$. In this case we have $k_\pm = 0$ and $\Lambda_\pm = 0$. The function $\psi^{(i)}_n(x)$ satisfying Eq.(2.1) for any $\lambda_n \equiv \lambda_0$ must be identified as a spherical

Bessel (Neumann) function [6]. Accordingly, the operators given in Eq.(A.18) correspond to the ladder operators acting on the order of the spherical Bessel (Neumann) function.

**Appendix B: Potential with $f_{-2}(i) \neq 0$, $\mathfrak{f}_{-1} = 0$ but $\mathfrak{f}_2 \neq 0$**

Similarly, in this appendix we are going to show the special results for those obtained in subsection 4.2 in the case of $\mathfrak{f}_2 \neq 0$. Let us consider the following potential

$$u(x;i) = \frac{1}{2}\left(i^2 - \frac{1}{4}\right)x^{-2} + f_0(i) + \frac{x^2}{2}. \tag{B.1}$$

From Eq.(3.30), one has

$$\Lambda_\pm = -d \pm 1. \tag{B.2}$$

By substituting Eq.(A.10) into Eqs.(4.10) and (4.11) we have

$$k_{+\pm} = 2\left(\pm i + 1 - f_0(i)\right), \tag{B.3}$$

$$k_{-\pm} = -2\left(\pm i + 1 - f_0(i)\right). \tag{B.4}$$

As a result, the superpotentials (4.15) and (4.16) are simplified as

$$W_{+\pm} = \left(\pm i + \frac{1}{2}\right)x^{-1} + x, \tag{B.5}$$

$$W_{-\pm} = \left(\pm i + \frac{1}{2}\right)x^{-1} - x. \tag{B.6}$$

By the way we are going to mention another potential with the form

$$u(x;i) = \frac{i(i-1)}{2}x^{-2} + f_0(i) + \frac{x^2}{2}. \tag{B.7}$$

Likewise, we have $\Lambda_\pm = -d \pm 1$. Substituting Eq.(A.17) into Eqs.(4.10) and (4.11) leads to

$$k_{+\pm} = (\mp 2i + 2 \pm 1) - 2f_0(i), \tag{B.8}$$

$$k_{-\pm} = -\left(\mp 2i + 2 \pm 1\right) - 2f_0(i). \tag{B.9}$$

The corresponding superpotentials (4.15) and (4.16) are now expressed as

$$W_{+\pm}(x) = -\left(\pm i - \frac{1}{2} \mp \frac{1}{2}\right)x^{-1} + x, \tag{B.10}$$

$$W_{-\pm}(x) = -\left(\pm i - \frac{1}{2} \mp \frac{1}{2}\right)x^{-1} - x. \tag{B.11}$$

Formally, these four operators correspond to radial ladder operators of N-dimensional isotropic harmonic oscillator defined by following effective potential

$$V_{eff}(r) = \frac{L(L-1)}{2r^2} + \frac{r^2}{2}; \quad L \equiv l + \frac{1}{2}(N-1), \tag{B.12}$$

in which $l$ is the quantum orbital number [7-9].

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