The Convexity Spectra of Graphs

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Abstract

Let $D$ be a connected oriented graph. A set $S \subseteq V(D)$ is convex in $D$ if, for every pair of vertices $x, y \in S$, the vertex set of every $x - y$ geodesic ($x - y$ shortest dipath) and $y - x$ geodesic in $D$ is contained in $S$. The convexity number $con(D)$ of a nontrivial oriented graph $D$ is the maximum cardinality of a proper convex set of $D$. Let $G$ be a graph and $S_C(G) = \{con(D) : D$ is an orientation of $G\}$ and $S_{SC}(G) = \{con(D) : D$ is a strongly connected orientation of $G\}$. In the paper, we show that, for any $n \geq 4$, $1 \leq a \leq n - 2$, and $a \neq 2$, there exists a 2-connected graph $G$ with $n$ vertices such that $S_C(G) = S_{SC}(G) = \{a, n - 1\}$ and there is not any connected graph $G$ of order $n \geq 3$ with $S_{SC}(G) = \{n - 1\}$. Then, we determine that $S_C(K_3) = \{1, 2\}$, $S_C(K_4) = \{1, 3\}$, $S_{SC}(K_3) = S_{SC}(K_4) = \{1\}$, $S_C(K_5) = \{1, 3, 4\}$, $S_C(K_6) = \{1, 3, 4, 5\}$, $S_{SC}(K_5) = S_{SC}(K_6) = \{1, 3\}$, $S_C(K_n) = \{1, 3, 5, 6, \ldots, n - 1\}$, $S_{SC}(K_n) = \{1, 3, 5, 6, \ldots, n - 2\}$ for $n \geq 7$. Finally, we prove that, for any integers $n, m$, and $k$ with $n \geq 5, n + 1 \leq m \leq \binom{n}{2} - 1, 1 \leq k \leq n - 1$, and $k \neq 2, 4$, there exists a strongly connected oriented graph $D$ with $n$ vertices, $m$ edges, and convexity $k$.

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1 Introduction

Convexity in graphs is discussed in the book by Buckley and Harary [1] and studied by Harary and Neiminen [5]. The concept of convexity number of an oriented graph was first introduced by Chartrand, Fink and Zhang in [3].

Here we introduce the definitions used in the paper. Graphs considered in the paper are finite, without loops or multiple edges. In a graph $G = (V, E)$, $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. A cut vertex $v$ is a vertex in a connected graph $G$ with $G - \{v\}$ being disconnected. A block of a graph $G$ is a maximal connected subgraph of $G$ without a cut vertex. A block $B$ of $G$ is an end block of a graph $G$ if $B$ contains exactly one cut vertex of $G$. An oriented graph is an orientation of some graph. In an oriented graph $D = (V, E)$, $V(D)$ and $E(D)$ denote the vertex set and the edge set of $D$, respectively. An oriented subgraph $D' = (V', E')$ of an oriented graph $D = (V, E)$ is an oriented graph with $V' \subseteq V$ and $E' \subseteq E$. An oriented graph is connected if its underlying graph is connected. A dipath is a sequence $(v_1, v_2, ..., v_k)$ of vertices of an oriented graph $D$ such that $v_1, v_2, ..., v_k$ are distinct and $(v_i, v_{i+1}) \in E(D)$ for $i = 1, 2, ..., k - 1$. An oriented graph is called strongly connected if for every ordered pair of vertices $u$ and $v$, there exists a dipath from $u$ to $v$. A strong component of an oriented graph $D$ is a maximal strongly connected oriented subgraph in $D$.

A $u - v$ geodesic in a digraph $D$ is a shortest $u - v$ dipath and its length is $d_D(u,v)$. The closed interval $I[u,v]$ between two vertices $u$ and $v$ of a digraph $D$ is the set of all vertices lying on a $u - v$ or $v - u$ geodesic (if it exists) in $D$. If there is no $u - v$ and $v - u$ geodesics, then we define that $I[u,v]_D = \{u,v\}$. A nonempty subset $S$ of the vertex set of a digraph $D$ is called a convex set of $D$ if, for every $u, v \in S$, every vertex lying on a $u - v$ or $v - u$ geodesic belongs to $S$. For a nonempty subset $A$ of $V(D)$, the convex hull $[A]$ is the minimal convex set containing $A$. Thus $[S] = S$ if and only if $S$ is convex in $D$. The convexity number $\text{con}(D)$ of a digraph $D$ is the maximum cardinality of a proper convex set of $D$. A maximum convex set $S$ of a digraph $D$ is a convex set with cardinality $\text{con}(D)$. Since every singleton vertex set is convex in a connected oriented graph $D$, $1 \leq \text{con}(D) \leq n - 1$. The degree $\text{deg}(v)$ of a vertex $v$ in an oriented graph is the sum of its indegree and outdegree; that is, $\text{deg}(v) = \text{id}(v) + \text{od}(v)$. A vertex $v$ is an end-vertex if $\text{deg}(v) = 1$. A source is a vertex having positive outdegree and indegree 0, while a sink is a vertex having positive indegree and outdegree 0. For a vertex $v$ of $D$, let $N^+(v) = \{x : (v, x) \in E(D)\}$ and $N^-(v) = \{x : (x, v) \in E(D)\}$. So if $v$ is a source, then $N^-(v) = \emptyset$, while if $v$ is a sink, then $N^+(v) = \emptyset$. A vertex $v$ of $D$ is a transitive vertex if $\text{od}(v) > 0$, $\text{id}(v) > 0$, and for every $u \in N^+(v)$ and $w \in N^-(v)$, $(w, u) \in E(D)$. For a nontrivial connected graph $G$, we define that the convexity-spectrum $S_C(G)$ of a graph $G$ is the set of
convexity numbers of all orientations of \( G \) and the \textit{strong convexity-spectrum} \( S_{SC}(G) \) of a graph \( G \) is the set of convexity numbers of all strongly connected orientations of \( G \). If \( G \) has no strongly connected orientation then \( S_{SC}(G) \) is empty. Then the \textit{lower orientable convexity number} \( \text{con}^{-}(G) \) of \( G \) is the minimum convexity number among the orientations of \( G \) and the \textit{upper orientable convexity number} \( \text{con}^{+}(G) \) is the maximum convexity number among the orientations of \( G \); that is, \( \text{con}^{-}(G) = \min S_{C}(G) \) and \( \text{con}^{+}(G) = \max S_{C}(G) \). Hence, for every nontrivial connected graph \( G \) of order \( n \), \( 1 \leq \text{con}^{-}(G) \leq \text{con}^{+}(G) \leq n - 1 \).

Chartrand et al. \cite{3} characterized the nontrivial connected oriented graphs of order \( n \) with convexity number \( n - 1 \), and showed that there is no connected oriented graph of order at least 4 with convexity number 2. They also showed that every pair \( k, n \) of positive integers with \( 1 \leq k \leq n - 1 \) and \( k \neq 2 \) is realizable as the convexity number and order, respectively, of some connected oriented graph.

In the paper, we show that for any \( n \geq 4 \), \( 1 \leq a \leq n - 2 \) and \( a \neq 2 \), there exists a 2-connected graph \( G \) with \( n \) vertices such that \( S_{C}(G) = S_{SC}(G) = \{a, n - 1\} \), and there is no connected graph \( G \) of order \( n \geq 3 \) with \( S_{SC}(G) = \{n - 1\} \). Then we prove that \( S_{C}(K_{3}) = \{1, 2\} \), \( S_{C}(K_{4}) = \{1, 3\} \), \( S_{SC}(K_{3}) = S_{SC}(K_{4}) = \{1\} \), \( S_{C}(K_{5}) = \{1, 3, 4\} \), \( S_{C}(K_{6}) = \{1, 3, 4, 5\} \), \( S_{SC}(K_{5}) = S_{SC}(K_{6}) = \{1, 3\} \), \( S_{C}(K_{n}) = \{1, 3, 5, 6, ..., n - 1\} \), \( S_{SC}(K_{n}) = \{1, 3, 5, 6, ..., n - 2\} \) for \( n \geq 7 \). Finally, for any integers \( n, m, \) and \( k \) with \( n \geq 5, n + 1 \leq m \leq \left(\binom{n}{2}\right) - 1, 1 \leq k \leq n - 1, \) and \( k \neq 2, 4 \), we prove that there exists a strongly connected oriented graph \( D \) with \( n \) vertices, \( m \) edges, and convexity \( k \).

## 2 Constructing oriented graphs with fixed lower orientable convexity number and upper orientable convexity number

For each connected graph \( G \) of order \( n \geq 2 \), there exists an acyclic orientation \( D \) of \( G \). Then \( D \) has a source \( v \) and \( V(D) - \{v\} \) is a convex set. This gives that \( n - 1 \in S_{C}(G) \). The following two useful results were proved by Chartrand et al. in \cite{3}.

\textbf{Theorem 1} \cite{3} Let \( D \) be a connected oriented graph of order \( n \geq 2 \). Then \( \text{con}(D) = n - 1 \) if and only if \( D \) contains a source, sink, or transitive vertex.

\textbf{Theorem 2} \cite{3} There is no connected oriented graph of order at least 4 with convexity number 2.
Farrugia [4] proved that a connected graph of order at least 3 has no end-vertex if and only if $\text{con}^-(G)$ and $\text{con}^+(G)$ are different.

**Theorem 3** [4] Suppose $G$ is a connected graph of order $n \geq 3$. Then $\text{con}^-(G) < \text{con}^+(G)$ if and only if $G$ has no end-vertex.

By Theorem 3, we have that,

**Theorem 4** Suppose $G$ is a connected graph of order $n \geq 3$. Then $|S_C(G)| \geq 2$ if and only if $G$ has no end-vertex.

By Theorem 4, for any connected graph $G$ of order $n \geq 3$, $|S_C(G)| = 1$ if and only if $G$ has an end-vertex.

If a connected graph $G$ has a cut vertex then there is a lower bound of $\text{con}^-(G)$ related to the cardinality of a minimum end block of $G$.

**Lemma 5** Suppose $G$ is a nontrivial connected graph of order $n \geq 3$ with a cut vertex and an end block $B$. Let $D$ be an orientation of $G$. Then $\text{con}(D) \geq n - |B| + 1$.

**Proof.** Suppose $D$ is an orientation of $G$ and $u$ is the cut vertex of $G$ with $u \in V(B)$. Let $D - \{u\} = D_1 \cup D_2 \cup ... \cup D_k$ where $D_i$ is a connected component of $D - \{u\}$ for $i = 1, 2, ..., k$. Without loss of generality, $V(D_1) = V(B) - \{u\}$. It is clear that $V(D) - V(D_1)$ is a convex set and $|V(D) - V(D_1)| = n - |B| + 1$. Then $\text{con}(D) \geq n - |B| + 1$.

According to Lemma 5 and $B$ being a minimum end block of $G$, we have that

**Theorem 6** For positive integer $n \geq 3$, there exists a connected graph $G$ with $\text{con}^-(G) \geq (n + 1)/2$.

Next, we show that for any $1 \leq a \leq n - 2$ with $a \neq 2$, there exists a 2-connected graph $G$ with $n$ vertices such that $\text{con}^-(G) = a$ and $\text{con}^+(G) = n - 1$.

**Theorem 7** For every pair of positive integers $n$ and $a$ with $n \geq 4$, $1 \leq a \leq n - 2$ and $a \neq 2$, there exists a 2-connected graph $G$ with $n$ vertices such that $S_C(G) = S_{SC}(G) = \{a, n - 1\}$.

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Proof. For \( a = 1 \), define a connected graph \( G_1 = (V_1, E_1) \) with \( V_1 = \{u, v, v_1, v_2, \ldots, v_{n-2}\} \) and \( E_1 = \{uv \} \cup \{uv_i, v_i v : i = 1, 2, \ldots, n-2\} \). Then \( G_1 \) is 2-connected. Let \( D \) be an orientation of \( G_1 \). Without loss of generality, \((u,v) \in E(D)\). If \( D \) has a source, sink, or transitive vertex then \( \text{con}(D) = n - 1 \). If \( D \) has no source, sink, or transitive vertex then \( D \) is strongly connected. Consider \( D \) is strongly connected, there exists \( i \) such that \((v, v_i, u)\) is a geodesic in \( D \). If there exists \( j \) such that \((u, v_j, v)\) is a dipath in \( D \) then \( v_j \) is a transitive vertex in \( D \). Thus, \( \text{con}(D) = n - 1 \). If, for every \( 1 \leq j \leq n - 2 \), \((v, v_j, u)\) is a geodesic in \( D \) then, for any two distinct vertices \( x \) and \( y \) of \( D \), \( u, v \in I[x, y] \) and \( \{x, y\} = V(D) \); that is, \( \text{con}(D) = 1 \). Hence we have that \( S_C(G_1) = S_S(G_1) = \{1, n - 1\} \).

Assume that \( 3 \leq a \leq n - 2 \) and define \( G_a \) to be the graph with \( V(G_a) = \{u, v, u_1, \ldots, u_{n-a}, v_1, \ldots, v_{a-2}\} \) and \( E(G_a) = \{uv, uu_1, vv_{n-a}\} \cup \{uv_i, v_i v : 1 \leq i \leq a - 2\} \cup \{u_i u_{i+1} : 1 \leq i \leq n - a - 1\} \). It is evident that \( G_a \) is 2-connected.

Let \( D_{n-1} \) be the orientation of \( G_a \) with \( E(D_{n-1}) = \{(u, v), (u, v_1), (v, u), vv_{n-a}\} \cup \{(v, v_i) : 2 \leq i \leq a - 2\} \cup \{(u_{i+1}, u_i) : 1 \leq i \leq n - a - 1\} \). For \( a \geq 3 \), \( D_{n-1} \) is a strongly connected graph and \( v_1 \) is a transitive in \( D_1 \). Then \( \text{con}(D_{n-1}) = n - 1 \); that is \( n - 1 \in S_C(G_a) \cap S_S(G_a) \).

Let \( D \) be an orientation of \( G_a \) with \( \text{con}(D) < n - 1 \). By Theorem 1, \( D \) has no sink, source, or transitive vertex. Without loss of generality, \((u,v) \in A(D)\). Then \((v, v_i, u)\) is a geodesic in \( D \) for \( i = 1, 2, \ldots, a - 2 \). By \( n - a \geq 2 \), the length of the path \((u, u_1, \ldots, u_{n-a}, v)\) in \( G_a \) is greater than 2. Since \( u_1, u_2, \ldots, u_{n-a} \) are not sources or sinks, either \((u, u_1, u_2, \ldots, u_{n-a}, v)\) or \((v, u, u_{n-a}, u_{n-a-1}, \ldots, u_{1}, u)\) is in \( D \). For either \((u, u_1, u_2, \ldots, u_{n-a}, v)\) or \((v, u, u_{n-a}, u_{n-a-1}, \ldots, u_{1}, u)\) being in \( D \), \( D \) is strongly connected and the set \( \{u, v_1, \ldots, v_{a-2}, v\} \) is a proper convex set in \( D \). If a convex set \( S \) contains vertices \( u_i \) and \( x \) for some \( x \in V(D) - \{u_i\} \) then \( I[u_i, x] \) contains vertices \( u, v, u_1, \ldots, u_{n-a} \). This implies that \( \{u_i, x\} = V(D) \). So, if \( S \) is a proper convex set in \( D \) then \( S \) does not contain vertices \( u_j \). Thus \( \{u, v_1, \ldots, v_{a-2}, v\} \) is the unique maximum proper convex set of \( D \). Hence \( \text{con}(D) = a \). The proof is complete.

Corollary 8 For every pair of positive integers \( n \) and \( a \) with \( n \geq 4 \), \( 1 \leq a \leq n - 1 \) and \( a \neq 2 \), there exists a connected graph \( G \) with \( n \) vertices such that \( S_C(G) = \{a, n - 1\} \).

Proof. For \( 1 \leq a \leq n - 2 \) and \( a \neq 2 \), by Theorem 7, there is a connected graph \( G \) such that \( S_C(G) = \{a, n - 1\} \). If \( a = n - 1 \) then we take \( G \) to be a tree; \( S_C(G) = \{n - 1\} \).

Corollary 9 For every pair of positive integers \( n \) and \( a \) with \( n \geq 4 \), \( 1 \leq a \leq n - 1 \) and \( a \neq 2 \), there exists a connected graph \( G \) with \( n \) vertices such that \( \text{con}^{-1}(G) = a \) and \( \text{con}^+(G) = n - 1 \).
By the above investigation, we need to study graphs \( G \) of order \( n \) with \( S_{SC}(G) = \{n-1\} \). But there is no such graph. The following theorem can be proved by Theorem 2 of [4]. We give another proof in the following.

**Theorem 10** There is no connected graph \( G \) of order \( n \geq 3 \) with \( S_{SC}(G) = \{n-1\} \).

**Proof.** Suppose \( G \) is a connected graph with \( S_{SC}(G) = \{n-1\} \). Then there exists a strongly connected orientation of \( G \). This implies that every block of \( G \) is 2-connected; that is, each block has at least 3 vertices. If every block of \( G \) has a strongly connected orientation without transitive vertex then there is a strongly connected orientation \( D \) without source, sink, transitive vertex. By Theorem 1, \( \text{con}(D) < n-1 \). It contradicts that \( S_{SC}(G) = \{n-1\} \). So, we construct a strongly connected orientation of \( G \) without transitive vertex in the next paragraph.

Suppose \( G' \) is 2-connected. Claim that there exists a strongly connected orientation \( D \) of \( G' \) without a transitive vertex. By Menger’s Theorem, we have that the property (*): for three distinct vertices \( x, y, z \) of \( G' \), there exist two paths \( P_1 \) from \( x \) to \( y \) and \( P_2 \) from \( x \) to \( z \) in \( G' \) such that \( V(P_1) \cap V(P_2) = \{x\} \).

Since \( G' \) is 2-connected, there exists a cycle \((v_1, v_2, ..., v_k, v_1)\) which is an induced subgraph of \( G' \). (i.e. there is not any chord in \((v_1, v_2, ..., v_k, v_1)\).) Define the directed cycle \((v_1, v_2, ..., v_k, v_1)\) in \( D \). We have that, for each \( i \), either \( v_i \) has an out neighbor and an in neighbor which are nonadjacent or \((v_1, v_2, v_3, v_1)\) is a directed cycle in \( D \). This implies that vertices \( v_1, v_2, ..., v_k \) are not source, sink, or transitive vertex. Let \( S_1 = \{v_1, v_2, ..., v_k\} \). If \( V(G') - S_1 \) is not empty then, by the property (*), there is a shortest path \((v_i, x_1, ..., x_r, v_j)\) of \( G' \) satisfying \((k \geq i > j \geq 1 \text{ and } (i,j) \neq (k,1))\) or \((i,j) = (1,k)\) such that \( r \geq 1 \) and \( x_1, ..., x_r \notin S_1 \). Define the directed path \((v_i, x_1, ..., x_r, v_j)\) is in \( D \). We observe that if \( r > 1 \) then each vertices of \( x_1, ..., x_r \) has an out neighbor and an in neighbor that are nonadjacent in \( G' \) by \((v_i, x_1, ..., x_r, v_j)\) being shortest. If \( r = 1 \) then either \( v_iv_j \notin E(G') \) or \((v_j, v_i) \in E(D) \). Let \( S_2 = S_1 \cup \{x_1, ..., x_r\} \). The other edges between two vertices of \( S_2 \) are assigned random directions in \( D \). Therefore each vertex of \( x_1, ..., x_r \) is not source, sink, or transitive vertex in \( D \). Repeatedly, we can find a shortest path \((x, y_1, ..., y_s, y)\) with \( xy \notin E(G') \) or \((y, x) \in E(D) \) such that \( s \geq 1 \) and \( y_1, ..., y_s \notin S_1 \), then define the directed path \((x, y_1, ..., y_s, y)\) is in \( D \), and let \( S_{i+1} = S_i \cup \{y_1, ..., y_s\} \). The other edges between two vertices of \( S_{i+1} \) are assigned random directions in \( D \). Until \( S_{n+1} = V(G') \), we obtained a strongly connected orientation \( D \) of \( G \) without source, sink, or transitive vertex. \( \blacksquare \)
3 Strong convexity spectra of complete graphs

In this section, we determine the convexity-spectra and the strong convexity-spectra of complete graphs. Since $SC(G) = \{ \text{con}(D) : D \text{ is an orientation of } G \}$ and $SCSC(G) = \{ \text{con}(D) : D \text{ is a strong orientation of } G \}$, $SCSC(G) \subseteq SC(G)$. An orientation of the complete graph of order $n$ is called a tournament of order $n$. In a strongly connected graph $D$, the diameter of $D$ is denoted by $diam(D)$.

We find some strongly connected tournaments $D$ of order $n \geq 3$ with $\text{con}(D) = 1$.

**Lemma 11** Suppose $n$ is a positive integer with $n \geq 3$ and $n \neq 4$. Then there exists a strongly connected tournament $D$ of order $n$ with $d(D) = 2$ and $\text{con}(D) = 1$ and every strongly connected tournament of order 4 has diameter 3.

**Proof.** If $n = 4$, then all strongly connected tournaments of order 4 are isomorphic. It is easy to check that the diameter of every strongly connected tournament of order 4 is 3.

Suppose $n$ is a positive integer with $n \geq 3$ and $n \neq 4$. For $n = 3$, a directed cycle $D_1$ with the vertex set $\{a, b, c\}$ has $d(D_1) = 2$ and $\text{con}(D_1) = 1$. For $n = 5$, Let $D_2$ be the oriented graph with $V(D_2) = V(D_1) \cup \{x, y\}$ and $E(D_2) = E(D_1) \cup \{(u, x), (y, u) : u \in V(D_1)\} \cup \{(x, y)\}$. We can find that $D_2$ has $d(D_2) = 2$ and $\text{con}(D_2) = 1$. For $n = 6$, let $D_3$ be the oriented graph with $V(D_3) = V(D_2) \cup \{z\}$ and $E(D_3) = E(D_2) \cup \{(a, z), (b, z), (x, z), (z, c), (z, y)\}$. We also can find that $D_3$ has $d(D_3) = 2$ and $\text{con}(D_3) = 1$. For $n \geq 7$, if we have a strongly connected tournament $D'$ with order $n - 1$, $d(D') = 2$, and $\text{con}(D') = 1$, then, let $D$ be the oriented strongly connected tournament with the vertex set $V(D') \cup \{s, t\}$ and the edge set $E(D') \cup \{(u, s), (t, u) : u \in V(D')\} \cup \{(s, t)\}$. We have that $D$ has $d(D) = 2$ and $\text{con}(D) = 1$. By induction, we can get the theorem.

**Lemma 12** If $D$ is a strongly connected tournament of order $n \geq 3$, then $\text{con}(D) \leq n - 2$.

**Proof.** Let $S$ be a set of vertices in $D$ with $|S| = n - 1$. Then there exists a vertex $v \in V(D) - S$. Let $X = \{x \in V(D) : (x, v) \in E(D)\}$ and $Y = \{y \in V(D) : (v, y) \in E(D)\}$. By $D$ being a strongly connected tournament, there exist $x \in X$ and $y \in Y$ such that $(y, x) \in E(D)$. Thus $(x, v, y)$ is a geodesic in $D$. Then $S$ is not a convex set; that is, $\text{con}(D) \leq n - 2$.

**Lemma 13** For $n \geq 6$ being a positive integer, $4 \notin SCSC(K_n)$.
Proof. Suppose $D$ be a strongly tournament with $S$ being a proper convex set of 4 vertices in $D$. Then the induced subgraph $H$ of $S$ is strongly connected. Since $\text{diam}(H) = 3$, there is a geodesic $(a, b, c, d)$ in the induced subgraph of $S$. Let $A = \{u : u \in V(D) - S \text{ and } (u, a) \in E(D)\}$ and $B = \{v : v \in V(D) - S \text{ and } (d, v) \in E(D)\}$. Take $u \in A$ and $v \in B$. Since $(a, b, c, d)$ is a dipath of $D$ and $S$ is convex, $(u, b), (c, v) \in E(D)$. If $(u, b), (c, v) \in E(D)$ then, by $S$ being convex, $(u, c), (b, v) \in E(D)$. Similarly, we have that $(u, d), (a, v) \in E(D)$. By $\text{diam}_D(a, d) = 3$ and $S$ being convex, $A$ and $B$ are disjoint and $(u, v) \in E(D)$ for $u \in A$ and $v \in B$. If $V(D) - (S \cup A \cup B)$ is empty, then $D$ is not strongly connected. So, there exists a vertex $x \in V(D) - (S \cup A \cup B)$ with $(a, x), (x, d) \in E(D)$. It contradicts that $\text{diam}_D(a, d) = 3$. Hence, $4 \notin S_{SC}(K_n)$ for $n \geq 6$.

We combine the ideas in Lemmas 11 12, and 13 to get the following Theorem.

**Theorem 14** $S_{SC}(K_3) = S_{SC}(K_4) = \{1\}$, $S_{SC}(K_5) = S_{SC}(K_6) = \{1, 3\}$, and $S_{SC}(K_n) = \{1, 3, 5, 6, \ldots, n-2\}$ for integer $n \geq 7$.

**Proof.** For $n = 3$ or 4, by Theorem 2, Lemma 11, and Lemma 12, $S_{SC}(K_3) = S_{SC}(K_4) = \{1\}$. For $n \geq 5$, by Lemma 11, $1 \in S_{SC}(K_n)$; and by Theorem 2 and Lemma 12 and 13, $2, 4, n-1 \notin S_{SC}(K_n)$. In the following paragraphs, we construct a strongly connected orientation of $K_n$ for $n \geq 5$ and $\text{con}(D) = k$ for $3 \leq k \leq n-2$ and $k \neq 4$.

Suppose $n \geq 5$, $3 \leq k \leq n-2$ and $k \neq 4$. Let the vertex set of $K_n$ be $\{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_{n-k}\}$, $V = \{v_1, v_2, \ldots, v_k\}$, and $U = \{u_1, u_2, \ldots, u_{n-k}\}$.

By Lemma 11, there exists a strongly connected oriented graph $D_1$ with the vertex set $V$, $(v_k, v_1) \in E(D_1)$, $d(D_1) = 2$, and $\text{con}(D_1) = 1$. Let $D_2$ be the oriented graph with the vertex set $U$ and the edge set $\{(u_i, u_j) : 1 \leq i < j \leq n-k\} - \{(u_1, u_{n-k})\}$ for $n-k \geq 3$. If $n-k = 2$ then the edge set of $D_2$ is defined by $\{(u_2, u_1)\}$. Let $D_3$ be the oriented graph with the vertex set $V \cup U$ and the edge set $\{(u_1, v_i) : 1 \leq i \leq k\} \cup \{(v_i, u_j) : 1 \leq i \leq k \text{ and } 2 \leq l \leq n-k\}$. Define that $D$ is a strongly connected orientation of $K_n$ with $E(D) = E(D_1) \cup E(D_2) \cup E(D_3)$.

By Lemma 11, $d_D(v_i, v_j) \leq 2$ for all $1 \leq i < j \leq k$. And $u_l \notin [v_i, v_j]$ in $D$ for all $i, j, k$. Then $V$ is a convex set of $D$.

Let $S$ be a convex set of $D$. If there exist $1 \leq l < m \leq n-k$ such that $u_l, u_m \in S$, then $I[u_m, u_l]$ contains vertices $u_l$ and $u_m$. Since $(u_1, u_m, u_{n-k})$ and $(u_l, u_{n-k})$ are geodesics in $V(D)$ for all $1 < p < n-k$ and $1 \leq i < k$, $S = V(D)$. If there exist $1 \leq l < n-k$ and $1 \leq m \leq k$ such that $u_l, u_m \in S$, then $I[u_l, v_m]$ contains vertices $u_1$ and $u_{n-k}$. By the same above reason, $S = V(D)$. So, we have that $V$ is a maximum convex set with $V \neq V(D)$. Thus, $\text{con}(D) = |V| = k$.

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Lemma 15 For positive integer \( n \geq 7 \), \( 4 \notin S_C(K_n) \).

Proof. Suppose \( D \) is a tournament of order \( n \geq 7 \). If \( D \) is strongly connected then, by Lemma 13, \( \text{con}(D) \neq 4 \). Assume that \( D \) is not strongly connected. Then there exists a strong component \( S \) of \( D \) such that \( (x, y) \in E(D) \) for each \( x \in V(D) - S \) and \( y \in S \). If \( |S| = 1 \) or \( |V(D) - S| = 1 \), then \( D \) has a sink or a source; that is, \( \text{con}(D) = n - 1 \neq 4 \). If \( |S|, |V(D) - S| > 1 \), then, for \( x \in V(D) - S \) and \( y \in S \), \( S \cup \{x\} \) and \( (V(D) - S) \cup \{y\} \) are proper convex sets in \( D \); that is, \( \text{con}(D) \geq n/2 + 1 > 4 \).

Theorem 16 \( S_C(K_3) = \{1, 2\}, S_C(K_4) = \{1, 3\}, S_C(K_5) = \{1, 3, 4\}, S_C(K_6) = \{1, 3, 4, 5\} \) and \( S_C(K_n) = \{1, 3, 5, 6, ..., n - 1\} \) for integer \( n \geq 7 \).

Proof. Since every acyclic orientation \( D \) has a source, \( \text{con}(D) = n - 1 \). Then \( n - 1 \in S_C(K_n) \) for \( n \geq 2 \). By \( S_{SC}(G) \subseteq S_C(G) \), Theorem 14, and Lemma 15, we have that \( S_C(K_3) = \{1, 2\} \), \( S_C(K_4) = \{1, 3\} \), \( S_C(K_5) = \{1, 3, 4\} \), and \( S_C(K_n) = \{1, 3, 5, 6, ..., n - 1\} \) for integer \( n \geq 7 \). For \( K_6 \), we have an orientation \( D \) with \( V(D) = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) and \( E(D) = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_4, v_5), (v_5, v_6), (v_6, v_4)\} \cup\{(v_i, v_j) : 1 \leq i \leq 3 \text{ and } 4 \leq j \leq 6\} \) with \( \text{con}(D) = 4(\{v_1, v_2, v_3, v_4\} \) is a maximum convex set). Thus, \( S_C(K_6) = \{1, 3, 4, 5\} \).

4 Constructing strongly connected oriented graphs with fixed order, size, and convexity number

Analyzing the orientation \( D \) in the proof of Theorem 14, we have the following observations. (We use the same notations in the following observations.)

Observation 17 For \( 1 \leq i < j \leq k \), \( \{v_i, v_k\} \subseteq \{v_i, v_j\} \) and \( \{v_1, v_k\} = V \). Then \( V \) is a convex set in \( D \).

Observation 18 (1) For \( 2 \leq l \leq n - k - 1 \), \( (u_1, u_l, u_{n-k}) \) is a geodesic in \( D \).

(2) For \( 1 \leq i \leq k \), if \( k \) is odd then \( (u_1, v_i, u_n-k) \) is a geodesic in \( D \), and if \( k \) is even and \( i \neq k - 1 \) then \( (u_1, v_i, u_{n-k}) \) is a geodesic in \( D \) and \( (v_k, v_{k-1}, v_1) \) is a geodesic in \( D \).

Observation 19 Every vertex of \( U \) except \( u_1 \) and \( u_{n-k} \) belongs to a unique \( u_1 - u_{n-k} \) geodesic in \( D \).
Observation 20  For $1 \leq i < j \leq n$, $\{u_1, u_{n-k}\} \subseteq \{u_i, u_m\}$ and $\{u_1, u_{n-k}\} = V(D)$. Then $\{u_l, u_m\} = V(D)$.

Observation 21  For $1 \leq i \leq k$ and $1 \leq l \leq n-k$, $\{u_1, u_{n-k}\} \subseteq \{v_i, u_l\}$. Then $\{v_l, u_i\} = V(D)$.

Lemma 22  Let $D$ be a connected oriented graph of order $n \geq 5$. If $V(D) = V \cup U$ where $V = \{v_1, v_2, ..., v_k\}$, $U = \{u_1, u_2, ..., u_{n-k}\}$, $k \geq 3$ and $n-k \geq 2$ satisfying two conditions:

1. $[\{v_i, v_j\}] = V$, for every $i < j$ and
2. $[\{u_1, u_{n-k}\}] = V(D)$ and $u_1, u_{n-k} \in \{x, y\}$ for every two vertices $x, y$ with $x \in U$ and $y \in V(D) - \{x\}$.

then $V$ is the unique maximum convex set in $D$ and $\text{con}(D) = |V|$.

Proof. By the condition (1), we have that $V$ is a convex set in $D$. According to the condition (2), the convex hull of every pair of vertices in $U$ is $V(D)$ and the convex hull of every pair $u, v$ with $u \in U, v \in V$ is also $V(D)$. Then we have $V$ is the unique maximum convex set in $D$ and $\text{con}(D) = |V|$. □

By Theorem 2, there is no connected graph $G$ of order $n \geq 4$ with $2 \in S_{SC}(G)$. In the first theorem of this section, we consider the existence of strongly connected oriented graphs of order $n$, size $m$, and convexity number $k$ where $n \geq 5$, $n+1 \leq m \leq \binom{n}{2}$, and $3 \leq k \leq n-2$.

Theorem 23  For integers $k, n, m$ with $n \geq 5$, $3 \leq k \leq n-2$, $k \neq 4$, and $n+1 \leq m \leq \binom{n}{2}$, there exists a strongly connected oriented graph with $n$ vertices, $m$ edges, and convexity number $k$.

Proof. Let $n, m$ and $k$ be positive integers with $n \geq 5$, $3 \leq k \leq n-2$, $n+1 \leq m \leq \binom{n}{2}$. In the following, we construct a strongly connected oriented graph with $n$ vertices, $m$ edges, and convexity number $k$ by examining different cases.

(a) First, for $m = \binom{n}{2}$, by Theorem 14, there exists a strongly connected tournament $D$ of order $n$ and convexity number $k$.

(b) For $\binom{n}{2} - k(n-k)+4 \leq m < \binom{n}{2}$, there is a strongly connected oriented graph $D$ with $E(D) = E(D_1) \cup E(D_2) \cup E(D_3)$ in the proof of Theorem 14. Let $S \subseteq E(D_3) - \{(u_1, v_1), (u_1, v_k), (v_1, u_{n-k}), (v_k, u_{n-k})\}$ with $|S| = \binom{n}{2} - m$ and $H_m = D - S$. By $V$ being a convex set of $D$ and $S \subseteq E(D_3)$, $V$ is still a convex set in $H_m$. By Lemma
11, for every $i < j$, $\{v_i, v_j\} \in H_m$. Since $(u_1, u_i, u_{n-k}), (u_1, v_i, u_{n-k}), (u_1, v_k, u_{n-k})$ are geodesics in $H_m$, $\{u_1, u_{n-k}\} \in H_m$. And, for $i < j$, $u_1, u_{n-k} \in I_{H_m}[u_1, u_i]$. This implies that $\{u_1, u_j\} \in H_m$. By Lemma 22, $V$ is the unique maximum convex set in $H_m$ and $\text{con}(H_m) = |V| = k$.

(c) For $\left(\frac{n}{2}\right) - k(n-k) + 4 - \binom{n-k-2}{2} < m < \left(\frac{n}{2}\right) - k(n-k) + 4$. Let $D' = D - E(D_3) \cup \{(u_1, v_i), (u_1, v_k), (v_i, u_{n-k}), (v_k, u_{n-k})\}$. Let $D' = \{(u_1, u_i), (u_1, u_k), (v_i, u_{n-k}), (v_k, u_{n-k})\}$, with $|S| = \left(\frac{n}{2}\right) - k(n-k) + 4 - m$. And $H_m = D' - S$. Similarly to part (b), by Theorem 22, $V$ is the unique maximum convex set in $H_m$ and $\text{con}(H_m) = |V| = k$.

(d) For $2n-2 \leq m < \left(\frac{n}{2}\right) - k(n-k) + 4 - \binom{n-k-2}{2}$. Define $D'$ to be a strongly connected oriented graph with $V(D') = V \cup U$ and $E(D') = E_1 \cup E_2 \cup \{(u_1, v_1), (u_1, v_k), (v_1, u_{n-k}), (v_k, u_{n-k})\}$. Then $|E(D')| = \left(\frac{n}{2}\right) - k(n-k) + 4 + 2(n-k-2) + 1$. Let $S \subseteq \{(v_j, v_i) : 2 \leq i < j \leq k-1\}$ with $|S| = \left(\frac{n}{2}\right) + 4 + 2(n-k-2) + 1 - m$ and $H_m = D' - S$. We have that $|E(H_m)| = m$, and $V$ is convex in $H_m$. For $x \in U$ and $y \in V(H_m) - \{x\}$, $u_1, u_{n-k} \in I_{H_m}[x, y]$, $v_1, v_k, u_1, u_2, \ldots, u_{n-k} \in I_{H_m}[u_1, u_k]$ and $I_{H_m}[v_1, v_k] = V$. By Lemma 22, $V$ is the unique maximum convex set in $H_m$ and $\text{con}(H_m) = |V| = k$.

(e) For $m = 2n-3$. (i) If $k = n-2$ then let $D = (V, E)$ be the digraph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{(v_i, v_j) : i = 2, \ldots, n-3\} \cup \{(v_n, v_1), (v_n, v_k), (v_1, v_{n-1}), (v_1, v_n), (v_1, v_{n-1})\}$. Then $|E| = 2(n-4) + 5 = 2n-3$ and $\{v_1, v_2, \ldots, v_{n-2}\}$ is a convex set. If $S$ is a convex set with $|S| \geq n-1$ then there exist $2 \leq i \leq n-2$ and $n-1 \leq j \leq n$ such that $v_i, v_j \in S$. We have that $v_1, v_{n-2}, v_{n-1}, v_n \in S$. Thus $S = V$. Hence $\text{con}(D) = n-2$. (ii) If $3 \leq k \leq n-3$ and $n \geq 6$ then let $D = (V, E)$ be the digraph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{(v_1, v_i), (v_1, v_k) : i = 2, \ldots, k-1\} \cup \{(v_k, v_1), (v_k, v_{k+1}), (v_{k+2}, v_{k+3})\} \cup \{(v_{k+1}, v_j), (v_j, v_i) : j = k+2, \ldots, n\}$. Then $|E| = 2(k-2) + 3 + 2(n-k-1) = 2n-3$ and $\{v_1, v_2, \ldots, v_k\}$ is a convex set. For $1 \leq i < j \leq n$, $v_i, v_k, v_{k+1} \in I[v_i, v_j]$; that is, $\{v_i, v_j\} = V$. For $k \leq i < j \leq n$, $v_i, v_k, v_{k+1} \in I[v_i, v_j]$; that is, $\{v_i, v_j\} = V$. So $\{v_1, v_2, \ldots, v_k\}$ is the unique maximum convex set. Therefore $\text{con}(D) = k$.

(f) For $n+1 \leq m \leq 2n-4$. Let $a, b$ be integers with $1 \leq a \leq k-2$ and $1 \leq b \leq n-k-1$. Define $H(a, b)$ to be a strongly connected oriented graph with $V(H(a, b)) = V \cup U$ and $E(H(a, b)) = E_1 \cup E_2$ where $V = \{v_1, v_2, \ldots, v_k\}$, $U = \{u_1, u_2, \ldots, u_{n-k}\}$, $E_1 = \{(v_1, v_i), (v_1, v_{n+2}) : 2 \leq a \leq i \leq a + 1\} \cup \{(v_j, v_{j+1}) : a + 2 \leq j \leq k - 1\} \cup \{(v_k, v_i)\}$ and $E_2 = \{(v_k, u_{n-k})\} \cup \{(u_j, u_{j-1}) : n - k \leq j \leq b + 2\} \cup \{(u_{b+1}, u_1), (u_1, v_1) : 1 \leq i \leq b\}$. In $H(a, b)$, $|E(H(a, b))| = n + a + b - 1$ and $V$ is a convex set. And $I_{H(a, b)}[v_1, v_k] = V$ and $u_1, u_2, \ldots, u_{n-k} \in I_{H(a, b)}[v_1, v_k]$. If $S$ is a convex set containing $u_i$ and $u_j$ with $i < j$ then $v_1, v_k, u_{n-k} \in I_{H(a, b)}[u_1, u_j]$; that is, $S = V(H(a, b))$. If $S$ is a convex
set containing \( v_i \) and \( u_j \) then \( v_1, v_k, u_{n-k} \in I_{H(a,b)}[v_i, u_j] \); that is, \( S = V(H(a, b)) \). Therefore \( V \) is the unique maximum convex set in \( H_m \) and \( \text{con}(H_m) = |V| = k \).

For strongly connected oriented graphs with convexity number 1, we have that:

**Theorem 24** For any integers \( n, m \) with \( n \geq 3, n \leq m \leq \left( \frac{n}{2} \right) \), there exists a strongly connected oriented graph \( D \) with \( n \) vertices, \( m \) arcs, and convexity 1.

**Proof.** First, we consider the case \( m = \left( \frac{n}{2} \right) \). Define \( D_0 \) to be an oriented graph with vertex set \( \{v_1, v_2, ..., v_n\} \) and arc set \( \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\} \cup \{(v_j, v_i) : 3 \leq i + 2 \leq j \leq n\} \). Since \( (v_k, v_{k+1}, v_{k+2}, v_k) \) is a directed cycle of length 3 for all \( 1 \leq k \leq n - 2 \), we have that \( (v_{k+1}, v_{k+2}, v_k) \) and \( (v_k, v_{k+2}, v_k) \) are geodesics in \( D_0 \). If a convex set of \( D_0 \) contains two consecutive vertices \( v_i, v_{i+1} \) for some \( 1 \leq i \leq n - 1 \), then it must be \( V(D_0) \). Suppose that \( \text{con}(D_0) > 1 \) and \( S \) is a convex set of \( D_0 \) with \( |S| > 1 \). Take \( v_i, v_j \in S \) with \( i < j \). Then the vertices of the geodesic \( (v_i, v_{i+1}, ..., v_j) \) belong to \( S \). Thus, \( S \) contains two consecutive vertices of \( V(D_0) \). By the above property, \( S = V(D_0) \). Hence, \( \text{con}(D_0) = 1 \).

Second, we consider the case of \( 2n - 2 \leq m \leq \left( \frac{n}{2} \right) - 1 \). Let \( T \) be a subset of \( \{(v_j, v_i) : 1 \leq i \leq j \leq 3 \leq n - 3\} \cup \{(v_n, v_1)\} \) with \( |T| = \left( \frac{n}{2} \right) - m \) and \( D' = D_0 - T \). If \( S \) is a convex set of \( D' \) containing two distinct vertices \( v_i \) and \( v_j \) with \( i < j \), then \( (v_i, v_{i+1}, ..., v_{j-1}, v_j) \) is a geodesic in \( D' \); that is, vertices \( v_i, v_{i+1}, ..., v_{j-1}, v_j \) are in \( S \). Since \( (v_k, v_{k+1}, v_{k-1}) \) and \( (v_{k+1}, v_{k-1}, v_k) \) are geodesics in \( D' \) for \( 2 \leq k \leq n - 1 \) and \( v_i, v_{i+1} \in S \), \( S = V(D') \). Hence \( D' \) is an oriented graph with \( m \) arcs and \( \text{con}(D') = 1 \).

Finally, the case \( n \leq m \leq 2n - 3 \). For \( 1 \leq k \leq n - 2 \), define \( D_k = (V, E) \) with \( V = \{u_0, u_1, ..., u_{n-1}\} \) and \( E = \{(u_0, u_i), (u_i, u_{i+1}) : 1 \leq i \leq k \} \cup \{(u_i, u_{i+1}) : k + 1 \leq i \leq n - 2\} \cup \{(u_{n-1}, u_0)\} \). Then \( D_k \) is strongly connected and \( |E| = n + k - 1 \). And, for each \( i < j \), we have that vertices \( u_0 \) and \( u_{n-1} \) are contained in \( u_j - u_i \) and \( u_i - u_j \) geodesics and \( I[u_0, u_{n-1}] = V \); that is, \( \text{con}(D_k) = 1 \).

Consider the strongly connected oriented graphs with convexity number \( n - 1 \). By Lemma 12, there is no strongly connected tournament of order \( n \geq 3 \) with convexity number \( n - 1 \); and, the convexity number of each directed cycle is 1. So we have the following theorem.

**Theorem 25** For any integers \( n \) and \( m \) with \( n \geq 4 \), there exists a strongly connected oriented graph \( D \) with \( n \) vertices, \( m \) arcs, and convexity \( n - 1 \) if, and only if, \( n + 1 \leq m \leq \left( \frac{n}{2} \right) - 1 \).

**Proof.** By Lemma 12, there is no strongly connected tournament \( D \) of order \( n \geq 3 \) with \( \text{con}(D) = n - 1 \). So, we assume that \( n + 1 \leq m \leq \left( \frac{n}{2} \right) - 1 \). If \( n + 1 \leq
If \( \binom{n}{2} - (n-4) \leq m \leq \binom{n}{2} - 1 \) and \( n \geq 5 \) then, let \( r = \binom{n}{2} - m \), define a strongly connected graph \( D = (V, E) \) with \( V = \{v_1, v_2, ..., v_n\} \) and \( E = \{(v_a, v_b) : 1 \leq a < b \leq n - 2\} \cup \{(v_c, v_d) : r + 1 \leq c \leq n - 2 \text{ and } n - 1 \leq d \leq n\} \cup \{(v_n, v_1) : 1 \leq i \leq r\} \cup \{(v_n, v_{n-1})\} \). We observe that \( D \) is a strongly connected oriented graph with \( n \) vertices and \( m \) edges, and \( v_n \) is a transitive vertex of \( D \). Therefore \( \text{con}(D) = n - 1 \).

For the remained cases, we have the following result.

**Theorem 26** Suppose \( D \) is a strongly connected oriented graph with \( n \) vertices and \( m \) edges. Then

1. if \( n = 3 \) and \( m = 3 \) then \( \text{con}(D) = 1 \);
2. if \( n = 4 \) and \( m = 4 \) then \( \text{con}(D) = 1 \);
3. if \( n = 4 \) and \( m = 5 \) then \( \text{con}(D) = 1 \) or 3;
4. if \( n = 4 \) and \( m = 6 \) then \( \text{con}(D) = 1 \).

**Proof.** (1) If \( D \) is a strongly connected oriented graph with 3 vertices and 3 edges then \( D \) is directed cycle of length 3. Thus \( \text{con}(D) = 1 \).

(2) If \( D \) is a strongly connected oriented graph with 4 vertices and 4 edges then \( D \) is directed cycle of length 4. Thus \( \text{con}(D) = 1 \).

(3) If \( D \) is a strongly connected oriented graph with 4 vertices and 5 edges then the underlying graph of \( D \) is isomorphic to \( K_4 - \{e\} \) for some edge \( e \in E(K_4) \). Let the underlying graph \( D \) be the \( G = (V, E) \) with \( V = \{x_1, x_2, x_3, x_4\} \) and \( E = \{x_ix_j : i < j\} \cup \{x_1x_4\} \). Without loss of generality, \( (x_2, x_1), (x_1, x_3) \) are in \( E(D) \). If \( x_1 \) is a transitive vertex in \( D \) then \( (x_2, x_3) \) is in \( E(D) \). Thus, \( (x_3, x_4), (x_4, x_2) \) are in \( E(D) \) and \( \text{con}(D) = 3 \). If \( x_1 \) is not a transitive vertex in \( D \) then \( (x_3, x_1) \) is in \( E(D) \). If \( x_4 \) is not a transitive vertex in \( D \) then \( \text{con}(D) = 1 \); otherwise, for \( x_4 \) being a transitive vertex, \( \text{con}(D) = 3 \).

(4) By Theorem 14, \( S_{SC}(K_4) = \{1\} \). Then \( \text{con}(D) = 1 \).
References


