Veering triangulations admit strict angle structures

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Joint work with
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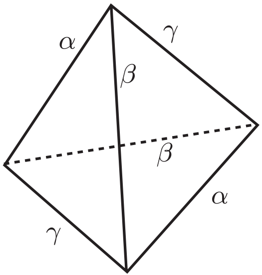
Geometric Triangulations

We want to understand the relationship between the geometry and combinatorics of triangulations of 3-manifolds. Today we look at the following:

Question. Given an ideal triangulation $\mathcal{T}$ of a orientable cusped hyperbolic 3-manifold M, is the triangulation *geometric*, i.e. realized by positively oriented ideal hyperbolic tetrahedra in the complete hyperbolic structure on M?

Ideal tetrahedra in $\mathbb{H}^3$

have shapes parametrized by 3 dihedral angles $\alpha, \beta, \gamma \in (0, \pi)$ as shown below.



Note that the sum of angles at each vertex is

$$\alpha + \beta + \gamma = \pi,$$

and opposite edges have the *same* dihedral angle.

To show that an ideal triangulation $\mathcal{T}$ is *geometric* we need to find dihedral angles for the ideal tetrahedra such that:

(1) the sum of dihedral angles around each edge is 2π,

(2) there is no translational holonomy ("shearing") along the edges,

(3) each "cusp" is complete (i.e. the holonomy of each peripheral curve is parabolic).

One approach, due to Rivin and Casson, divides this non-linear problem into two steps.

Angle structures

Step 1. Solve the linear angle sum equations (1).
The set of solutions is called the space $\mathcal{A}$ of *angle structures* on $\mathcal{T}$.

In other words, an *angle structure* on $\mathcal{T}$ is an assignment of dihedral angles $\in (0, \pi)$ to the edges of each tetrahedron σ so that

- angles attached to opposite edges of σ are equal,
- angles add up to π at each vertex of σ, and $(*)$
- the sum of all angles around each edge of $\mathcal{T}$ is 2π.

Remarks

1. Finding such an angle structure is a linear programming problem.

2. An angle structure gives a hyperbolic structure on $M \setminus \{\text{the edges of the triangulation}\}$, but with possible shearing type singularities around these edges.

3. If an angle structure exists, then Casson showed that M is irreducible, atoroidal and not Seifert fibred, so M admits a hyperbolic structure by the uniformization theorem of Thurston.

Step 2. Solve the *non-linear* equations (2) and (3) by a volume maximization procedure.

We define a volume function

$$V : \mathcal{A} \to \mathbb{R}$$

by adding up the *hyperbolic volumes* of the ideal tetrahedra in $\mathbb{H}^3$ given by a point in $\mathcal{A}$.

For an ideal tetrahedron in $\mathbb{H}^3$ with dihedral angles α, β, γ we have:

$$\mathrm{Vol} = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where

$$\Lambda(\alpha) = -\int_0^\alpha \log(2\sin t)\, dt$$

is the "Lobachevsky function". (See [Milnor].)

This volume function is smooth and strictly concave on $\mathcal{A}$ and extends to a continuous function on the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ (a compact convex polytope), so attains a maximum on $\overline{\mathcal{A}}$.

Theorem [Rivin, Casson, Ken Chan-H]. If $V : \overline{\mathcal{A}} \to \mathbb{R}$ attains its maximum at a point of $\mathcal{A}$, then this gives the complete hyperbolic structure on M; in particular the triangulation $\mathcal{T}$ is geometric.

The proof uses elementary properties of the Lobachevsky function together with some of the combinatorics of ideal triangulations developed by Neumann and Zagier.

This program has ben carried out very successfully by François Guéritaud for the standard layered triangulations of once-punctured torus bundles, and by Dave Futer for 2-bridge knot and link complements.

We would like to extend this to other classes of 3-manifolds!

A year or so ago, Ian Agol introduced a class of “veering taut triangulations” with nice combinatorial properties and showed that every hyperbolic bundle over the circle admits such an ideal triangulation, possibly after removing a suitable knot or link.

Veering triangulations

Last year, Rubinstein, Segerman and Tillmann and I introduced a new class of "veering triangulations", which includes the class of triangulations considered by Agol, and proved the following:

Main Theorem [HRST]
Veering triangulations admit angle structures.

(Gueritaud and Futer have recently given another proof of this.)

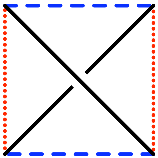
Remarks: Experimental evidence suggests that the veering triangulations are quite special but occur reasonably often. Further, every veering triangulation we've studied so far seems to be geometric (according to SnapPea). This suggests the following:

Open Question. Is every veering triangulation geometric?

Some definitions

Definition: A *taut angle structure* on an ideal triangulation $\mathcal{T}$ is a solution to the angle equations $(*)$ where each angle is 0 or π.

Then each tetrahedron has 2 opposite edges with angle π and the other 4 edges with angle 0. For an oriented tetrahedron we have the following standard picture with the 0 angles at the sides of the square, and the π angles at the diagonals.



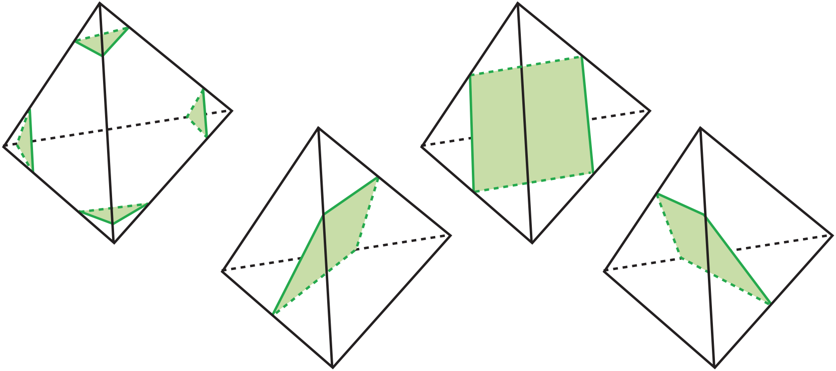
Definition: We colour the *zero angle* edges *red* and *blue* as shown. Then the red edges are called *right-veering* and the blue edges are called *left veering*.

Note:
As viewed from a red edge: the triangles at the edge move to the *right* going from bottom to top. As viewed from a blue edge: the triangles at the edge move to the *left* going from bottom to top.

Definition: A *veering structure* on an oriented ideal triangulation $\mathcal{T}$ is a choice of taut angle structure together with an assignment of a colour to every edge of the triangulation so that the zero angle edges of each tetrahedron are coloured as above. We also say that $\mathcal{T}$ is a *veering triangulation* for short.

Normal surfaces and angle structures

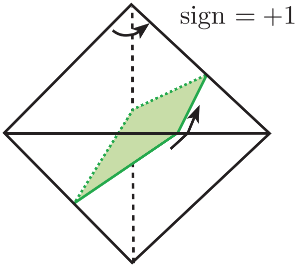
A *normal surface* in a triangulation $\mathcal{T}$ of M is a surface that intersects each tetrahedron in a collection of *normal triangles* and *normal quadrilaterals (quads)* as shown below. There are 4 types of normal triangles and 3 types of normal quads in each tetrahedron.

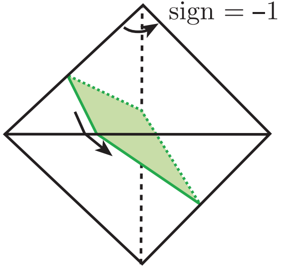


The normal surfaces are parametrized by the numbers of normal triangles and quads of each type. These satisfy a simple system of linear inequalities and linear equations: the number of arcs of each type in a face of a tetrahedron is a sum of numbers of appropriate quads and triangles in the two adjacent tetrahedra.

In fact, all the essential information is contained in the number of quads of each type q. These quad coordinates x_q satisfy the *Q-matching equations* of Tollefson:

Each quad type q meeting an edge e has a sign $\varepsilon(q) = +1$ or -1 as shown below, where e is the dotted vertical edge.





The Q-matching equations say that the sum of the numbers $\varepsilon(q)x_q$ around each edge e is zero.

The set of all solutions to the Q-matching equations is a vector subspace $Q(\mathcal{T}) \subset \mathbb{R}^{3n}$ where n is the number of tetrahedra in $\mathcal{T}$.

Any *admissible* solution to these equations in non-negative integers with only one quad type per tetrahedron gives a *spun normal surface* made up of finitely many quads and (possibly infinitely many) triangles.

Convex duality (from linear programming) gives the following:

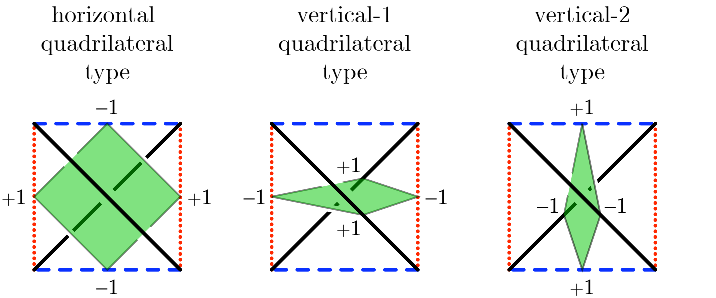
Proposition. [Rubinstein-Kang, Luo-Tillmann] Assume that $\mathcal{T}$ has a taut angle structure. Then $\mathcal{T}$ admits an angle structure if there is no non-negative solution x to the Q-matching equations with $\chi^*(x) = 0$ and at least one quad coordinate positive, where

$$2\pi\chi^*(x) = \sum_{\text{quads } q} -2\alpha(q)x_q$$

and $\alpha(q)$ is the angle on the two edges opposite q in the tetrahedron containing the quad q.

Note: The generalised Euler characteristic $\chi^* : Q(\mathcal{T}) \to \mathbb{R}$ is a linear function which agrees with the Euler characteristic for embedded and immersed normal surfaces in $\mathcal{T}$.

For a triangulation with taut angle structure the quads are of three types:



$\alpha(q) = \pi$ $\qquad$ $\alpha(q) = 0$ $\qquad$ $\alpha(q) = 0$

The quad types in an angle-taut tetrahedron, with angles $\alpha(q)$ of opposite edges and associated signs in the Q-matching equations.

Since any horizontal quad gives a negative contribution to χ^*, the previous result can be reformulated as follows:

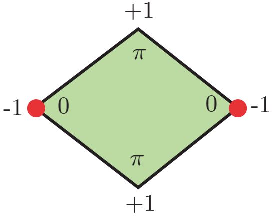
Proposition. Assume that $\mathcal{T}$ has a taut angle structure. Then $\mathcal{T}$ admits an angle structure if there is no non-negative solution x to the Q-matching equations *consisting of only vertical quads* and with at least one quad coordinate positive.

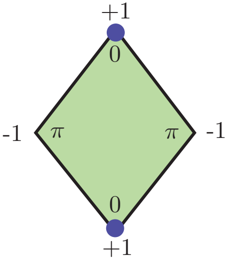
Sketch of the proof of Main Theorem

Let $\mathcal{T}$ is a veering triangulation and assume (for a contradiction) we have a solution to the Q-matching equations where all $x_q \geq 0$, at least one $x_q > 0$ and q is a vertical quad type whenever $x_q > 0$.

From above, there are two types of vertical quads:

- type 1: zero angle edges are red and contribute -1 to the Q-matching equations
- type 2: zero angle edges are blue and contribute $+1$ to the Q-matching equations





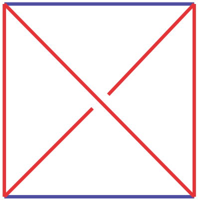
Now consider the Q-matching equations:

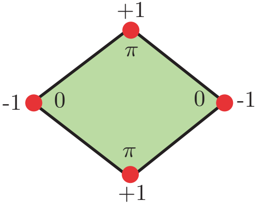
At red edges, we get a negative contribution from angle 0 edges of vertical type 1 quads. This cannot be compensated for by vertical type 2 quads so must cancel with contributions from vertical type 1 quads.

Thus

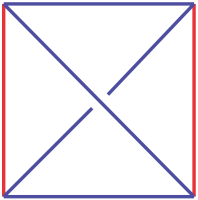
(the sum of negative contributions from type 1 vertical quads along all red edges)
$+$ (the sum of positive contributions from type 1 vertical quads along all red edges) $= 0$.

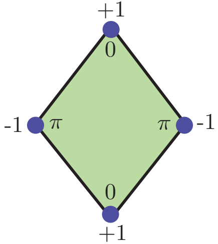
This means that if q is a type 1 vertical quad with $x_q > 0$ then all corners of the quad are on red edges, and the quad lies in a tetrahedron with 4 red edges and 2 blue edges as shown below.





Similarly for type 2 vertical quads with $x_q > 0$, all 4 corners are on blue edges.





It follows that the Q-matching equations split into two subsets:

- the equations for red edges and type 1 vertical quads
- the equations for blue edges and type 2 vertical quads

Thus, without loss of generality, we can assume *we have a solution with only type 1 vertical quads*.

Then there is *only one quad type in each tetrahedron*, so we have an "admissible solution" x to the Q-matching equations. By standard results about these equations we can assume x gives

an *embedded spun normal surface S, connected and oriented,* made up of type 1 vertical quads and triangles.

Now each quad occurring in S lies in a tetrahedron with 4 red edges and has 4 red corners:

- 2 corners with angle π and sign $+1$ in the Q-matching equations, and
- 2 corners with angle 0 and sign -1 in the Q-matching equations.

This implies that in the cell decomposition of S into normal quads and triangles:

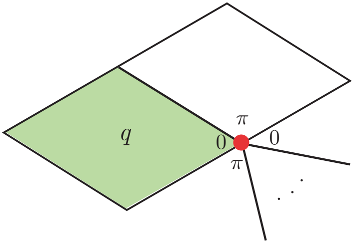
at each vertex we have at most 2 quads with angle π (since angle sum is 2π) and each of these contributes sign $+1$ to Q-matching equations so needs another quad corner with sign -1 to cancel.

So we have *at most 4 quads at each vertex* (and perhaps some triangles).

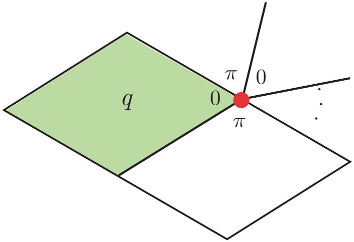
Now the veering condition gives:

Lemma. Let e be an edge with angle 0 in a tetrahedron with 4 red edges. Then the two neighbouring tetrahedra around e have angle π at e.

This implies if q is a quad in S with angle 0 and sign -1 at a vertex v, then both adjacent cells around v have angle π at v and at least one of these is a quad with sign $+1$ at v.

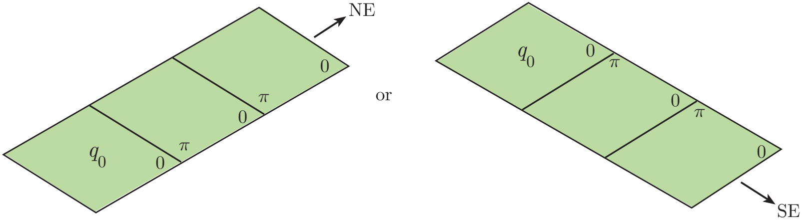


or

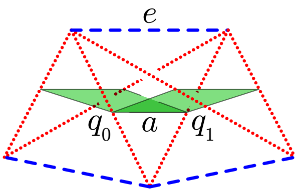


This leads to the following:

Lemma. For some starting quad q_0, we can find an arbitrarily long strip of adjacent quads in S continuing in the NE direction or in the SE direction.



Orientability of S implies that this strip must consist of quads facing a single edge e of the triangulation:



The dual edge remains the same as we walk across quads.

But then the angle sum around e would be zero, contradicting the definition of a taut angle structure. This completes the proof of the main theorem. $\square$

Some references

Ian Agol, *Ideal triangulations of pseudo-Anosov mapping tori*, arXiv:1008.1606.

David Futer, François Guéritaud, *Explicit angle structures for veering triangulations*, arXiv:1012.5134.

François Guéritaud, *On canonical triangulations of once-punctured torus bundles and two-bridge link complements*, with an appendix by David Futer, Geometry & Topology **10** (2006), 1239–1284.

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