Robust stabilization of discrete-time nonlinear Lur’e systems with sector and slope restricted nonlinearities

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Abstract

In this paper, we present a novel method for the robust control problem of uncertain nonlinear discrete-time linear systems with sector and slope restrictions. The nonlinear function considered in this paper is expressed as convex combinations of sector and slope bounds. Then the equality constraint is derived by using convex properties of the nonlinear function. A stabilization criterion for the existence of the state feedback controller is derived in terms of linear matrix inequalities (LMIs) by using Finsler’s lemma. The proposed method is demonstrated by a system with saturation nonlinearity.

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1. Introduction

Sector bounded nonlinearities are commonly encountered in practice such as saturation, quantization, backlash, deadzones, and so on. The existence of sector bounded nonlinearities is a source of degradation or instability in system performance.

Systems with a feedback connection of a linear dynamical system and a sector bounded nonlinear element have been extensively studied since as early as the 1940’s when Lur’e and Postnikov [1] first proposed the concept of absolute stability and it has been addressed extensively in the nonlinear systems and control literature [2–9]. Using the concept of absolute stability theory, there have been various different criteria for the systems with the nonlinearities which is additionally considered slope-restricted [7–11,16], but stabilization technique for the systems has not been considered yet.

In this paper, we investigate the state feedback controller synthesis problem for the Lur’e systems with sector and slope restricted nonlinearities in discrete-time domain. Recently, Johansson et al. [12] proposed the observer-based control system design method for Lur’e systems using strictly positive real lemma. Arcak

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et al. [13] presented an observer design technique for systems with slope-restricted nonlinearities. Also, as an other control technique, the variable structure control scheme has been proposed for the systems with sector nonlinearities [14,15].

This paper presents an asymptotic stabilization criteria for sector and slope restricted Lur'e systems by using the convexity. The nonlinear functions is expressed as convex combinations of sector and slope bounds, so that the equality constraint is derived by using convex properties of the nonlinear function. As a result, a stabilization criterion for the existence of the state feedback controller is derived in terms of LMIs. A numerical example is illustrated to demonstrate the proposed method.

2. Problem statement

Consider a discrete-time system with parametric nonlinear uncertainty:

\[ x(k+1) = Ax(k) + B\phi(q(k)) + B_n u(k), \]

\[ q(k) = Cx(k), \]

\[ \phi(q(k)) \triangleq \begin{bmatrix}
    \phi_1(q_1(k)) \\
    \vdots \\
    \phi_m(q_m(k))
\end{bmatrix} \]

(1)

where \( x(k) \in \mathbb{R}^n \) is the state vector, \( u(k) \in \mathbb{R}^n \) is the control input vector, \( q(k) \in \mathbb{R}^m \) is the output vector, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{m \times n} \), \( A \) is asymptotically stable (Hurwitz), and \( \phi(\cdot) \) is memoryless time-invariant nonlinearities with sector bound and slope restriction such as

\[ b_1 \leq \frac{\phi_i(\sigma)}{\sigma} \leq a_i, \]

\[ b_i' \leq \frac{\phi_i(\sigma_1) - \phi_i(\sigma_2)}{\sigma_1 - \sigma_2} \leq a_i', \quad \sigma_1 \neq \sigma_2, \]

(2)

(3)

where \( b_i \) an \( a_i \) are lower and upper sector bound, respectively, and \( b_i' \) and \( a_i' \) are lower and upper slope bound, respectively.

The nonlinear function \( \phi(\cdot) \) can be represented by following equation as shown in Fig. 1: (See Fig. 2):

\[ \phi(q) = A(q)q = \begin{bmatrix}
    A_1(q_1)q_1 \\
    \vdots \\
    A_m(q_m)q_m
\end{bmatrix} = \begin{bmatrix}
    \{ A_{11}(q_1)a_1 + A_{12}(q_1)b_1 \}q_1 \\
    \vdots \\
    \{ A_{m1}(q_m)a_m + A_{m2}(q_m)b_m \}q_m
\end{bmatrix}, \]

(4)

where

\[ A_{11}(q_1) \triangleq \frac{w_{12}(q_1)}{w_{11}(q_1) + w_{12}(q_1)} = \frac{\phi_1(q_1) - b_1q_1}{(a_1 - b_1)q_1}, \]

\[ A_{12}(q_1) \triangleq \frac{w_{11}(q_1)}{w_{11}(q_1) + w_{12}(q_1)} = \frac{a_1q_1 - \phi_1(q_1)}{(a_1 - b_1)q_1}, \]

\[ A_{11}(q_1) + A_{12}(q_1) = 1 \quad \text{and} \quad A_{11}(q_1), \quad A_{12}(q_1) \geq 0. \]

Let us define

\[ A' \triangleq \text{diag}\{b_1, b_2, \ldots, b_m\}, \quad A'' \triangleq \text{diag}\{a_1, a_2, \ldots, a_m\}, \]

\[ \overline{A'} \triangleq \text{diag}\{b_1', b_2', \ldots, b_m'\}, \quad \overline{A''} \triangleq \text{diag}\{a_1', a_2', \ldots, a_m'\}, \]

(5)

where diag\{\ldots\} denotes the diagonal matrix.

Then the functions \( A(q) \) and \( \overline{A}(q) \) can be represented by:

\[ A(q) \in \text{Co}\{A', A''\}, \quad \overline{A}(q) \in \text{Co}\{\overline{A'}, \overline{A''}\}, \]

(6)

where Co denotes the convex hull.
Also, the parameter $A(q)$ belongs to the following set:

$$
\mathbf{A} := \{ A(q) \in \mathbb{R}^{m \times m} | A(q) \in \text{Co}\{ A', A'' \}, \, \overline{A}(q) \in \text{Co}\{ \overline{A}', \overline{A}'' \} \}. 
$$

(7)

Define $\nabla := \text{diag}\{ A(q), \overline{A}(q) \}$ in the following set:

$$
\mathbf{\Phi} := \{ \text{diag}(A, \overline{A}) | A \in \text{Co}\{ A', A'' \}, \, \overline{A} \in \text{Co}\{ \overline{A}', \overline{A}'' \} \}. 
$$

(8)
Let now consider the state feedback controller
\[ u(k) = Kx(k). \] (9)
Then the closed loop system is written by
\[ x(k + 1) = (A + B_uK)x(k) + B\phi(q(k)), \]
\[ q(k) = Cx(k). \] (10)
For simplicity, define the following augmented matrices
\[
\begin{bmatrix}
C & 0 \\
C(A + B_uK - I) & CB \\
0 & I \\
0 & -I
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
I
\end{bmatrix}.
\] (11)
The following lemmas will be used for deriving our main results.

**Lemma 1** (Finler’s Lemma [10,11]). Let matrices \( Q = Q^T, F, \) and a compact subset of real matrices \( \mathcal{H} \) be given. The following statements are equivalent:

- for each \( H \in \mathcal{H} \)
  \[ \xi^TQ\xi < 0 \quad \forall \xi \neq 0 \quad \text{such that} \quad HF\xi = 0, \] (12)
- there exists a matrix \( \Theta = \Theta^T \) such that
  \[ Q + F^T\Theta F < 0, \quad \mathcal{N}_H^T\Theta\mathcal{N}_H \geq 0, \quad \forall H \in \mathcal{H}, \] (13)
where \( \mathcal{N}_H \) is a matrix belong to a null space of \( H \).

**Lemma 2** (Dualization lemma [17]). Let \( P \) be a non singular symmetric matrix in \( \mathbb{R}^{n \times n} \), and let \( U, V \) be two complementary subspaces whose sum equals \( \mathbb{R}^n \). Then
\[ x^TPx < 0 \quad \text{for all} \quad x \in U \setminus \{0\} \quad \text{and} \quad x^TPx \geq 0 \quad \text{for all} \quad x \in V \] (14)
is equivalent to
\[ x^TP^{-1}x > 0 \quad \text{for all} \quad x \in U \cap \{0\} \quad \text{and} \quad x^TP^{-1}x \leq 0 \quad \text{for all} \quad x \in V^\perp. \] (15)

3. Main results

From the description and preliminary results in Section 2, we have the following main result for stabilization of Lur’e system (1).

**Theorem 1.** Consider the system (1). There exists a state feedback controller that stabilizes the system (1) if there exist \( Q = Q^T > 0, X, Y, H, \) and \( \Pi \) such that the following LMIs hold:
\[
\begin{bmatrix}
\Phi_{11} & 0 & \Phi_{13} \\
\ast & \Phi_{22} & \Phi_{23} \\
\ast & \ast & \Phi_{33}
\end{bmatrix} > 0,
\begin{bmatrix}
-\nabla^T \\
I
\end{bmatrix}^T \Pi \begin{bmatrix}
-\nabla^T \\
I
\end{bmatrix} \leq 0, \quad \forall \ \nabla \in \Phi
\] (16)
where
\[
\Phi_{11} = Q, \quad \Phi_{13} = \begin{bmatrix} AX + B_uH & BY \\ 0 & 0 \end{bmatrix}, \quad \Phi_{22} = \Pi, \\
\Phi_{23} = \begin{bmatrix} CX \\ C(AX + B_uH - X) \\ 0 \\ 0 \end{bmatrix}, \quad \Phi_{33} = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} - Q, \quad H = KX.
\]

**Proof.** Construct the following Lyapunov function candidate
\[
V(x(k)) = x_T^T(k)Px_a(k),
\]
where \(x_T^T(k) = [x^T(k)\phi^T(q(k))]\) and \(P = Q^{-1}\).

Let us define the following variables:
\[
x = x(k), \quad p = \phi(k), \quad \omega = \phi(k + 1).
\]

Then the difference of the candidate of Lyapunov function can be written as:
\[
\begin{bmatrix} x^T \\
p \\
\omega \end{bmatrix}^T \begin{bmatrix} A + B_uK \\ B^T \\ 0 \
I \end{bmatrix} \begin{bmatrix} A + B_uK & B & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\
p \\
\omega \end{bmatrix} < 0.
\]

Using convex properties of the nonlinear function, we obtain the following constraints:
\[
\phi(k) = Aq(k) = ACx(k),
\]
\[
\phi(k + 1) = \overline{A}(q(k + 1) - q(k)) + \phi(k),
\]
\[
= \overline{A}\{C(A + B_uK - I) + CB\phi(k)} + \phi(k),
\]
and
\[
\begin{bmatrix} AC \\ \overline{A}(A + B_uK - I) \\ \overline{A}CB + I \\ -I \end{bmatrix} \begin{bmatrix} x \\
p \\
\omega \end{bmatrix} = 0.
\]

Hence, the Lyapunov inequality (21) is equivalent to
\[
\zeta^T \begin{bmatrix} I \\ 0 \end{bmatrix} P \begin{bmatrix} \tilde{A} & \tilde{B} \\ 0 & I \end{bmatrix} \zeta < 0,
\]
for all nonzero vector \(\zeta = [x^T p^T \omega^T]^T\), such that
\[
[\nabla -I][\tilde{C} \quad \tilde{D}] \zeta = 0,
\]
for all \(\nabla \in \Phi\).

By Lemma 1, the inequality (21) and the equality constraint (23) hold if there exist positive symmetric \(P\) and \(\Theta\) subject to
\[
\begin{bmatrix} \tilde{A} & \tilde{B} \\ 0 & I \\ \tilde{C} & \tilde{D} \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & \Theta & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} \\ 0 & I \\ \tilde{C} & \tilde{D} \\ I & 0 \end{bmatrix} < 0
\]
and
\[
\begin{bmatrix} I \\ \nabla \end{bmatrix}^T \Theta \begin{bmatrix} I \\ \nabla \end{bmatrix} \geq 0 \quad \forall \nabla \in \Phi
\]
where \(\Theta = \Pi^{-1}\).
Using Lemma 2, the stability condition of the closed-loop system is equivalent to

\[
\begin{bmatrix}
I & 0 \\
-\overline{B}^T & -\overline{D}^T \\
0 & I \\
-\overline{A}^T & -\overline{C}^T
\end{bmatrix}^T
\begin{bmatrix}
P^{-1} & 0 & 0 \\
0 & \Theta^{-1} & 0 \\
0 & 0 & -P^{-1}
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
-\overline{B}^T & -\overline{D}^T \\
0 & I \\
-\overline{A}^T & -\overline{C}^T
\end{bmatrix}
> 0,
\]

(28)

\[
\begin{bmatrix}
-\nabla^T & I \\
I & \Theta^{-1} - \nabla^T
\end{bmatrix} \leq 0, \quad \forall \nabla \in \Phi.
\]

(29)

By the elimination lemma [18], the inequality (28) is equivalent to

\[
\begin{bmatrix}
P^{-1} & 0 & 0 \\
0 & \Theta^{-1} & 0 \\
0 & 0 & -P^{-1}
\end{bmatrix}
+ \begin{bmatrix}
\bar{A} & \bar{B} \\
0 & I \\
\bar{C} & \bar{D}
\end{bmatrix} G \begin{bmatrix}
0 & 0 & I \\
0 & 0 & I
\end{bmatrix} G^T \begin{bmatrix}
\bar{A}^T & \bar{B}^T & \bar{C}^T \\
0 & I & 0
\end{bmatrix} > 0,
\]

(30)

where

\[
G = \begin{bmatrix}
X & 0 \\
0 & Y \\
0 & 0
\end{bmatrix}.
\]

Finally, the inequality (16) is easily derived from Eq. (30) by some matrix manipulation. This completes the proof. \(\square\)

4. Numerical example

Consider the following system with saturation nonlinearity

\[
\begin{align*}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= \text{sat}(x_1(k) + x_2(k)) + u(k)
\end{align*}
\]

(31)

The equivalent system can be expressed as the form (1)

\[
\begin{align*}
x(k+1) &= \begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix} x(k) - \begin{bmatrix}
0 \\
1
\end{bmatrix} \phi(q(k)) + \begin{bmatrix}
0 \\
1
\end{bmatrix} u(k) \\
q(k) &= \begin{bmatrix}
1 & 1
\end{bmatrix} x(k)
\end{align*}
\]

(32)

where

\[
\phi(q(k)) = q(k) - \text{sat}(q(k)),
\]

(33)

with sector and slope condition such as

\[
0 \leq \frac{\phi(q(k))}{q(k)} \leq \delta,
\]

\[
\frac{\phi(q(k+1)) - \phi(q(k))}{q(k+1) - q(k)} = \begin{cases}
0, & |q(k)| < q_{\text{max}} \\
1, & |q(k)| > q_{\text{max}}
\end{cases}
\]

(34)

By applying Theorem 1, the state feedback gain can be obtained with sector bound 1 and slope bound 1:

\[
K = [-0.9825 \quad -1.2380].
\]

(35)
Since the feasible solution is obtained with sector bound 1, it implies that the system can be globally stabili-
able by the controller obtained. Fig. 3 shows simulation result with initial condition $x(0) = \begin{bmatrix} 10 & 10 \end{bmatrix}^T$. One can see that the proposed controller well stabilizes the system (32).

5. Conclusions

In this paper, we propose a robust asymptotic stabilization method for Lur’e systems with sector and slope restrictions. Equality constraint is derived by using convex properties of the nonlinear functions. Then a stabiliz-
ation condition for the systems is derived by the dualization lemma. In order to demonstrate the effective-
ess of the proposed method, a numerical example for a system with saturation nonlinearity is provided.

References

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