Queue layouts on folded hypercubes FQ_n with $n \leqslant 7$

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Abstract

A queue layout of a graph consists of a linear order of its vertices and a partition of its edges into queues, such that no two edges in the same queue are nested. The minimum number of queues in a queue layout of a graph G, denoted by $qn(G)$, is called the queuenumber of G. An n-dimensional folded hypercube, denoted by FQ_n, is an enhanced n-dimensional hypercube with one extra edge between vertices that have the furthest Hamming distance. In this paper, we deal with queue layout of folded hypercubes and contribute some results as follows:

(1) $qn(FQ_n) = 2$ *if* $n \in \{2, 3\}$.
(2) $2 \leqslant qn(FQ_4) \leqslant 4$.
(3) $2 \leqslant qn(FQ_5) \leqslant 6$.
(4) $2 \leqslant qn(FQ_6) \leqslant 7$.
(5) $3 \leqslant qn(FQ_7) \leqslant 12$.

Keywords: Combinatorial problems; Queue layout; Folded hypercube; Interconnection networks;

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex ordering σ of G is a bijection from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$. For $u, v \in V(G)$, we write $u <_\sigma v$ if $\sigma(u) < \sigma(v)$. A *k-queue layout* of a graph G consists of a vertex ordering σ and a partition of its edges into k queues such that no two edges in the same queue are nested (i.e., two edges $(u, v), (x, y) \in E(G)$ are *nested* if $u <_\sigma x <_\sigma y <_\sigma v$). The minimum k for which G has a k-queue layout under a given vertex ordering σ is denoted by $qn(G, \sigma)$. The *queuenumber* of a graph G is defined as $qn(G) = \min\{qn(G, \sigma) \mid \sigma$ is a vertex ordering of $G\}$. A graph G is a *k-queue graph* if $qn(G) \leqslant k$.

Queue layouts were first introduced by Heath et al. [7, 10]. There are many applications of queue layout in computer science, including sorting permutations, parallel process scheduling, matrix computations, and graph drawing (see [1, 10, 16] and references therein quoted). In particular, queue layouts of interconnection networks have applications to the Diogenes approach to testable fault-tolerant arrays of processors [16]. Heath and Rosenberg [10] showed that the problem of recognizing k-queue graphs is NP-complete even if $k = 1$. Thus, further investigations tended to study bounds on queuenumber for certain families of graphs [1, 2, 4–10, 12–15, 17, 18].

The *n-dimensional hypercube* Q_n is a graph with 2^n vertices in which each vertex corresponds to an n-tuple $(b_{n-1}, b_{n-2}, \ldots, b_0)$ on the set $\{0, 1\}^n$ and two vertices are linked by an edge if and only if they differ in exactly one coordinate [11]. For simplicity, we write $b_{n-1}b_{n-2}\cdots b_0$ instead of $(b_{n-1}, b_{n-2}, \ldots, b_0)$. In 1992, Heath and Rosenberg [10] first showed that the n-dimensional hypercube can be laid out using at most $n - 1$ queues.

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Hasunuma and Hirota [6] recently revisited the queue layout problem on hypercubes and showed that $qn(Q_n) \leqslant n-2$ for all $n \geqslant 5$. Pai et al. [12] showed that the same upper bound also holds for $n = 4$. Subsequently, Pai et al. [14] showed that $qn(Q_n) \leqslant n-3$ for all $n \geqslant 8$. Note that all these bounds are derived by induction, and barriers to improving those bounds are encountered in the base step.

As a variation of the hypercube Q_n, the n-dimensional folded hypercube, denoted by FQ_n, is defined as follows: FQ_n is an $(n+1)$-regular graph with vertex set $V(FQ_n) = V(Q_n)$ and edge set $E(FQ_n) = E(Q_n) \cup E_0$, where $E_0 = \{(x, \bar{x}) \mid x \in V(Q_n)\}$ [3]. In other words, FQ_n is a strengthened variation of Q_n by adding extra edges between any pair of vertices whose binary strings are complements of each other. For example, Figure 1 depicts FQ_2 and FQ_3, where thick lines represent complement edges and thin lines represent normal edges. In this paper, we deal with queue layout of folded hypercubes.

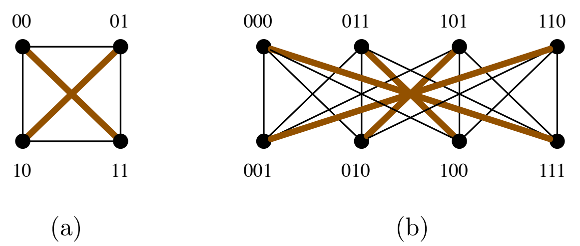

Figure 1: (a) FQ_2 ; (b) FQ_3.

2 Queue layout of FQ_n

Let K_n denote the complete graph with n vertices, and let $K_{m,n}$ denote the complete bipartite graph with partite sets of m vertices and n vertices, respectively. The following lemma is due to Heath and Rosenberg [10].

Lemma 1 (Heath and Rosenberg [10]) *$qn(K_n) = \lfloor \frac{n}{2} \rfloor$ and $qn(K_{m,n}) = min(\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil)$.*

From the isomorphism of FQ_2 and K_4, and the isomorphism of FQ_3 and $K_{4,4}$, the following corollary directly follows from Lemma 1.

Corollary 2 *$qn(FQ_n) = 2$ if $n \in \{2, 3\}$.*

Given a vertex ordering σ of a graph $G = (V, E)$, the *length* of an edge $(u, v) \in E$ is defined to be $\ell_\sigma(u, v) = |\sigma(u) - \sigma(v)|$. Note that if $|\ell_\sigma(u, v) - \ell_\sigma(x, y)| \leqslant 1$, then (u, v) and (x, y) do not nest. Let σ_1 and σ_2 be two vertex orderings with no common vertex. The *concatenation* of σ_1 and σ_2, written $\sigma_1 \circ \sigma_2$, is the ordering σ_1 followed by the ordering σ_2.

For Q_n, a vertex ordering is called the *complementary vertex ordering* (CVO for short) if all vertices of $V(Q_n)$ are arranged in a particular way $u_1 v_1 \cdots u_i v_i \cdots u_{2^{n-1}} v_{2^{n-1}}$ where $u_i = \bar{v}_i$. For FQ_n, we denote FQ_n^i, $i \in \{0, 1\}$, as the subgraph of FQ_n induced by the set of vertices with i as the leading bit in their labels. Clearly, each subgraph FQ_n^i is isomorphic to Q_{n-1} under the isomorphism $\varphi(x) = x'$, where $x = i x_{n-1} \cdots x_1$ is a vertex in FQ_n^i and $x' = x_{n-1} \cdots x_1$ is a vertex in Q_{n-1}. To simplify notation, if a vertex v is contained in a subgraph FQ_n^i, we write $v \in FQ_n^i$ instead of $v \in V(FQ_n^i)$. A normal edge $(u, v) \in E(FQ_n)$ is called the *leading edge* provided $u \in FQ_n^0$ and $v \in FQ_n^1$.

The following lemma provides an induction to derive our upper bounds.

Lemma 3 *For $n \geqslant 3$, if σ is a CVO of Q_{n-1}, then $qn(FQ_n) \leqslant qn(Q_{n-1}, \sigma) + 2$.*

Proof. Let σ be a CVO of Q_{n-1}. For $i \in \{0, 1\}$, since FQ_n^i is isomorphic to Q_{n-1}, let σ_i denote the vertex ordering of FQ_n^i that is corresponding to σ in Q_{n-1}. Let $\Pi = \sigma_0 \circ \sigma_1$ be a particular vertex ordering of FQ_n. We now show that the vertex ordering Π of FQ_n permits a $(qn(Q_{n-1}, \sigma) + 2)$-queue layout.

Let $E_0 = E(FQ_n^0)$ and $E_1 = E(FQ_n^1)$. Also, let E_n and E_c denote the set of leading edges and the set of complement edges of FQ_n, respectively. Obviously, $E(FQ_n) = E_0 \cup E_1 \cup E_n \cup E_c$. Since vertices of each subgraph FQ_n^i for $i \in \{0, 1\}$ are arranged in the ordering as σ and E_0 and E_1 occur in the sequence of Π, all edges of $E_0 \cup E_1$ can be partitioned into $qn(Q_{n-1}, \sigma)$ queues without nested edges. Moreover, since all leading edges have the same length 2^{n-1}, edges of E_n can be assigned to a queue without producing nested edges. Also, since the arrangement of Π is a double portion of a CVO, the length of a complement edge is either $2^{n-1}+1$ or $2^{n-1}-1$. Thus, edges of E_c with length $2^{n-1}+1$ can be placed in the same queue as leading edges placed. Finally, edges of E_c with length $2^{n-1}-1$ can be placed in an additional queue. Consequently, we assign edges of FQ_n to $qn(Q_{n-1},\sigma)+2$ queues and the lemma follows □

For example, 000, 111, 110, 001, 101, 010, 100, 011 is a CVO of Q_3. Figure 2 shows the result of two queues for placing the leading edges and the complement edges of FQ_4, where vertices in each subgraph Q_4^i for $i \in \{0,1\}$ are arranged in such a CVO. In this figure, edges above the vertices are assigned to the first queue and edges below the vertices are assigned to the second queue. Also, thin lines represent leading edges and thick lines represent complement edges.

Lemma 4 *For $3 \leqslant n \leqslant 6$, there exists a CVO of Q_n, say σ, such that the following upper bounds are fulfilled.*

(1) $qn(Q_3,\sigma) \leqslant 2$.

(2) $qn(Q_4,\sigma) \leqslant 4$.

(3) $qn(Q_5,\sigma) \leqslant 5$.

(4) $qn(Q_6,\sigma) \leqslant 10$.

Proof. Note that Q_3 contains 12 edges. To show the upper bound stated in (1), it suffices to find a CVO of Q_3 and then partition its edges into two edge-disjoint spanning subgraphs so that all edges in a subgraph do not nest under such a CVO. Figure 3 shows the desired CVO $000, 111, 001, 110, 101, 010, 100, 011$ and the partition of edges, where V_{Dec} (respectively, V_{Bin}) denotes the decimal (respectively, binary) representation of vertices, and a vertical line between two vertices means an edge of Q_3. It is not hard to verify the validity of the drawings.

For showing the validity of (2)-(4), Figures 4, 5 and 6 provide the desired CVO and the partition of edges for Q_4, Q_5 and Q_6, respectively. Thus, the correctness directly follows. □

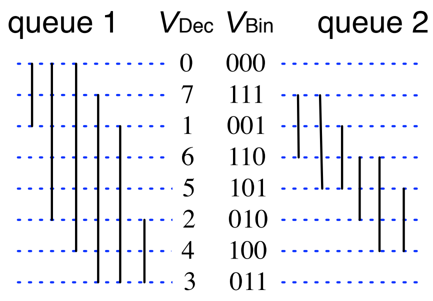

Figure 3: A 2-queue layout of Q_3.

Lemma 5 (Heath and Rosenberg [10]) *For any graph $G=(V,E)$, $qn(G) \geqslant \lceil \frac{|E|}{2|V|-3} \rceil$.*

Since FQ_n contains 2^n vertices and $(n+1)2^{n-1}$ edges, by Lemma 5, $qn(FQ_n) \geqslant \lceil \frac{(n+1)2^{n-1}}{2^{n+1}-3} \rceil$. In particular, $qn(FQ_n) \geqslant 2$ if $n \in \{4,5,6\}$ and $qn(FQ_7) \geqslant 3$.

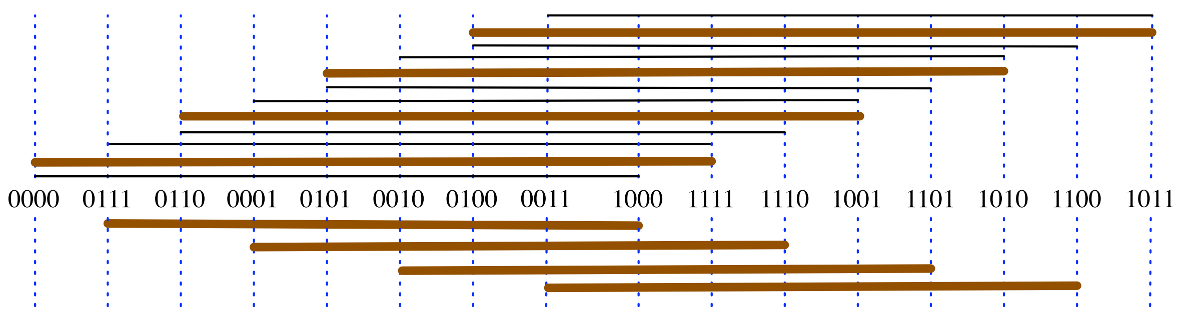

Figure 2: Two queues place the leading edges and the complement edges of FQ_4.

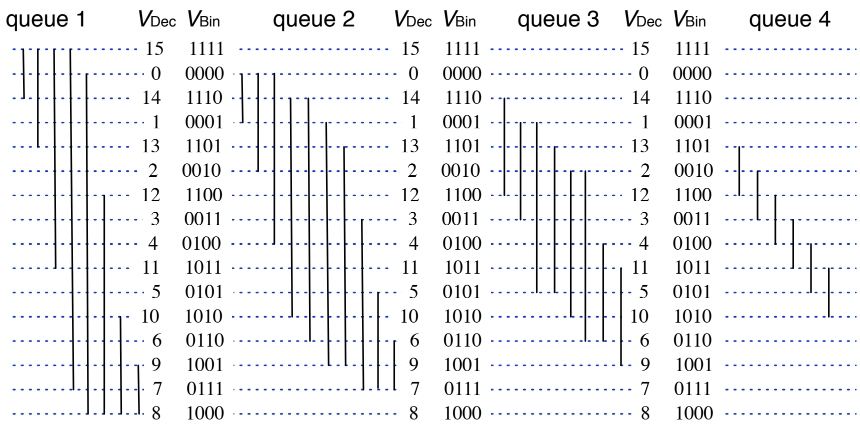

Figure 4: A 4-queue layout of Q_4.

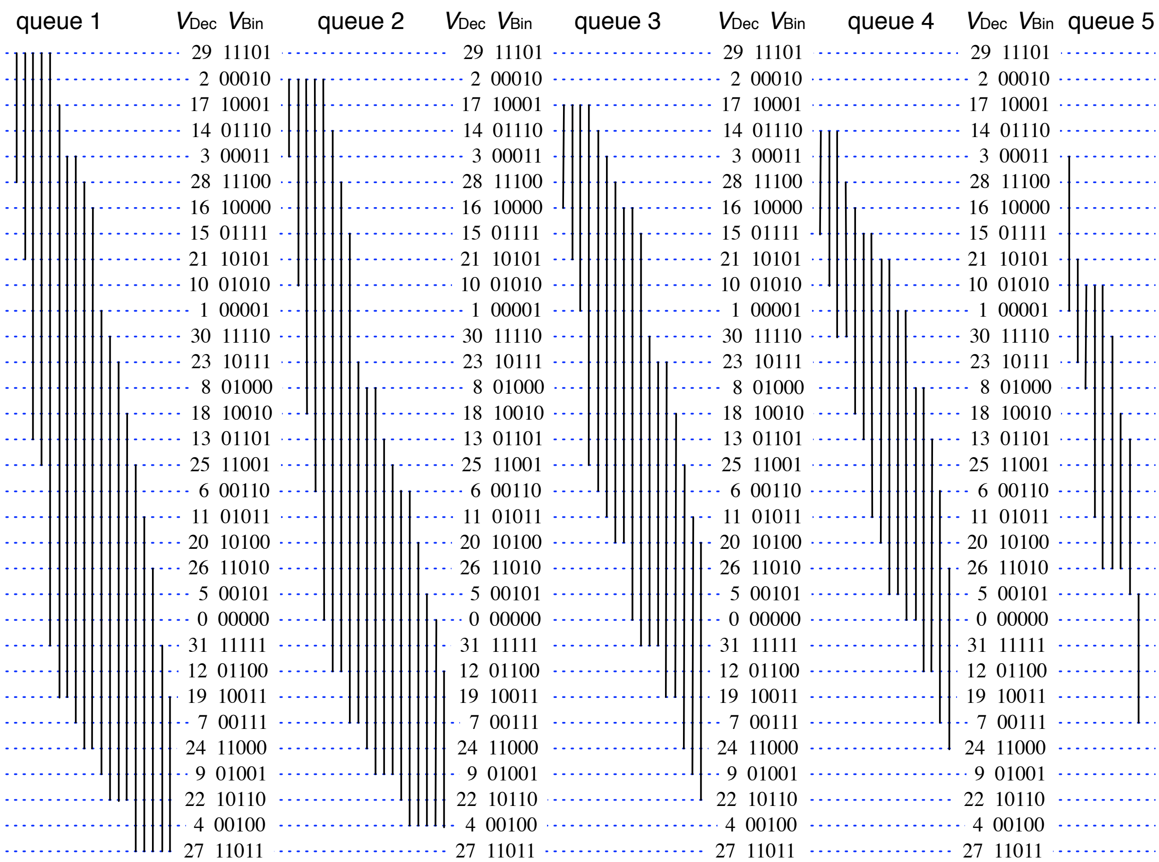

Figure 5: A 5-queue layout of Q_5.

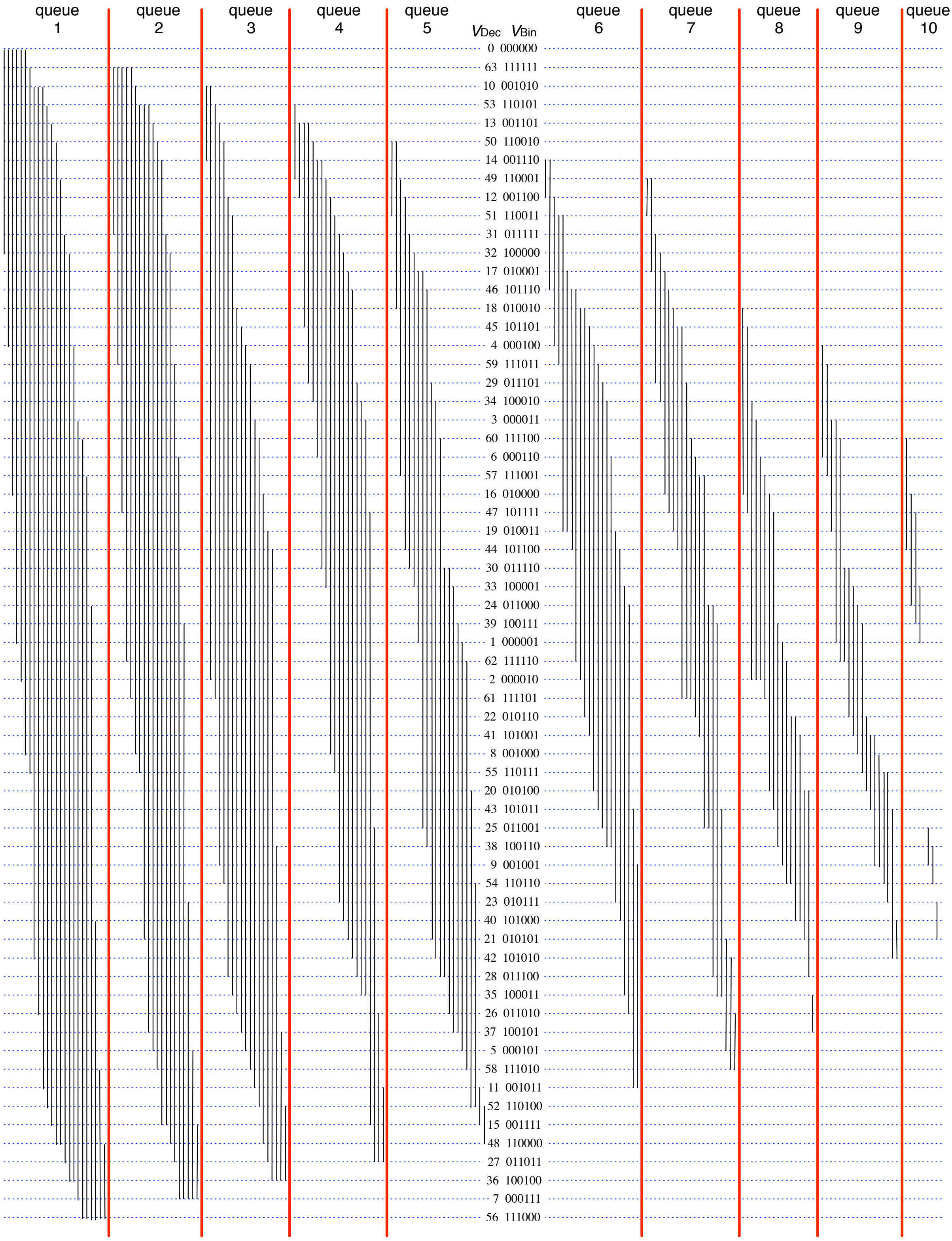

Figure 6: A 10-queue layout of Q_6.

Consequently, combining the results of Corollary 2 and Lemmas 3, 4 and 5, we immediately obtain the following theorem.

Theorem 6 *The bounds of* $qn(FQ_n)$ *for* $2 \leqslant n \leqslant 7$ *are the following:*

(1) $qn(FQ_n) = 2$ *if* $n \in \{2, 3\}$,

(2) $2 \leqslant qn(FQ_4) \leqslant 4$,

(3) $2 \leqslant qn(FQ_5) \leqslant 6$,

(4) $2 \leqslant qn(FQ_6) \leqslant 7$,

(5) $3 \leqslant qn(FQ_7) \leqslant 12$.

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