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Representations of the Magnitudes of Fractions

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We tested whether adults can use integrated, analog, magnitude representations to compare the values of fractions. The only previous study on this question concluded that even college students cannot form such representations and instead compare fraction magnitudes by representing numerators and denominators as separate whole numbers. However, atypical characteristics of the presented fractions might have provoked the use of atypical comparison strategies in that study. In our 3 experiments, university and community college students compared more balanced sets of single-digit and multi-digit fractions and consistently exhibited a logarithmic distance effect. Thus, adults used integrated, analog representations, akin to a mental number line, to compare fraction magnitudes. We interpret differences between the past and present findings in terms of different stimuli eliciting different solution strategies.

Keywords: numerical fractions, distance effect, magnitude representation, mental number line, comparison strategies

On a wide range of tasks, people represent natural number magnitudes in a form akin to a mental number line. For example, when people compare two natural numbers, their solution times decrease logarithmically with increasing numerical distance between the numbers (Dehaene, Dupoux, & Mehler, 1990). The continuous function suggests that people use analog magnitude representations to compare natural numbers (Moyer & Landauer, 1967). The assumption of an integrated analog representation for natural numbers, akin to a mental number line, also explains interference between the numerical content of tasks and visuospatial task characteristics (Fias & Fischer, 2005) as well as overlaps between representations of natural numbers and representations of space in the parietal cortex of the human brain (Hubbard, Piazza, Pinel, & Dehaene, 2005).

The present study extends research on numerical representations to the domain of fractions. In particular, it asked whether the magnitudes of fractions are represented in an integrated, analog format, similar to the mental number line used to represent natural numbers. The one published study that has examined this issue (Bonato, Fabbri, Umiltà, & Zorzi, 2007) concluded that even educated adults do not represent the magnitudes of fractions in any integrated form. Instead, Bonato et al. (2007) argued that adults use strategies involving comparisons of the whole number components of fractions (numerators or denominators) to circumvent their inability to think about fractions as integrated magnitudes. However, the results of the three experiments reported in the present study argue against this conclusion. Our findings indicate that students at both highly selective universities and nonselective community colleges can represent the magnitudes of both single- and multi-digit fractions on the same type of mental number line as they use to represent natural numbers.

Mental Number Line Representations of Natural Numbers

Several recent literature reviews indicate broad agreement about many characteristics of mental number line representations of natural numbers (Ansari, 2008; Izard & Dehaene, 2008). The representation is used with multi-digit as well as single-digit numbers (Dehaene et al., 1990), although additional factors seem to come into play for multi-digit numbers (Nuerk, Weger, & Willmes, 2001). Its use is evident by age 4 years, and some findings suggest that it is used even in infancy (Cantlon, Brannon, Carter, & Pelphry, 2006; Lipton & Spelke, 2003). Neural as well as behavioral data indicate that such numerical representations have a great deal in common with spatial representations of quantities; quantitative information expressed in either numerical or spatial form triggers activation of highly overlapping areas of the intraparietal sulcus, and some individual neurons even respond to both numerical and spatial stimuli (Tudusciuc & Nieder, 2007).

The mental number line construct is of more than theoretical importance. Although the representation appears to be very widely used during both childhood and adulthood, the precision with which it is used is related to a variety of important mathematical skills as well as to overall mathematical competence. Individual differences in the precision of children's whole number magnitude representations are closely related to individual differences in arithmetic, several types of estimation, magnitude comparison, numerical categorization, and overall mathematics achievement test scores (Booth & Siegler, 2008; Gilmore, McCarthy, & Spelke, 2007; Halberda & Feigenson, 2008; Holloway & Ansari, in press; Laski & Siegler, 2007; Ramani & Siegler, 2008). The relation is

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causal as well. Experimental manipulations that improve the precision of whole number magnitude representations also improve proficiency at magnitude comparison, numerical categorization, estimation, and subsequent ability to learn answers to arithmetic problems (Booth & Siegler, 2008; Siegler & Ramani, 2008; Whyte & Bull, 2008). More generally, the mental number line provides an intuitive feel for the magnitudes of whole numbers and their interrelations, sometimes referred to as *number sense* (Dehaene, 1997; Jordan, Kaplan, Olah, & Locuniak, 2006; Jordan, Kaplan, Locuniak, & Ramineni, 2007; Schneider et al., 2008).

Are Fractions Represented on a Mental Number Line?

Although these and other studies provide ample evidence that natural numbers are represented on a mental number line, there are both mathematical and empirical reasons to question whether fractions are represented in the same way. Fractions differ from whole numbers in many mathematical ways that could easily affect how they are represented. Fractions are infinitely divisible. They are not linked by successor relations; no fraction comes immediately before or after another fraction, and between any two fractions are an infinite number of other fractions. This makes it impossible to count fractions directly, which precludes one of the main processes through which people learn about whole numbers. The magnitudes of fractions do not increase in any consistent way with the size of their components; 7 is greater than 4, but a fraction with a numerator of 7 may or may not be larger than a fraction with a numerator of 4. All of these properties seem likely to interfere with formation of a mental number line for fractions.

Empirical evidence of a variety of types provides further reason to doubt whether fraction magnitudes are represented on a mental number line. Whole number magnitudes are represented fairly accurately across diverse species, cultures, and age groups, leading to proposals that mental number line representations of whole numbers are innate (e.g., Feigenson, Dehaene, & Spelke, 2004; Geary, 2005). Clearly, none of these properties hold true for representations of fractions. Consider just the developmental data. Infants accurately approximate whole number magnitudes (Feigenson, Carey, & Spelke, 2002), yet preadolescents and adolescents have considerable difficulty approximating fraction magnitudes. This difficulty representing fraction magnitudes is evident in people's errors on tasks such as numerical magnitude comparison, arithmetic, and estimation—tasks that with whole numbers involve use of a mental number line representation (Ansari, 2008). It would be hard to find a U.S. fourth grader who did not know that 345 is larger than 67, yet the large majority of U.S. fourth graders err in choosing the larger of .345 and .67 (Resnick & Omanson, 1987; Rittle-Johnson, Siegler, & Alibali, 2001).

The same discrepancy between knowledge of whole numbers and fractions is evident on arithmetic problems: A long search would be required to find a second grader who would claim that $3 + 3 = 3$, yet middle school students frequently claim the equivalent when they write that $1/3 + 1/3 = 2/6$ (National Mathematics Advisory Panel, 2008). Estimation provides another such case. Approximation of whole number magnitudes between 0 and 100 is very accurate even among second graders (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007), yet when a large, nationally representative sample of U.S. eighth graders was asked to estimate the closest of four answers to $12/13 + 7/8$, both 19 and 21 were chosen more often than 2 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981). Even in ninth grade, roughly half of students believe that fractions form a countable series, rather than being infinitely divisible (Vamvakoussi & Vosniadou, 2004). In all of these cases and many others, children confuse the characteristics of fractions with those of whole numbers, a characteristic that has been labeled the *whole number bias* (Gelman, 1991; Ni & Zhou, 2005).

Empirical Evidence About the Mental Representation of Fractions

In the only previously published article that directly examined whether adults use the mental number line to represent fractional magnitudes, Bonato et al. (2007) concluded that they do not. Indeed, Bonato et al. went further and claimed that even educated adults do not represent the magnitude of fractions at all. Thus, they wrote that in university students' comparison of fractional magnitudes, "the real numerical value of the fraction was not accessed" (p. 1410). Instead, Bonato et al. concluded that even educated adults use strategies involving whole number components (numerators and denominators) to circumvent their inability to understand fractions.

The evidence on which Bonato et al. (2007) based this conclusion came from experiments on numerical comparisons of fractions. When people compare the magnitudes of pairs of whole numbers, their mean reaction time decreases as a logarithmic function of the distance between the magnitudes. The usual explanation is that the approximate magnitudes of the two numbers are activated on a mental number line where the representations of more distant numbers overlap to a lesser degree (Ansari, 2008; Izard & Dehaene, 2008; Stoianov, Kramer, Umilta, & Zorzi, 2008). This distance effect with whole numbers has been replicated so often with different age groups, nationalities, and species, and with both behavioral and brain imaging data, that it has been called "the litmus test for determining the nature of basic representations of numerical quantity" (Ansari, 2008, p. 279).

Bonato et al. (2007) reasoned that if people represent the numerator and denominator of a fraction as independent components, then the distances between numerators or between denominators should predict reaction times better than the distances between the fractions' magnitudes. In four experiments with Italian university students, Bonato et al. found that either distances between numerators or distances between denominators (depending on the stimulus set) did predict reaction times better than distances between the fractions' magnitudes. They concluded, "In line with results on the whole number bias in children, these findings suggest that the understanding of fractions is rooted in the ability to represent discrete numerosities (i.e., integers) rather than real numbers" (p. 1410).

This conclusion was consistent with the data of Bonato et al. (2007). However, their selection of stimuli raises doubts about its general validity. Simply put, their stimuli lend themselves to comparisons of whole number components in ways that unrestricted samples of fractions would not. In their Experiments 1 and 2, they presented participants with the standard fraction 1/5 and asked them to compare it with fractions of the form $1/n$. Thus, only the denominator of the presented fractions varied and contributed to differences between the fractions' magnitudes. Faced with this set of problems, it is not surprising that the participants only focused on the denominator when solving the magnitude comparison task. After all, denominator values were perfectly predictive of the correct answer, and the denominators were given in the problem rather than requiring an extra step of approximating the comparison fraction's magnitude before performing the comparison.

In their Experiment 3, Bonato et al. (2007) asked participants to compare to a standard of 1 the magnitudes of fractions of the form *n*/4, *n*/5, and *n*/6, with *n* assuming all values from 1 to 9 (excluding 4/4, 5/5, and 6/6). All of these problems could be solved correctly by applying the simple rule, "If numerator $>$ denominator, the answer is 'greater'; if numerator $<$ denominator, the answer is 'smaller.'" Distance between numerator and denominator accounted for considerable variance, but not as much variance as distance between the numerator and 5, which Bonato et al. concluded was the strategy that participants used (i.e., they converted the standard fraction to 5/5, even when the presented standard was 4/4 or 6/6). In addition to this hypothesis being ad hoc, another problem with it is that without understanding fraction magnitudes, participants would have no obvious reason to substitute 1 for 4/4, 5/5, or 6/6. Only the knowledge that these fraction magnitudes are equivalent justifies such a transformation. Yet another problem with the interpretation is that this strategy either yields no answer or an error on two of the problems, the ones where the numerator of the comparison problem was also 5, yet the error rate for the experiment made it extremely unlikely that the error rate on those problems was 50–100% (error rates on specific problems were not reported). In the stimulus set used in this experiment, distance between numerator and denominator of the comparison fraction correlated $r = 1.00$ with distance between the numerators of the comparison and standard fractions, which accounts for how distance between the numerators could correlate highly with solution times even if the numerators were not being compared.

In Experiment 4 of Bonato et al. (2007), fractions of the form used in Experiments 1 and 2 were interleaved with fractions of the form used in Experiment 3. The results were more complex than in the first three experiments, but overall were consistent with their description that participants did not access the value of the fraction as a whole to perform the comparison. Again, it was reasonable to use these simple approaches, given that they yielded fast and accurate performance on this problem set. Doing so required adaptive choices between the two useful strategies, but both children and adults are highly adept at making such choices among numerical strategies (Bisanz, 2003; Geary, 2006; Siegler, 1996).

To summarize, Bonato et al. (2007) provided compelling evidence that people do not compare fraction magnitudes when problems can be solved quickly and accurately without doing so. However, the applicability of this conclusion to fractions in general is open to question. When comparison of whole number components of fractions does not lead to consistently correct performance, as is the case with the overwhelming majority of possible fraction comparison problems, people may well compare the fraction magnitudes.

The Present Experiments

There are at least two reasons to expect that people can compare fraction magnitudes, as well as using the whole number comparison strategies documented by Bonato et al. (2007). One reason is that on problem-solving tasks in general, and numerical tasks in particular, people typically know and use multiple representations and strategies (Siegler, 1996). The numerical tasks on which people have been found to use varied representations and strategies include all four arithmetic operations, at least four types of estimation, number conservation, probability, and inversion problems (Campbell & Epp, 2005; Booth & Siegler, 2006; Geary, 2006; LeFevre, Sadesky, & Bisanz, 1996; Siegler & Stern, 1998). Whole number magnitudes are among the examples of this variability of representations and strategies. In addition to the linear representations implied by the mental number line construct, adults use circular internal representations of whole number magnitudes when asked to do so (Bächtold, Baumüller, & Brugger, 1998). Similarly, Siegler and Opfer (2003) found that most second graders used a linear representation to estimate the magnitudes of whole numbers on 0–100 number lines but a logarithmic representation on 0–1,000 number lines. The variability that is evident in representations of whole number magnitudes seems likely to characterize representations of fraction magnitudes as well.

A second argument in the same direction is that people generally choose among alternative representations and strategies in adaptive ways (Lemaire, Arnaud, & Lecacheur, 2004; Luwel, Verschaffel, Onghena, & De Corte, 2003; Siegler, 1996). Fast and accurate approaches are used when they are available, and slower or less accurate approaches are used when no better alternative is known. The consistent availability of fast and accurate approaches in Bonato et al. (2007) seems likely to have obscured the simultaneous availability of the slower and probably less accurate approach of comparing fraction magnitudes.

The present study tested four main hypotheses regarding representations of fraction magnitudes. First, when a set of stimuli is used in which comparisons of whole number components do not lead to consistently correct performance, participants will show a stronger distance effect for fraction magnitudes than for numerator magnitudes or denominator magnitudes. Second, adults will generate highly accurate performance on comparison problems where neither consistent reliance on numerators nor consistent reliance on denominators would yield high accuracy. Third, as with comparisons of whole number magnitudes, the distance effect with fraction magnitudes will more closely approximate a natural logarithmic function than a linear one. Fourth, the first three hypotheses will be general across elite university and community college populations and across fractions with single-digit and multi-digit numerators and denominators. All of these hypotheses follow from the assumption that adults can represent the magnitudes of fractions on a similar type of mental number line as they use to represent the magnitudes of whole numbers, with task characteristics determining whether adults choose solution strategies that make use of mental number line representations of fractions or whether they rely on alternative strategies.

Experiment 1

Experiment 1 paralleled the experiments of Bonato et al. (2007) in that the participants were university students and the stimuli were fractions with single-digit numerators and denominators. The main difference between the Bonato et al. experiments and our Experiment 1 was that most of the fractions being compared in the present experiment did not have equal numerators or equal denominators, nor could the answer be consistently found by deciding whether the numerator or the denominator was larger. The hypotheses were the first three stated in the immediately preceding paragraph.

Method

Participants. We tested 66 undergraduate students (24 women) at Carnegie Mellon University, Pittsburgh, who participated for course credit. The quantitative proficiency of students at this university tends to be quite high. In the most recent year for which data were available, 71% of entering freshmen had math SAT scores over 700, and 98% of entering freshmen had scores over 600.

Problems. On each item, participants were asked to compare a single-digit fraction with 3/5. The comparison fractions were 2/9, 2/7, 3/8, 4/9, 1/2, 2/3, 3/4, 7/9, 5/6, and 6/7. This set of 10 fractions included substantial variation in numerators (1–7), denominators (2–9), and magnitudes (.22–.86), thus avoiding the potential problem of restriction of range in some variables contributing to variables with greater range being better predictors of performance. Half of the comparison fractions were larger than the standard (3/5), and half were smaller. Of the 10 fractions, 6 could be correctly compared with the standard by focusing exclusively on numerators (i.e., judging the fraction with the larger numerator to be larger) and 6 by focusing exclusively on denominators (i.e., judging the fraction with the smaller denominator to be larger).

To describe the stimulus set in a form that can be compared across the present and future experiments, we computed correlations among numerators, denominators, and fractions for the 10 problems. Numerators and denominators correlated minimally with the corresponding fraction: $r = .02$ and $r = .13$, respectively. The correlation between the logarithms of each variable also were minimal: for numerator and fraction value, $r = .03$, and for denominator and fraction value, $r = .07$.

Procedure. Small groups of participants were tested in a computer cluster for sessions of about 15 min; each student worked at a different computer. The problems that they were presented included four introductory trials and 200 fraction comparison trials. Introductory trials were designed to familiarize participants with the response requirements; on each such trial, a fraction with a letter as numerator and a letter as denominator (e.g., "*a*/*b*") was presented in the center of the screen, and participants responded by arbitrarily pressing either the *a* or *l* button. Comparison trials were presented immediately after the introductory trials. The 200 comparison trials included 20 blocks of the 10 problems. Order of presentation of the 10 problems was randomized within each block and was the same on that block for all participants.

On each comparison problem, the comparison fraction was printed in white font in the middle of a black computer screen. The numbers were about 0.6 in. high and were shown under a screen resolution of 1,024 × 768 pixels. Participants were asked to push the *a* key quickly when the comparison fraction's value was below 3/5, and to do the same with the *l* key when the value exceeded 3/5. As a reminder, the phrase "a: below 3/5" was printed in the bottom left corner of the screen and "l: greater than 3/5" was printed in the bottom right corner. Except for the comparison fraction and the two reminders, the screen was empty.

On each trial, participants saw a fixation cross in the center of the screen for 500 ms, a blank screen for 150 ms, and finally a fraction that remained present at the center of the screen until the participant pressed *a* or *l*. The time from stimulus onset to response was measured in milliseconds. If participants erred more than 3 times in a block, a window popped up, asking them to try harder.

Results

One student was removed from the analyses because his percentage errors were more than 4 standard deviations higher than those of any other participant. For the remaining participants, mean error rate was 5%, $SD = 7.2$, range = 0–20%, and median solution time was 0.7 s, $SD = 0.3$ s, range = 0.5–2.6 s, on the level of individual trials. Trials on which participants answered incorrectly were excluded from the analyses of solution times.

We first analyzed the data aggregated at the group level, computing for each fraction the mean error rate and the median solution time across all persons and trials. These were used as criterion variables in six separate regression analyses, with the predictors being linear distance between the numerators of the comparison fraction and the standard, distance between their denominators, and distance between the fraction magnitudes. Only the distance between the fraction magnitudes was a significant predictor of sample median solution times and sample mean percentage errors (see Table 1, left side). Distance between the magnitudes of the standard and comparison fractions explained 58% of the variance in solution times and 48% of the variance in error rates on the 10 problems. In contrast, distance between the numerators of the comparison and standard fractions and between their denominators explained 0% of the variance.

Parallel regression analyses using as independent variables the logarithm of the distance between numerators, between denominators, and between fraction magnitudes showed even stronger effects (see Table 1, right side). The logarithmic distance between fraction magnitudes explained 63% of the variance in median solution times for each fraction and 74% of the variance in error rates for each fraction, more in both cases than explained by the linear distances. In contrast, logarithmic distance between numerators and between denominators accounted for 0% of the variance in both solution times and errors. As with the analyses of linear distance, the analyses in which logarithmic distance was the independent variable were based on all 10 fractions for the analyses of denominators and of fraction magnitudes. For the analysis of logarithmic distance between numerators, the analysis was based on eight rather than 10 fractions, because on the other two problems, the numerators of the fractions being compared were equal, and the natural logarithm of zero is not defined.

Figure 1 shows the sample's median solution times and mean percentage errors for each problem as functions of the comparison fraction. Consistent with the analyses reported above, the two graphs show approximately logarithmically decreasing patterns of solution times and errors with increasing distance of the comparison fraction from 3/5.

The one notable exception to this pattern was the lower than predicted solution time for 1/2. Solution times for 1/2 had the largest deviation from the best fitting logarithmic equation among the 10 fractions. This could not be explained by the small sizes of the whole number components of 1/2; a regression analysis

Table 1
Results of the Regressions of Response Time (RT) or Error Rate on Numerical Distance in Experiment 1

Level of analysis, criterion, and coefficients	Linear regression: Numerator value	Linear regression: Denominator value	Linear regression: Fraction value	Logarithmic regression: Numerator value	Logarithmic regression: Denominator value	Logarithmic regression: Fraction value
Group level						
Criterion: RT						
Data points	10	10	10	8	10	10
β	−.25	.15	−.79	−.37	.15	−.82
Adj. R^2	.00	.00	.58**	.00	.00	.63**
Criterion: Error rate						
Data points	10	10	10	8	10	10
β	−.18	−.14	−.73	−.35	−.07	−.88
Adj. R^2	.00	.00	.48**	.00	.00	.74**
Person level						
Criterion: RT						
Data points	200	200	200	160	200	200
Median adj. R^2	.00	.00	.01	.00	.00	.01
SD of adj. R^2	.01	.01	.03	.02	.01	.04
% of the sample with significant effect	9	3	48	15	6	48
Criterion: Error rate						
Data points	200	200	200	160	200	200
Median adj. R^2	.00	.00	.02	.01	.00	.04
SD of adj. R^2	.01	.03	.04	.02	.04	.06
% of the sample with significant effect	14	14	55	34	11	65

** $p < .01$.

showed no relation of the sum of numerator and denominator to the solution time and error rate on these fractions. Instead, it seemed likely that the especially fast and accurate performance on 1/2 was due at least in part to the magnitude of that fraction being known far better than other fractional magnitudes. Even preschoolers usually know the magnitude of 1/2 (Miller, 1984; Singer-Freeman & Goswami, 2001), and in a recent pilot study, many adults explained their comparisons by saying that they just knew that the comparison was greater than 1/2, a type of explanation rarely cited with other fractions.

In a second step, we analyzed the data at the level of individual participants. We repeated all analyses separately for each person, using the individual trials as data points. The results were highly similar to those at the group level (see Table 1). Distance between fractions explained higher percentages of variance and yielded more significant effects than distances between numerators or denominators. This held true using either linear distance or logarithmic distance between the comparison fraction and the standard.

Wilcoxon signed ranks tests, which can be used on repeated measures data with nonnormal distributions, revealed that, with distance between fractions as predictor, the logarithmic regressions explained higher proportions of variance than the linear regressions. This was true for error rates, $Mdn = .035$ versus .018, $Z = -5.971$, $p < .001$, as well as for solution times, $Mdn = .014$ versus .012, $Z = -3.668$, $p < .001$.

Discussion

The results of Experiment 1 supported all three hypotheses that motivated the experiment. The presence of a substantial distance

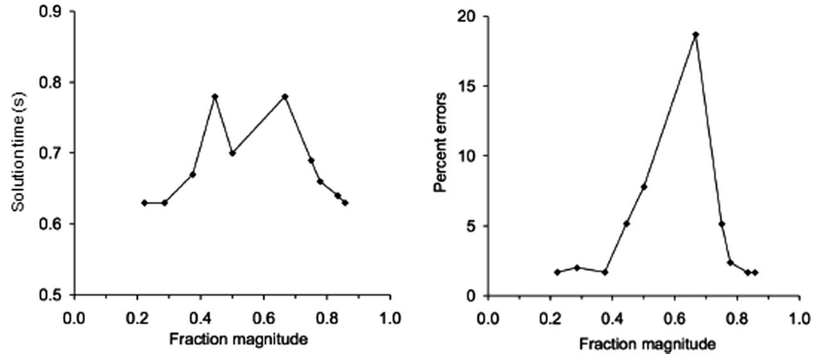

Figure 1. Median solution time (left) and mean percentage errors (right) of comparisons against 3/5 as functions of fraction magnitude in Experiment 1.

effect for fraction magnitudes for both solution times and error rates, and the absence of any effect for distance between numerators and between denominators, indicated that when problems are not constrained to make whole number components straightforward predictors of correct answers, adults rely on fraction magnitudes to solve comparison problems. The percentage correct was 95%, far higher than could have been obtained by comparing only the numerators or only the denominators of these fractions. In addition, numerator comparisons and denominator comparisons would have yielded 100% errors on some problems. As shown in Figure 1, this clearly did not occur. Finally, as is typically the case on tasks involving comparisons of whole numbers, the natural logarithm of the distance between fraction magnitudes was a better predictor of performance than was the linear distance. All of these findings indicate that participants relied on the magnitudes of fractions to perform the comparisons, rather than on the whole number components.

One finding that was not anticipated was the especially fast and accurate performance that participants generated in comparing 1/2 to 3/5. This comparison produced the largest deviation from the overall logarithmic pattern of solution times shown in Figure 1. The likely sources of this effect were participants' knowledge of the absolute magnitude of 1/2 and their ability to quickly and accurately code the magnitudes of other fractions (in this case 3/5) relative to it. The finding indicates that in addition to being able to solve fraction comparison problems by comparing fraction magnitudes, numerator magnitudes, and denominator magnitudes, adults can solve such problems via knowledge of specific fractions and coding of other fractions in terms of them. Whether fractions other than 1/2 can be used in this way remains an open question.

Experiment 2

All fractions presented in Experiment 1 had single-digit numerators and denominators. However, many fractions are more complex, and the challenge of representing the magnitudes of these fractions is much greater. University students might have enough experience with single-digit fractions to form number line representations of their magnitudes, even if they cannot form such representations of fractions in general.

In Experiment 2, we tested the generality of the Experiment 1 findings by presenting participants with a problem set in which more than 80% of fractions had two-digit numerators and denominators. This preponderance of two-digit fractions led to Experiment 2 presenting a different and considerably more challenging task than that in Experiment 1 or in previous research on fraction comparisons. The Experiment 2 problem set also included a greater number of fractions than the problem set used in Experiment 1, so that we could discriminate more precisely between logarithmic and linear functions for the distance effect.

The basic hypothesis of Experiment 2 was that participants would use an integrated analog representation, akin to a mental number line, to represent multi-digit as well as single-digit fractions. Therefore, all three predictions from Experiment 1 were expected to hold true on this more challenging set of fraction comparison problems.

Method

Participants were 30 undergraduate students (9 women) at Carnegie Mellon University, none of whom participated in Experiment 1. The fractions that they were asked to compare with 3/5 were 20/97, 1/4, 26/89, 30/91, 28/71, 31/72, 32/69, 1/2, 25/49, 23/44, 33/62, 5/9, 29/51, 24/41, 22/37, 27/43, 37/58, 35/54, 2/3, 36/53, 38/55, 40/57, 41/56, 39/50, 47/59, 6/7, 43/48, 49/52, and 46/47. These 29 fractions are evenly distributed around the comparison standard of 3/5, each numerator and each denominator is only used in one fraction, and all fractions are in simplest form. The correlation of linear distance between the comparison and standard fractions with linear distance between numerators was $r = .13$. Its correlation with the linear distance between denominators was $r = .26$. The corresponding correlations involving distances between the natural logarithms were $r = .04$ and $r = .01$, respectively.

Each fraction was presented only once, leading to a total of 29 trials. As in Experiment 1, solution times on error trials were excluded from the analyses.

During Experiment 1, we observed that participants sometimes took their fingers off the answer keys between trials, which could distort their solution times. To prevent this in Experiment 2, the participants were required to push both answer keys to trigger the display on each trial. All other aspects of the procedure were identical to those in Experiment 1.

Results

The mean error rate was 6%, $SD = 5$, ranging from 0% to 17% on the 29 problems. The median solution time on individual trials was 4.8 s, $SD = 3.5$, ranging from 1.6 s. to 17.5 s on the 29 problems.

At the group level, distances between the standard (3/5) and the comparison fraction accounted for significant variance in both sample median solution time (48%) and sample mean percentage errors (39%) on each problem (see Table 2). In contrast, distances between the numerators of the standard and comparison fractions and between their denominators were unrelated to median solution time and error rate on the corresponding problems.

As shown in Figure 2, mean solution times were much longer with the multi-digit fractions presented in this experiment than with the single-digit fractions presented in Experiment 1 (4.8 s vs. 0.7 s). Solution times on different problems also varied far more than in Experiment 1 (SDs = 3.5 s vs. 0.3 s).

Figure 2 also suggests that the distance effect again followed a logarithmic function. Consistent with this hypothesis, the logarithmic predictor accounted for considerably more variance in solution times on each problem than did the best fitting linear function ($R^2 = .70$ vs. .48). Similarly, the logarithmic function accounted for considerably more variance in errors on each problem than did the best fitting linear function ($R^2 = .63$ vs. .39). As in Experiment 1, the solution time for 1/2 showed the largest negative deviation from the logarithmic function of any of the 29 fractions, 2.8 s versus the predicted 6.2 s.

We repeated all regressions separately for each person. As in Experiment 1, the results on the person level were very similar to the results on the group level. Distances between fraction values explained higher proportions of variance and yielded more signif-

Table 2
Results of the Regressions of Response Time (RT) or Error Rate on Numerical Distance in Experiment 2

Level of analysis, criterion, and coefficients	Linear regression: Numerator value	Linear regression: Denominator value	Linear regression: Fraction value	Logarithmic regression: Numerator value	Logarithmic regression: Denominator value	Logarithmic regression: Fraction value
Group level						
Criterion: RT						
β	.03	−.06	−.70	.16	.16	−.84
Adjusted R^2	.00	.00	.48***	.00	.00	.70***
Criterion: Error rate						
β	−.06	−.12	−.64	.02	.03	−.80
Adjusted R^2	.00	.00	.39***	.00	.00	.63***
Person level						
Criterion: RT						
Median adjusted R^2	.00	.00	.18	.00	.00	.29
SD of adjusted R^2	.05	.03	.12	.04	.04	.19
% of the sample with significant effect	7	3	70	7	7	73
Criterion: Error rate						
Median adjusted R^2	.00	.00	.01	.00	.00	.01
SD of adjusted R^2	.02	.02	.04	.02	.01	.11
% of the sample with significant effect	0	0	8	4	0	28

Note. All analyses were carried out with the 29 fractions as data points.
*** $p < .001$.

icant effects than distances between numerators or denominators. This held true for error rates as well as solution times.

As with the simpler fractions in Experiment 1, Wilcoxon signed ranks tests indicated that the logarithmic function explained higher proportions of variance than the linear regressions for both percentage errors, *Mdn* = .070 versus .027, $Z = -2.606$, $p = .009$, and solution times, *Mdn* = .270 versus .189, $Z = -3.579$, $p < .001$.

Discussion

The results with this much more challenging set of fraction comparison problems replicated and extended the findings of Experiment 1. Again, we found strong effects for the distance between fraction magnitudes but not for the distance between numerators or denominators. Almost all numerators and almost all denominators were much larger than the numerators and the denominator in the standard, which meant that comparing the numerators of the standard and the comparison fraction or comparing their denominators would have led to very high error rates, far higher than the 6% that was observed.

The accuracy of the fraction comparisons was especially striking, because the participants would have been very unlikely to have much experience with the specific fractions that were presented. People encounter individual two-digit fractions (e.g., 26/89) far less often than single-digit fractions (e.g., 3/4). It seems extremely unlikely that people could recall the magnitudes of such two-digit fractions from long-term memory and use these memorized relations on the experimental task. Instead, participants needed to estimate the magnitude of the comparison fraction and relate it to the magnitude of the standard. This interpretation was consistent with the sample's median solution time of 4.8 s, much longer than the solution times of less than 1 s in Experiment 1. The unfamiliarity of the specific two-digit fractions, the long solution times, and the relatively low error rate (6%) were consistent with the view that participants solved the two-digit fraction comparison

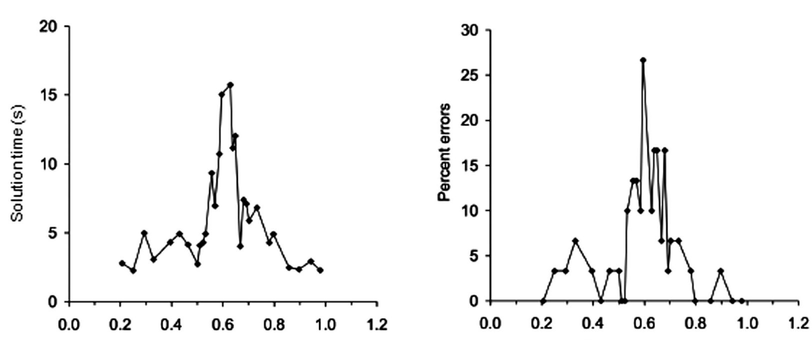

Figure 2. Median solution time (left) and mean percentage errors (right) of comparisons against 3/5 as functions of fraction magnitude in Experiment 2.

problems by estimating the fractions' magnitudes and then comparing the estimate with the standard.

Experiment 3

The participants in Experiments 1 and 2 were students at a very selective university with high mathematics achievement test scores. This raised the question of whether the results of Experiments 1 and 2 are limited to adults with unusually strong mathematical skills. The mathematics skills of the participants in Bonato et al. (2007) were not described, but they might well have been different from those of the students in the present Experiments 1 and 2. If so, some or all of the difference between the present findings and those previous ones might be due to the mathematics skills of the participants, rather than to differences in the problem sets.

Experiment 3 addressed this issue. It was identical to Experiment 2, except that the participants were students at a community college with open admissions. These students were presumed to be far less mathematically skilled than the students in Experiments 1 and 2. The goal was to determine whether only mathematically outstanding adults compare fractions on the basis of fraction magnitudes or whether other adults also do.

Method

The procedures, problems, and other aspects of Experiment 3 were identical to those in Experiment 2. The one difference was that the 22 participants (11 women) were students at Community College of Allegheny County, a junior college at which the only entrance requirements are that the students be at least 18 years old and have either a high school diploma or GED.

Results

As expected, the absolute level of performance among the community college students was considerably lower than that of the more mathematically knowledgeable participants in Experiment 2. The mean error rate of 30% (SD = 20.8, range = 4–69%) was much higher than the rate of 6% on the same problems in Experiment 2, and the median solution time of 10.7 s (SD = 7.0, range = 2.9–29.5 s) was considerably higher than the median of 4.8 s in the earlier experiment.

Despite these differences in participants' proficiency, the patterns of solution times and errors paralleled those in the previous experiment. As shown in Table 3 and Figure 3, on the group level, neither distance between numerators nor distance between denominators predicted either solution times or errors on the 29 fraction comparison problems. In contrast, the distance between fraction magnitudes accounted for substantial percentages of variance in both solution times (36%) and percentage errors (26%).

As in Experiments 1 and 2, we examined the fit of the natural logarithm of the distance between the comparison and standard fractions' numerators, denominators, and magnitudes to speed and accuracy on the 29 problems. As shown in Table 3, the fit of the logarithmic function to the percentage errors on each problem was somewhat better than the fit of the linear function to them, R^2 = .35 versus .26. Also as shown, the fit of the linear function to the median solution time on each problem was slightly better than that of the logarithmic function, R^2 = .36 versus .32. As in the prior experiments, the solution time for 1/2 was considerably faster (5.6 s) than would have been expected from the best fitting natural logarithmic function (10.4 s). The negative deviation from the value predicted by the logarithmic equation was the fourth largest among the 29 problems.

We repeated the regression analyses separately for each person. Again, the results showed a substantial distance effect for fraction values and almost no effects of numerators and denominators as indicated by the sample median R^2 values of 0 for these regressions.

Wilcoxon signed ranks test indicated that the distance effect with fraction values as predictor and solution rates as criterion

Table 3
Results of the Regressions of Response Time (RT) or Error Rate on Numerical Distance in Experiment 3

Level of analysis, criterion, and coefficients	Linear regression: Numerator value	Linear regression: Denominator value	Linear regression: Fraction value	Logarithmic regression: Numerator value	Logarithmic regression: Denominator value	Logarithmic regression: Fraction value
Group level						
Criterion: RT						
β	.03	−.06	−.62	.06	.08	−.59
Adjusted R^2	.00	.00	.36***	.00	.00	.32***
Criterion: Error rate						
β	.23	−.10	−.53	.24	.14	−.61
Adjusted R^2	.02	0	.26**	.02	0	.35***
Person level						
Criterion: RT						
Median adjusted R^2	.00	.00	.03	.00	.00	.08
SD of adjusted R^2	.09	.11	.12	.12	.13	.12
% of the sample with significant effect	5	9	45	9	18	36
Criterion: Error rate						
Median adjusted R^2	.00	.00	.05	.00	.00	.07
SD of adjusted R^2	.02	.01	.07	.02	.01	.12
% of the sample with significant effect	0	0	32	0	0	41

Note. All analyses were carried out with the 29 fractions as data points.
** $p < .01$. *** $p < .001$.

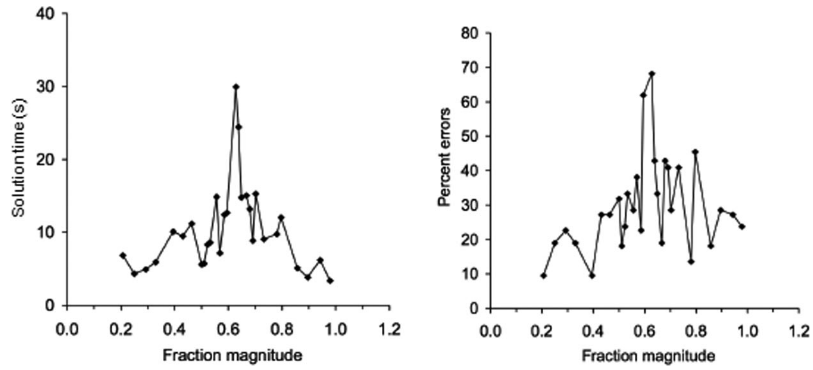

Figure 3. Median solution time (left) and mean percentage errors (right) of comparisons against 3/5 as functions of fraction magnitude in Experiment 3.

is fit better by a logarithmic function than by a linear function (*Mdn* = .066 vs. .053, $Z = -2.047$, $p = .041$). With solution times as criterion, the logarithmic function also explained higher proportions of variance than the linear function (*Mdn* = .083 vs. .031), but the difference was not significant, $Z = -1.374$, $p = .170$.

To test whether the strength of the distance effect differs between mathematically more and less skilled individuals, we compared the logarithmic distance effects found with reaction times on the person level in Experiment 2 and Experiment 3. As indicated by a Mann–Whitney test, the distance effect explained a much higher variance proportion for the highly skilled individuals in Experiment 2 (*Mdn* = .294) than for the less skilled students in Experiment 3 (*Mdn* = .084), $U = 176.5$, $p = .004$. The distance effect with percentage correct could not be compared across experiments because of ceiling effects among the highly skilled participants in Experiment 2, who answered 94% of problems correctly.

Discussion

The results of Experiment 3 replicated the main findings of Experiments 1 and 2 with a less mathematically skilled, and more representative, sample. The replication is noteworthy because there were large differences in mathematical knowledge between the university students in Experiment 2 and the community college students in Experiment 3. Consistent with these differences in knowledge, the university students were much faster and more accurate in comparing the fractions. Nonetheless, for the community college sample, as for the university sample, distance between fraction magnitudes was a much better predictor of solution times and error rates than distance between numerators or distance between denominators.

General Discussion

Can Adults Represent Fractional Magnitudes?

In the only previously published study on adults' representations of fractions, Bonato et al. (2007) found distance effects for the magnitudes of numerators and denominators but not for the magnitudes of the fractions being compared. They concluded that even educated adults are unable to represent fraction magnitudes and therefore resort to separately processing the fractions' whole number components.

The present results cast serious doubt on this conclusion. The findings indicate that adults can represent the magnitudes of fractions, that the representation resembles the mental number line used with whole numbers, and that adults use this representation to compare fraction magnitudes when simpler approaches would not yield accurate performance.

All four of the hypotheses of the present study were based on the belief that adults can represent fraction magnitudes. The results were consistent with each of them. The distance effect was far stronger for distances between fractions than for distances between numerators or denominators. Comparisons were consistently correct on problem sets where comparisons of whole number components could not yield high degrees of accuracy. As with whole numbers, a logarithmic function fit the data better than a linear function. Finally, all of these conclusions held true over a range of populations and stimuli: for students at both highly selective universities and nonselective community colleges, for both single-digit and multi-digit fractions, and for a small set of fractions presented 20 times and for a larger set of fractions presented once.

The consistency of use of representations of fraction magnitudes to perform the fraction comparison task is evident in its being unnecessary to aggregate over many trials for strong distance effects to emerge. Extensive aggregation is normally used to obtain the distance effect with whole numbers. For example, Dehaene et al. (1990) used more than 240 comparison trials per person in each of their three experiments. By contrast, in the present Experiments 2 and 3, reliable distance effects emerged with only 29 trials per person. The distance effects were not only found when data were aggregated over persons but also for considerable numbers of individual participants in all three experiments. Given the small number of trials, this is a conservative test of our hypotheses and shows the reliability of the distance effect for fractions. In all, the distance effect with solution times was stronger and more stable across the three experiments than the distance effect with error rates, which might be due to the small numbers of trials and to the even smaller number of errors the university students made on these trials.

Two very recent fMRI studies provide further evidence for the integrated magnitude representations for fractions that were implied by our results. Both studies found that brain activation in regions specialized for magnitude representation systematically covaries with the distance between fractions but not with the distance between their numerators or denominators (Ischebeck, Schocke, & Delazer, in press; Jacob & Nieder, 2009).

Although these results indicate important similarities between whole number and fraction magnitude representations, the results also indicate differences between them. Comparisons of fraction magnitudes are generally slower and less accurate than comparisons of whole numbers, with the single exception of mathematically sophisticated adults' solution times on single-digit fractions. Comparisons of fraction magnitudes also vary far more with individual and stimulus differences than do comparisons of whole number magnitudes. Experiments 1 and 2 indicate that adults take much more time to compare two-digit than one-digit fractions (median times of 0.7 s vs. 4.8 s). Experiments 2 and 3 indicate that adults at a highly selective university were both much faster and much more accurate than students at a community college (median solution times of 4.8 vs. 10.7 s, and mean error rates of 6% vs. 30%). To our knowledge, such large differences have never been found with adults' comparisons of whole numbers.

The lower absolute level of performance for fractions than for whole numbers and the larger differences between individuals and between stimuli for fractions than for whole numbers appear to reflect the same mental number line representation being used with differing degrees of noise in the representations of magnitudes. In a number of ways, adults' performance with fractions resembles elementary school children's performance with whole numbers. For example, just as adults with greater mathematical sophistication generate more precise representations of fraction magnitudes, children with higher mathematical achievement test scores generate more precise representations of whole number magnitudes (Booth & Siegler, 2006; Geary et al., 2007). Decreasing noise in whole number magnitude representations was particularly evident in Siegler and Booth's (2004) finding that younger children's repeated estimates of the same number's magnitude were less variable than those of older children. Similarly, just as adults represent single-digit fractions much more precisely than multi-digit fractions, children much more precisely represent magnitudes of smaller whole numbers than larger ones (Feigenson et al., 2002; Siegler & Opfer, 2003). Thus, the large literature on children's (and adults') representations of whole numbers provides a useful base for thinking about adults' (and children's) representations of fractions.

Strategy Choices and Representations of Fractional Magnitudes

Integrating the present results with those of Bonato et al. (2007) shows that people's mental magnitude representations of fractions cannot be investigated separately from their strategy choices. Bonato et al. showed that small variations in the types of fractions being compared led to different strategies for using the whole number components of fractions (i.e., comparing numerators only or denominators only). The present study showed that variations in the type of fractions being compared led to a fundamentally different solution strategy, one that involves integrated magnitude representations of fractions.

These findings demonstrate striking similarities between people's strategy choices with whole numbers and fractions. The extensive literature on strategy choices with whole numbers has shown that people usually know and use alternative strategies for solving a task, that use of these strategies varies within sessions and even within problems, and that people flexibly adapt strategy choices to problem characteristics (cf. Bisanz, 2003; Geary, 2006; Siegler, 1996). All of these properties also seem to hold true for fractions. When a task can be solved by a simple strategy (i.e., comparing numerators or denominators alone), people do so. They only use the more demanding strategy of estimating fraction magnitudes and then comparing them when that approach is necessary for answering accurately.

These findings also suggest that better understanding representations of fractional magnitudes will require trial-by-trial assessments of the strategies that are used to represent and compare fractions. The combination of observations of overt behavior and immediately retrospective verbal reports have been found to be useful for producing valid trial-by-trial strategy assessments with whole numbers, and given the similar time required to compare fractional magnitudes, the same seems likely to be true for this task.

Ignoring strategic variability and the relations between problem characteristics and strategy choices can produce misleading conclusions about underlying cognitive processes (Siegler, 1987). In contrast, attending to strategic and representational variability and the ways that people choose among alternative approaches can provide insights into the processes underlying obtained results (LeFevre et al., 1996; Nuerk, Kaufmann, Zoppoth, & Willmes, 2004; Siegler, 1987). Further research that directly examines strategy use and strategy choice with fraction problems thus seems warranted.

References

Ansari, D. (2008). Effects of development and enculturation on number representation in the brain. *Nature Reviews Neuroscience, 9,* 278–291.

Bächtold, D., Baumüller, M., & Brugger, P. (1998). Stimulus-response compatibility in representational space. *Neuropsychologia, 36,* 731–735.

Bisanz, J. (2003). Arithmetical development: Commentary on Chapters 1 through 7 and reflections on directions. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 435–452). Mahwah, NJ: Erlbaum.

Bonato, M., Fabbri, S., Umiltà, C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology: Human Perception and Performance, 33,* 1410–1419.

Booth, J. L., & Siegler, R. S. (2006). Developmental and individual differences in pure numerical estimation. *Developmental Psychology, 41,* 189–201.

Booth, J. L., & Siegler, R. S. (2008). Numerical magnitude representations influence arithmetic learning. *Child Development, 79,* 1016–1031.

Campbell, J. I. D., & Epp, L. J. (2005). Architectures for arithmetic. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 347–360). New York: Psychology Press.

Cantlon, J. F., Brannon, E. M., Carter, E. J., & Pelphry, K. A. (2006). Functional imaging of numerical processing in adults and 4-year-old children. *PLoS Biology, 4,* e125.

Carpenter, T. P., Corbitt, M. K., Kepner, H. S., Lindquist, M. M., & Reys, R. E. (1981). *Results from the second mathematics assessment of the National Assessment of Educational Progress.* Washington, DC: National Council of Teachers of Mathematics.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics.* New York: Oxford University Press.

Dehaene, S., Dupoux, E., & Mehler, J. (1990). Is numerical comparison digital? Analogical and symbolic effects in two-digit number comparison. *Journal of Experimental Psychology: Human Perception and Performance, 16,* 626–641.

Feigenson, L., Carey, S., & Spelke, E. (2002). Infants' discrimination of number vs. continuous extent. *Cognitive Psychology, 44,* 33–66.

Feigenson, L., Dehaene, S., & Spelke, E. (2004). Core systems of number. *Trends in Cognitive Sciences, 8,* 307–314.

Fias, W., & Fischer, M. H. (2005). Spatial representations of numbers. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 43–54). New York: Psychology Press.

Geary, D. C. (2005). *The origin of mind: Evolution of brain, cognition, and general intelligence.* Washington, DC: American Psychological Association.

Geary, D. C. (2006). Development of mathematical understanding. In W. Damon & R. M. Lerner (Series Eds.) & D. Kuhn & R. S. Siegler (Vol. Eds.), *Handbook of child psychology: Volume 2: Cognition, perception, and language* (6th ed., pp. 777–810). Hoboken, NJ: Wiley.

Geary, D. C., Hoard, M. K., Byrd-Craven, J., Nugent, L., & Numtee, C. (2007). Cognitive mechanisms underlying achievement deficits in children with mathematical learning disability. *Child Development, 78,* 1343–1359.

Gelman, R. (1991). Epigenetic foundations of knowledge structures: Initial and transcendent constructions. In S. Carey & R. Gelman (Eds.), *The epigenesis of mind: Essays on biology and cognition* (pp. 293–322). Hillsdale, NJ: Erlbaum.

Gilmore, C., McCarthy, S. E., & Spelke, E. (2007, May 31). Symbolic arithmetic without instruction. *Nature, 447,* 589–591.

Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology, 44,* 1457–1465.

Holloway, I. D., & Ansari, D. (in press). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children's math achievement. *Journal of Experimental Child Psychology.*

Hubbard, E. M., Piazza, M., Pinel, P., & Dehaene, S. (2005). Interactions between number and space in parietal cortex. *Nature Reviews Neuroscience, 6,* 435–448.

Ischebeck, A., Schocke, M., & Delazer, M. (in press). The processing and representation of fractions within the brain: An fMRI investigation. *NeuroImage.*

Izard, V., & Dehaene, S. (2008). Calibrating the mental number line. *Cognition, 106,* 1221–1247.

Jacob, S. N., & Nieder, A. (2009). Notation-independent representation of fractions in the human parietal cortex. *Journal of Neuroscience, 29,* 4652–4657.

Jordan, N. C., Kaplan, D., Locuniak, M. N., & Ramineni, C. (2007). Predicting first-grade math achievement from developmental number sense trajectories. *Learning Disabilities Research & Practice, 22,* 36–46.

Jordan, N. C., Kaplan, D., Olah, L. N., & Locuniak, M. N. (2006). Number sense growth in kindergarten: A longitudinal investigation of children at risk for mathematics difficulties. *Child Development, 77,* 153–175.

Laski, E. V., & Siegler, R. S. (2007). Is 27 a big number? Correlational and causal connections among numerical categorization, number line estimation, and numerical magnitude comparison. *Child Development, 76,* 1723–1743.

LeFevre, J., Sadesky, G. S., & Bisanz, J. (1996). Selection of procedures in mental addition: Reassessing the problem-size effect in adults. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 22,* 216–230.

Lemaire, P., Arnaud, L., & Lecacheur, M. (2004). Adults' age-related differences in adaptivity of strategy choices: Evidence from computational estimation. *Psychology and Aging, 10,* 467–481.

Lipton, J. S., & Spelke, E. S. (2003). Origins of number sense: Large-number discrimination in human infants. *Psychological Science, 14,* 396–401.

Luwel, K., Verschaffel, L., Onghena, P., & De Corte, E. (2003). Analyzing the adaptiveness of strategy choices using the choice/no-choice method: The case of numerosity judgment. *European Journal of Cognitive Psychology, 15,* 511–537.

Miller, K. (1984). Child as the measurer of all things: Measurement of procedures and the development of quantitative concepts. In C. Sophian (Ed.), *Origins of cognitive skills: The eighteenth annual Carnegie symposium on cognition* (pp. 193–228). Hillsdale, NJ: Erlbaum.

Moyer, R. S., & Landauer, T. K. (1967, September 30). Time required for judgments of numerical inequality. *Nature, 215,* 1519–1520.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel.* Washington, DC: U.S. Department of Education.

Ni, Y., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of the whole number bias. *Educational Psychologist, 40,* 27–52.

Nuerk, H.-C., Kaufmann, L., Zoppoth, S., & Willmes, K. (2004). On the development of the mental number line: More, less, or never holistic with increasing age? *Developmental Psychology, 40,* 1199–1211.

Nuerk, H.-C., Weger, U., & Willmes, K. (2001). Decade breaks in the mental number line? Putting the tens and units back in different bins. *Cognition, 82,* B25–B33.

Ramani, G. B., & Siegler, R. S. (2008). Promoting broad and stable improvements in low-income children's numerical knowledge through playing number board games. *Child Development, 79,* 375–394.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 3, pp. 41–95). Hillsdale, NJ: Erlbaum.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93,* 346–362.

Schneider, M., Heine, A., Thaler, V., Torbeyns, J., De Smedt, B., Verschaffel, L., . . . Stern, E. (2008). A validation of eye movements as a measure of elementary school children's developing number sense. *Cognitive Development, 23,* 424–437.

Siegler, R. S. (1987). The perils of averaging data over strategies: An example from children's addition. *Journal of Experimental Psychology: General, 117,* 258–275.

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking.* New York: Oxford University Press.

Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child Development, 75,* 428–444.

Siegler, R. S., & Opfer, J. E. (2003). The development of numerical estimation: Evidence for multiple representations. *Psychological Science, 14,* 237–243.

Siegler, R. S., & Ramani, G. B. (2008). Playing linear numerical board games promotes low-income children's numerical development. *Developmental Science, 11,* 655–661.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General, 127,* 377–397.

Singer-Freeman, K. E., & Goswami, U. (2001). Does half a pizza equal half a box of chocolates? Proportional matching in an analogy task. *Cognitive Development, 16,* 811–829.

Stoianov, I., Kramer, P., Umilta, C., & Zorzi, M. (2008). Visuospatial priming of the mental number line. *Cognition, 106,* 770–779.

Tudusciuc, O., & Nieder, A. (2007). Neuronal population coding of continuous and discrete quantity in the primate posterior parietal cortex. *Proceedings of the National Academy of Sciences, USA, 104,* 14513–14518.

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction, 14,* 453–467.

Whyte, J. C., & Bull, R. (2008). Number games, magnitude representation, and basic number skills in preschoolers. *Developmental Psychology, 44,* 588–596.

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