

Algebraic Geometry over Free Metabelian Lie Algebra I: U -Algebras and Universal Classes

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Abstract

This paper is the first in a series of three, the aim of which is to lay the foundations of algebraic geometry over the free metabelian Lie algebra F . In the current paper we introduce the notion of a metabelian Lie U -algebra and establish connections between metabelian Lie U -algebras and special matrix Lie algebras. We define the Δ -localisation of a metabelian Lie U -algebra A and the direct module extension of the Fitting's radical of A and show that these algebras lie in the universal closure of A .

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1 Introduction

This paper is the first in a series of three, the object of which is to lay the foundations of algebraic geometry over free metabelian Lie algebra F . In papers [4], [8] G. Baumslag, A. Myasnikov and V. Remeslennikov introduced and studied group-theoretic counterparts to the main notions of classical algebraic geometry. The results of [4], [8] can be extended to an arbitrary algebraic system.

The main aim of the current series of our papers is to create a structural theory of Lie algebras from the quasivariety $\mathbf{qvar}(F)$ and the universal closure $\mathbf{ucl}(F)$ generated by the free metabelian Lie algebra F . This motivation is due to the following circumstance: finitely generated Lie algebras from the quasivariety $\mathbf{qvar}(F)$ (the universal closure $\mathbf{ucl}(F)$) are exactly the coordinate algebras of (irreducible) algebraic sets over F .

This paper holds some preliminary results, which will play an important role for algebraic geometry over F . In Section 2 we list some preliminary results on metabelian Lie algebras. Some of these results are known, but we give new proofs and further use not only the statements itself, but the methods of proofs.

In papers of V. A. Artamonov [1] and [2] the author gives a presentation of the free metabelian Lie algebra in a module over the ring of polynomials. Note that this presentation is a particular case of a more general construction of embedding a Lie algebras of special type into verbal wreath product of Lie algebras. The latter construction is due to A. L. Shmel'kin (see [9]). In Section 3 we borrough the ideas of the authors mentioned to construct special matrix metabelian Lie algebras. There we introduce the notion of a *U*-algebra, which is very important for the use of algebraic geometry over the

free metabelian Lie algebra F and establish their connections with special matrix Lie algebras. In particular (see Theorem 3.2), every finitely generated U -algebra is a subalgebra of a special matrix Lie algebra.

In Section 4 for a fixed Lie algebra A we introduce the first order language L_A and study several universal classes in this language (the universal closure, the quasivariety, etc.). In this section we also introduce the notions of the Δ -localisation of a U -algebra A and of the direct module extension of the Fitting's radical of the algebra A . We show (see Propositions 4.1 and 4.4) that the new algebras lie in the universal closure of the algebra A . We refer to [3] for preliminaries of Lie algebras and to [5] and [7] for commutative algebra.

Some of the results of this paper can be found in author's preprint [6].

2 Metabelian Lie Algebras

In this section we give an exposition of preliminary results, which will be used further. Some of these results are well-known.

Recall that a vector space A over a field k equipped with a bilinear multiplication satisfying the following universal axioms

- The anti-commutativity identity $a \circ a = 0$ (notice that the regular anti-commutativity axiom $a \circ b = -b \circ a$ is implied by $a \circ a = 0$),
- The Jacoby identity $(a \circ b) \circ c + (b \circ c) \circ a + (c \circ a) \circ b = 0$,

is termed a Lie algebra.

In the event that the Lie algebra A satisfies

- The metabelian identity $(a \circ b) \circ (c \circ d) = 0$;

it is termed a metabelian Lie algebra.

The regular notation of multiplication of elements a and b from A is 'o', $a \circ b$. For the sake of brevity, we sometimes omit 'o' and use the notation ab .

Left-normed products $a_1 a_2 a_3 \cdots a_n$, of $a_1, a_2, a_3, \dots, a_n \in A$ are defined as

$$(\dots((a_1 \circ a_2) \circ a_3) \circ \dots) \circ a_n.$$

We term such products by left-normed words or monomials of the degree or of the length n . Similar, right-hand side definition of such words gives

rise to the notion of a right-normed monomial. As it is well-known in the theory of Lie algebras every element of degree n from letters a_1, \dots, a_n can be presented as a linear combination of left-normed (right-normed) monomials of length n from these letters.

2.1 The Fitting's Radical and the Commutant

Two special ideals of a metabelian Lie algebra A are of great importance for us.

Definition 2.1 (The commutant) *The ideal generated by the set $\{a \circ b \mid a, b \in A\}$ is termed the commutant of the algebra A and denoted by A^2 . In the event that A is a metabelian Lie algebra, resulting from the metabelian identity, the commutant A^2 is abelian.*

Definition 2.2 (The Fitting's radical) *The ideal generated by the elements from nilpotent ideals of the algebra A is termed the Fitting's radical and denoted by $\text{Fit}(A)$.*

Since the commutant of a metabelian Lie algebra is an Abelian ideal, it is always contained in the Fitting's radical, $A^2 \subseteq \text{Fit}(A)$.

Remark 2.1 *For this notation, there exists a metabelian Lie algebra A such that $A^2 \neq \text{Fit}(A)$. If A is Abelian then $A = \text{Fit}(A)$ while $A^2 = 0$. If $A \neq 0$, then $\text{Fit}(A) \neq 0$.*

Metabelian Lie algebras have the following useful property. Every left-normed monomial $abc_1 \cdots c_n$, where here $n \geq 2$ equals $abc_{\tau(1)} \cdots c_{\tau(n)}$, where τ is a permutation on the set of indices $\{1, \dots, n\}$.

To prove that a permutation of the far co-factors preserves the element, it suffices to show that a permutation of two adjacent co-factors c_i and c_{i+1} preserves the element. Consider the following equation

$$abc_1 \cdots c_i c_{i+1} \cdots c_k - abc_1 \cdots c_{i+1} c_i \cdots c_k = (abc_1 \cdots c_i c_{i+1} - abc_1 \cdots c_{i+1} c_i) \cdots c_k.$$

Using the anti-commutativity identity: $abc_1 \cdots c_{i+1} c_i = -c_{i+1}(abc_1 \cdots) c_i$. By the metabelian identity: $c_i c_{i+1}(abc_1 \cdots) = 0$. The initial expression, therefore, rewrites as follows

$$((abc_1 \cdots) c_i c_{i+1} + c_{i+1}(abc_1 \cdots) c_i + c_i c_{i+1}(abc_1 \cdots)) \cdots c_k.$$

The latter equals zero by the Jacoby identity.

We shall make use of some facts on nilpotent subalgebras of metabelian Lie algebras.

Lemma 2.1 *Let A be a metabelian Lie algebra. And let I_1 and I_2 be nilpotent ideals of A of nilpotency classes n_1 and n_2 . Then the ideal $I = \langle I_1, I_2 \rangle$ is also nilpotent.*

Proof. It is fairly easy to see that the nilpotency class of I is lower than or equals $2n$, where $n = \max \{n_1, n_2\}$. ■

Corollary 2.1 *Let I_1, I_2, \dots, I_n be nilpotent ideals of A . Then the ideal $I = \langle I_1, I_2, \dots, I_n \rangle$ is nilpotent.*

Corollary 2.2 *An element $x \in A$ lies in the Fitting's radical $\text{Fit}(A)$ if and only if $I = \langle x \rangle$ is nilpotent. Or, which is equivalent, if x lies in some nilpotent ideal of A .*

Lemma 2.2 *Let C be a nilpotent metabelian Lie algebra of nilpotency class $n \geq 2$. Then C contains a 2-generated nilpotent subalgebra D of nilpotency class 2.*

Proof. Let

$$\{0\} = Z_0(C) < Z_1(C) < Z_2(C) < \dots < Z_{n-1}(C) < Z_n(C) = C$$

be the upper central series for C (see [3]). Since C is non-Abelian, the set $Z_2(C) \setminus Z_1(C)$ is non-empty. Take $c_1 \in Z_2(C) \setminus Z_1(C)$ and $c_2 \in C$ so that $c_1 \circ c_2 \neq 0$. The algebra $D = \langle c_1, c_2 \rangle$ is a 2-generated nilpotent Lie algebra of class 2. ■

Lemma 2.3 *If every nilpotent ideal of A is Abelian then $\text{Fit}(A)$ is Abelian.*

Proof. Assume the converse. There, therefore, exist two elements $c_1, c_2 \in A$ and two nilpotent ideals I_1 and I_2 so that $c_1 \in I_1$ and $c_2 \in I_2$ and so that $c_1 \circ c_2 \neq 0$. According to Lemma 2.1, the ideal $I = \langle I_1, I_2 \rangle$ is nilpotent but not Abelian. This derives a contradiction. ■

Lemma 2.4 *Let A be a metabelian Lie algebra and suppose that the element $a \in A$ commutes with every element from A^2 . Then $a \in \text{Fit}(A)$.*

Proof. Straightforward checking shows that under the assumptions of the lemma the principal ideal $I = \langle a \rangle$ is Abelian. Consequently, by Corollary 2.2, $a \in \text{Fit}(A)$. \blacksquare

This series of lemmas allows to introduce a structure of a module over the ring of polynomials on the commutant A^2 and on the Fitting's radical $\text{Fit}(A)$ (in the event that it is Abelian). Or, more generally, a structure of a module on an arbitrary abelian ideal I of A , containing A^2 .

In that case A/I is also Abelian, since $A^2 \subseteq I$. Take a maximal linearly independent modulo I system of elements $\{a_\alpha, \alpha \in \Lambda\}$ from A . The ideal I admits a structure of a module over the ring of polynomials $R = k[x_\alpha, \alpha \in \Lambda]$. The addition and the multiplication by elements of k is correctly defined, since I is a vector space over k . By the definition, for an element $b \in I$, set

$$b \cdot x_\alpha = b \circ a_\alpha, \quad \alpha \in \Lambda.$$

To define the action of a singleton f from R on the element $b \in I$ we use induction on the degree $\deg f$. By the definition, $b \cdot (fx_\alpha) = (b \cdot f) \cdot x_\alpha$. Let f_1, \dots, f_m be a tuple of singletons from R and let $\gamma_1, \dots, \gamma_m \in k$. We set

$$b \cdot (\gamma_1 f_1 + \dots + \gamma_m f_m) = \gamma_1 b \cdot f_1 + \dots + \gamma_m b \cdot f_m. \quad (1)$$

Equation (1) defines a structure of a module on the ideal I over the ring of polynomials with non-commuting variables. We next show that this definition is correct for the ring of commutative polynomials R . It is sufficient to show that $b \cdot (x_\alpha x_\beta) = b \cdot (x_\beta x_\alpha)$, where here $b \in I$ and $\alpha, \beta \in \Lambda$ is an arbitrary pair of indices. Applying the Jacoby identity we obtain

$$b \cdot (x_\alpha x_\beta) - b \cdot (x_\beta x_\alpha) = b \circ a_\alpha \circ a_\beta - b \circ a_\beta \circ a_\alpha = a_\beta \circ a_\alpha \circ b.$$

Since A/I and I are Abelian, $a_\beta \circ a_\alpha \in I$ and $a_\beta \circ a_\alpha \circ b = 0$, from the foregoing argument, we obtain that I is a module over the ring R .

Remark 2.2 *How does the module structure of I change if instead of the initial maximal linearly independent modulo I system $\{a_\alpha, \alpha \in \Lambda\}$ of elements from A we choose another such system $\{a'_\alpha, \alpha \in \Lambda\}$? Since the ideal I is Abelian such transformation implies only a k -linear change of the variables from the ring of polynomials R .*

Remark 2.3 *Denote by V the k -linear span of the set $\{a_\alpha, \alpha \in \Lambda\}$. Then the algebra A , treated as a vector space, decomposes into the following direct sum $V \oplus I$ of vector spaces over k .*

Remark 2.4 *Furtherafter we shall sometimes abuse the notation $b \cdot f$, where $f \in R$ and $b \in A$ (not necessarily $b \in I$). By this notation we mean that if*

$$f = \gamma + x_{\alpha_1} f_1 + \cdots + x_{\alpha_l} f_l, \quad \gamma \in k, f_i \in R, \alpha \in \Lambda$$

then

$$b \cdot f = \gamma b + (b a_{\alpha_1}) \cdot f_1 + \cdots + (b a_{\alpha_l}) \cdot f_l. \quad (2)$$

This equation is correctly defined, since $b a_{\alpha_i} \in I$. Notice that Equation (2) depends on the choice of designated letters $x_{\alpha_1}, \dots, x_{\alpha_l}$. We use this notation in the context when this choice is not significant.

2.2 Generators and Defining Relations of Metabelian Lie Algebras and Finitely Generated Metabelian Lie Algebras

Below we describe a convenient set of generators of an arbitrary metabelian Lie algebra A .

As above, let A be a metabelian Lie algebra and let I be an ideal of A such that A/I is Abelian. We usually think of I as the commutant or the Fitting's radical (in the event that it is Abelian). Take maximal linearly independent modulo I system of elements $\{a_\alpha, \alpha \in \Lambda\}$ from A . Denote by V the k -linear span of $\{a_\alpha, \alpha \in \Lambda\}$. Suppose that $\{b_\beta, \beta \in \mathcal{B}\}$ generates I as an R -module and is minimal.

Lemma 2.5 *The union of the sets $\{a_\alpha, \alpha \in \Lambda\}$ and $\{b_\beta, \beta \in \mathcal{B}\}$ generates the algebra A .*

Proof. On account of Remark 2.3, every element $a \in A$ can be written as $a = c + b$, where $c \in V$ and $b \in I$. The element $b \in I$ has a presentation:

$$b = b_{\beta_1} \cdot f_1 + \cdots + b_{\beta_l} \cdot f_l,$$

where here $f_i \in R$. Every summand $b_{\beta_i} \cdot f_i$, treated as an element of A , is a Lie polynomial from $\{a_\alpha, \alpha \in \Lambda\}$, b_{β_i} . Consequently, b and, therefore, a are Lie polynomials from $\{a_\alpha, \alpha \in \Lambda\} \cup \{b_\beta, \beta \in \mathcal{B}\}$. ■

Remark 2.5 *The system of generators from Lemma 2.5 is termed the canonical system of generators of a metabelian Lie algebra. We shall use it in the following, when working with metabelian Lie algebras.*

Lemma 2.6 *If A is a finitely generated metabelian Lie algebra then its canonical set of generators is finite.*

Proof. Let $\{s_1, \dots, s_n\}$ be a finite set of generators of the algebra A . We show that the canonical set of generators is also finite.

First of all observe that A/I is a finite dimensional vector space over k . This is immediate, since every element of A is a Lie polynomial from the letters $\{s_1, \dots, s_n\}$. Every polynomial admits the following presentation

$$a = \alpha_1 s_1 + \dots + \alpha_n s_n + b.$$

Here $\alpha_i \in k$, and $b \in A^2$ and therefore $b \in I$. In what follows that $\dim(A/I) = r$ is finite and, moreover, is lower than n .

We next show that the ideal I , regarded as a module over the ring $R = k[x_1, \dots, x_r]$ is finitely generated. The commutant A^2 , treated as a k -vector space, is generated by left-normed monomials $s_{i_1} \circ s_{i_2} \circ s_{i_3} \circ \dots \circ s_{i_m}$ of the length ≥ 2 . Write $s_i, i = 1, \dots, n$, as a sum: $s_i = c_i + b_i$, $c_i \in V$, $b_i \in I$. Substituting and evaluating we see that $s_{i_1} \circ s_{i_2} \circ \dots \circ s_{i_m} = (s_{i_1} \circ s_{i_2}) \cdot f$, where $f \in R$. So A^2 as a module over R is generated by its finite subset $\{s_i \circ s_j \mid i < j = 1, \dots, n\}$. Since I/A^2 is also a finite dimensional k -vector space, the ideal I is a finitely generated R -module.

We thereby proved that both parts of the canonical system of generators are finite. ■

In the category of all metabelian Lie algebras over the field k every finitely generated k -algebra A is finitely presented. The proof of the next theorem establishes the form of the defining relations of a finitely generated metabelian Lie algebra A .

Theorem 2.1 *Every finitely generated metabelian Lie k -algebra A is finitely presented in the category of all metabelian Lie k -algebras.*

Proof. Consider the vector space A/A^2 . Suppose that $\dim A/A^2 = r$. The commutant A^2 is a finitely generated module over the ring $R = k[x_1, \dots, x_r]$. Let b_1, \dots, b_l be its module generators. We shall use the canonical system of generators of A : $a_1, \dots, a_r, b_1, \dots, b_l$.

Since the ring R is Noetherian and since A^2 is a finitely generated R -module, the module A^2 is finitely presented (see [5, 7]). Fix a finite presentation of the module A^2 . An arbitrary relation of the module A^2 has the

form:

$$b_1 \cdot f_1 + \cdots + b_l \cdot f_l = 0, \quad f_i \in R.$$

This relation rewrites into the signature of metabelian Lie algebras. This way, we obtain a relation, which involves letters $a_1, \dots, a_r, b_1, \dots, b_l$ and which is true in the algebra A . By the definition, the relations of this kind form the first type of defining relations of A .

There are three types of relations in A . The second type are the obvious relations, which show that the commutant is Abelian:

$$b_i \circ b_j = 0, \quad b_i \circ a_s \circ b_j = 0, \quad i < j = 1, \dots, l, \quad s = 1, \dots, r.$$

The third type of the relations is to show that for every pair of indices (i, j) the product $a_i \circ a_j$ lies in the commutant:

$$a_i \circ a_j = b_1 \cdot g_{ij}^1 + \cdots + b_l \cdot g_{ij}^l, \quad i < j = 1, \dots, r, \quad g_{ij}^p \in R.$$

Again, these relations rewrite into the signature of metabelian Lie algebras and then form the third type of defining relations for A .

We are now left to show that the algebra A is completely defined by the relations of these three types. Let F be the free metabelian Lie algebra with the base $x_1, \dots, x_r, y_1, \dots, y_l$. And let K be the ideal of F generated by the relations of A , which are one of the three types described above. Denote by $\varphi : F \rightarrow A$ the canonical epimorphism, defined by the following equalities: $\varphi(x_i) = a_i, \quad i = 1, \dots, r, \quad \varphi(y_j) = b_j, \quad j = 1, \dots, l$. We next show that $\ker \varphi = K$. The inclusion $\ker \varphi \supseteq K$ is obvious. Only the converse inclusion is at issue.

Consider an arbitrary Lie polynomial u from the letters $x_1, \dots, x_r, y_1, \dots, y_l$. Since u is an element of a Lie k -algebra, it rewrites as a k -linear combination of left-normed monomials from the same letters. As shown above, a permutation of the c_i 's in left-normed monomials of the form $abc_1 \cdots c_n$ preserves the element (here $a, b, c_1, \dots, c_n \in \{x_1, \dots, x_r, y_1, \dots, y_l\}$).

Using the relations of the third type and the anti-commutativity identity we write every monomial of the length ≥ 2 as a linear combination of monomials, the initial letter of which is one of the y_j 's. Now, if there are at least two occurrences of the y_j 's in a monomial then, on account of the second type of the relations, this monomial lies in K . In what follows that an arbitrary element $u \in F$ can be presented in the following form:

$$u = \alpha_1 x_1 + \cdots + \alpha_r x_r + y_1 \cdot f_1 + \cdots + y_l \cdot f_l + v,$$

where $v \in K$, $\alpha_i \in k$, $f_j \in R$. Suppose that $u \in \ker \varphi$. We, therefore, obtain:

$$\alpha_1 a_1 + \cdots + \alpha_r a_r + b_1 \cdot f_1 + \cdots + b_l \cdot f_l = 0.$$

Consequently, $\alpha_1 = \dots = \alpha_r = 0$ and $b_1 \cdot f_1 + \cdots + b_l \cdot f_l = 0$. The latter equality and the first type of the relations together imply that $y_1 \cdot f_1 + \cdots + y_l \cdot f_l \in K$. And we, therefore, have shown that $u \in K$. ■

Remark 2.6 *We suppose that the result of Theorem 2.1 is known, although we have not found any reference. Furthermore, in the following we use the particular form of the relations given in the proof of Theorem 2.1.*

Corollary 2.3 *In the event that the Fitting's radical $\text{Fit}(A)$ is Abelian the presentation given in the proof of Theorem 2.1 can be refined. The new presentation is obtained using the argument of Theorem 2.1 with A^2 replaced by $\text{Fit}(A)$.*

Remark 2.7 *For infinitely generated metabelian Lie algebras the three types of relations introduced in the proof of Theorem 2.1 are defining. Though, any of the three types may be infinite.*

We next introduce a class of endomorphisms of a metabelian Lie algebra which will play an important role in the following. Suppose that I is either the commutant A^2 or the Fitting's radical $\text{Fit}(A)$ (in the event that it is Abelian).

Let $f \in R$ be a polynomial with zero free term. We define the map φ as the multiplication of elements of the algebra A by the polynomial $(1 + f)$. Recall that this map is correctly defined only for the elements of I . To extend the action of φ to the algebra A we write the polynomial f in the form $f = x_{j_1} f_1 + \cdots + x_{j_q} f_q$ (see Remark 2.4). For the canonical system of generators of the algebra A , set:

$$\varphi(b_\beta) = b_\beta \cdot (1 + f), \quad \varphi(a_\alpha) = a_\alpha + h_\alpha, \quad \alpha \in \Lambda, \quad (3)$$

where here $h_\alpha \in A^2$, $h_\alpha = a_\alpha a_{j_1} f_1 + \cdots + a_\alpha a_{j_q} f_q$. This defines the images of the canonical set of generators of A under φ . Extend the action of φ to A agreeing with the definition of a homomorphism.

Proposition 2.1 *In this notation, the map $\varphi : A \rightarrow A$ is a homomorphism. Moreover if the ideal I is a torsion-free module over R then φ is injective.*

Proof. We check that Equation (3) gives rise to a homomorphism on A . We use the presentation of A given in Theorem 2.1. Every element $b \in I$ can be written in the form

$$b = b_{\beta_1} f^{(1)} + \cdots + b_{\beta_l} f^{(l)},$$

where $f^{(k)} \in R$. Using induction on the degree of the polynomials $f^{(k)}$ one verifies that $\varphi(b) = b \cdot (1 + f)$:

$$\varphi(b_{\beta} a_{\alpha}) = \varphi(b_{\beta}) \varphi(a_{\alpha}) = (b_{\beta}(1 + f))(a_{\alpha} + h_{\alpha}) = b_{\beta} a_{\alpha} (1 + f).$$

Now it is fairly obvious that φ preserves the relations of A of the first type (the ones that reflect the R -module structure of I) and of the second type (the ones that express the fact that I is an Abelian ideal). The relations of the third type take the form:

$$a_i a_j = b_{\beta_1} f_{ij}^{(1)} + \cdots + b_{\beta_l} f_{ij}^{(l)},$$

where $f_{ij}^{(k)} \in R$, $i, j \in \Lambda$ is an arbitrary pair of indices. To verify that φ preserves these relations it suffices to check that $\varphi(a_i) \varphi(a_j) = a_i a_j (1 + f)$:

$$\begin{aligned} \varphi(a_i) \varphi(a_j) &= (a_i + a_i a_{j_1} f_1 + \cdots + a_i a_{j_q} f_q)(a_j + a_j a_{j_1} f_1 + \cdots + a_j a_{j_q} f_q) = \\ &= a_i a_j + (a_i a_{j_1} a_j f_1 + \cdots + a_i a_{j_q} a_j f_q) - (a_j a_{j_1} a_i f_1 + \cdots + a_j a_{j_q} a_i f_q) = \\ &= a_i a_j + a_i a_j a_{j_1} f_1 + \cdots + a_i a_j a_{j_q} f_q = a_i a_j (1 + f). \end{aligned}$$

Since the action of φ on the ideal I is the multiplication of elements of the module by the polynomial $(1 + f)$. The homomorphism φ (in the event that I is a torsion-free R -module), is injective. Any element $a \in A$ can be written in the form $a = c + b$, where $c \in V$, $b \in I$. If $c \neq 0$ then, obviously, $\varphi(a) \neq 0$ and if $c = 0$, then $a = b$, i.e. $a \in I$. Consequently, in any case the homomorphism φ is injective on A . ■

2.3 Free Metabelian Lie Algebras

In this section we list some of the properties of free metabelian Lie algebras and find defining relations of its Fitting's radical treated as a module over the ring of polynomials R .

Let F be the free metabelian Lie algebra over the field k . And let $\{a_{\alpha}, \alpha \in \Lambda\}$ be a free base of F . Suppose that the set of indices Λ is totally ordered. Then left-normed monomials $a_{i_1} a_{i_2} \cdots a_{i_m}$ satisfying $i_1 > i_2 \leq i_3 \leq \dots \leq i_m$ are termed normalised. In [1] V. A. Artamonov proves the following

Theorem 2.2 *The set of all normalised monomials of the free metabelian Lie algebra F forms its linear basis over k .*

Corollary 2.4

- *The images of the free base $\{a_\alpha, \alpha \in \Lambda\}$ of the free Lie algebra F in the factor-algebra F/F^2 form its additive basis over k .*
- *If $|\Lambda| > 1$ then the algebra F is not Abelian. And in the event that $|\Lambda| = 1$ the algebra F is a one-dimensional vector space.*
- *Every collection of linearly independent modulo F^2 elements $\{b_\beta, \beta \in \mathcal{B}\}$ of the algebra F generates a free metabelian Lie algebra of the rank $|\mathcal{B}|$.*

The commutant of an arbitrary metabelian Lie algebra lies in its Fitting's radical. It turns out to be that if F is non-Abelian then this inclusion is in fact an equality.

Proposition 2.2 *If F is a non-Abelian free metabelian Lie algebra then $\text{Fit}(F) = F^2$.*

Proof. Take an arbitrary element $a \notin F^2$ and show that $a \notin \text{Fit}(F)$. Write a as a linear combination of normalised words:

$$a = \alpha_1 a_{i_1} + \cdots + \alpha_n a_{i_n} + b,$$

where $b \in F^2$. Since $a \notin F^2$, we have $c = \alpha_1 a_{i_1} + \cdots + \alpha_n a_{i_n} \notin F^2$. It suffices to show that $c \notin \text{Fit}(F)$. Assume the converse: $c \in \text{Fit}(F)$. The element c , therefore, can be embodied into a free base of the algebra F . The new base is obtained from the initial one by a linear transformation. Since $c \in \text{Fit}(F)$, the element c lies in a nilpotent ideal I of F . For F is non-Abelian, there exists an element d from the free base of F so that $d \circ c \neq 0$, $dc \in I$, where $c \in I$. Consequently, multiple multiplication of dc by c results zero, i.e. $dccc \cdots c = 0$. This derives a contradiction with Theorem 2.2. ■

On account of Proposition 2.2, the free base $\{a_\alpha, \alpha \in \Lambda\}$ of a free non-Abelian Lie algebra F is a maximal linearly independent modulo $\text{Fit}(F)$ system of elements. The system of generators of $\text{Fit}(F)$, treated as a module over $R = k[x_\alpha, \alpha \in \Lambda]$, (denote it $C(F)$) is constructed as follows. The set

$\{a_\alpha a_\beta \mid \alpha, \beta \in \Lambda\}$ generates the R -module $\text{Fit}(F)$. For each pair of indices (α, β) either $a_\alpha a_\beta$ or $a_\beta a_\alpha$ only, is included in $C(F)$. Consequently, the canonical system of generators of the algebra F is the union of $\{a_\alpha, \alpha \in \Lambda\}$ and $C(F)$. According to the Jacoby identity, the generators of $\text{Fit}(F)$ satisfy relations of the type:

$$a_\alpha a_\beta x_\gamma + a_\beta a_\gamma x_\alpha + a_\gamma a_\alpha x_\beta = 0. \quad (4)$$

Lemma 2.7 below shows that Relations (4) define the module $\text{Fit}(F)$.

Lemma 2.7 *The system of generators $C(F)$ and the set of Relations (4) given above define an R -module presentation of the Fitting's radical $\text{Fit}(F)$ of the free metabelian Lie algebra F .*

Proof. Let M be the R -module given by the generators and relations described above. Consider the following formal direct sum $V \oplus M$, where V is a k -vector space with the base $\{a_\alpha, \alpha \in \Lambda\}$. The set $V \oplus M$ is a k -vector space in the obvious way. We define a multiplication in $V \oplus M$ by setting:

$$(a_\alpha \oplus m_1) \circ (a_\beta \oplus m_2) = (0 \oplus a_\alpha a_\beta + m_1 x_\beta - m_2 x_\alpha).$$

The reader can check that the space $V \oplus M$ with the above operations is a metabelian Lie algebra with the generators $\{a_\alpha, \alpha \in \Lambda\}$. The commutant of $V \oplus M$ is the module M . Since F is a free algebra, there exists the canonical epimorphism $\varphi : F \rightarrow V \oplus M$. The image of the commutant of F coincides with M and the statement of the lemma follows. \blacksquare

3 U -algebras

In this section we construct special matrix metabelian Lie algebras.

Every matrix metabelian Lie algebra is constructed by a ring of polynomials and a free module T over R . In Section 3.1 we give a description of these algebras and prove that free metabelian Lie algebra of an arbitrary rank can be embedded into a matrix metabelian Lie algebra. The idea of this embedding arises from papers [1] and [2].

In Section 3.2 we introduce the notion of a U -algebra and establish connections between U -algebras and matrix metabelian Lie algebras. In particular, every finitely generated subalgebra of a U -algebra also embeds into a special matrix algebra.

Matrix metabelian Lie algebras, therefore, provide a convenient presentation of free metabelian Lie algebras and U -algebras. We shall use this presentation further, when instead of working with metabelian Lie algebras we investigate commutative rings and modules.

3.1 Matrix Metabelian Lie Algebras

Let k be a field, let R be the ring of polynomials from commuting variables $\{x_\alpha, \alpha \in \Lambda\}$ with coefficients from k and let T be the free module over the ring R with the base $\{u_i, i \in I\}$. By this ring R and module T we construct a matrix metabelian Lie algebra. Denote by $M_{I,\Lambda}$ the set of all matrices of the form $\begin{pmatrix} f & u \\ 0 & 0 \end{pmatrix}$, where $f \in R$ and $u \in T$. We turn the set $M_{I,\Lambda}$ into a Lie k -algebra by setting:

$$\begin{aligned} \alpha \cdot \begin{pmatrix} f & u \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} \alpha \cdot f & \alpha \cdot u \\ 0 & 0 \end{pmatrix}, \alpha \in k; \\ \begin{pmatrix} f & u \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} g & v \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} f+g & u+v \\ 0 & 0 \end{pmatrix}; \\ \begin{pmatrix} f & u \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} g & v \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & ug - vf \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The set $M_{I,\Lambda}$ is closed under the operations introduced above and forms a metabelian Lie algebra. The correctness of the anti-commutativity identity, the metabelian identity, the axioms of a vector space and the distributivity of multiplication in $M_{I,\Lambda}$ is straightforward. We next show that $M_{I,\Lambda}$ satisfies the Jacoby identity. For the sake of brevity we write elements from $M_{I,\Lambda}$ as pairs: $\{(f, u)\}$.

The product of arbitrary three elements from $M_{I,\Lambda}$ equals:

$$(f_1, v_1) \circ (f_2, v_2) \circ (f_3, v_3) = (0, v_1 f_2 - v_2 f_1) \circ (f_3, v_3) = (0, v_1 f_2 f_3 - v_2 f_1 f_3).$$

So the Jacoby identity takes the form:

$$\begin{aligned} (f_1, v_1) \circ (f_2, v_2) \circ (f_3, v_3) + (f_2, v_2) \circ (f_3, v_3) \circ (f_1, v_1) + (f_3, v_3) \circ (f_1, v_1) \circ (f_2, v_2) = \\ = (0, v_1 f_2 f_3 - v_2 f_1 f_3) + (0, v_2 f_3 f_1 - v_3 f_2 f_1) + (0, v_3 f_1 f_2 - v_1 f_3 f_2) = (0, 0). \end{aligned}$$

We thereby have proven that the set of matrices $M_{I,\Lambda}$ with the operations above is a metabelian Lie k -algebra. Such algebras and all their subalgebras are called *matrix metabelian Lie algebras*.

We allow for the possibility that $\Lambda = \emptyset$, i.e. $R = k$, in which case T is a vector space over k .

In general case, the algebra $M_{I,\Lambda}$ is not Abelian:

$$(0, u) \circ (f, u) = (0, uf) \neq (0, 0),$$

provided that $f \neq 0$ and $u \neq 0$. Although $M_{I,\Lambda}$ contains Abelian subalgebras. For instance, the commutant or any one-generated subalgebra are Abelian. The multiplication in $M_{I,\Lambda}$ is defined so that all the elements (matrices) of its commutant have zero principal diagonal, i.e. the commutant consists of elements of the form $(0, u)$. It turns out to be that the collection of all such elements forms the Fitting's radical $\text{Fit}(M_{I,\Lambda})$. The foregoing discussion is summarised and generalised in the lemma below.

Lemma 3.1 *Let A be a non-Abelian subalgebra of matrix metabelian Lie algebra $M_{I,\Lambda}$. Then the Fitting's radical of A consists of all the elements from A with zero principal diagonal:*

$$\text{Fit}(A) = \{(f, u) \in A, f = 0\}.$$

Proof. Let B be the following set $\{(f, u) \in A, f = 0\}$. We show that $B = \text{Fit}(A)$. Take an arbitrary element $(0, u) \in B$ and let I be the principal ideal of A generated by $(0, u)$. Clearly $I \subset B$, and so I is an Abelian ideal and, thus $(0, u) \in \text{Fit}(A)$. To show the reverse inclusion take an arbitrary element $z = (f, u)$ from A . Suppose that $z \notin B$, in particular $f \neq 0$. Assume the converse: $z \in \text{Fit}(A)$. Since $z \in \text{Fit}(A)$, by Corollary 2.2, z lies in a nilpotent ideal I . Since A is non-Abelian, there exist two elements in A such that their product is non-zero. By the definition, the product of two arbitrary elements lies in B . Consequently, there is to exist a non-zero element $w = (0, v) \in B, v \neq 0$. Then, in the above notation, $w \circ z = (0, vf) \in I$ and $w \circ z \neq 0$. For I is a nilpotent ideal, there exists a positive integer n such that $w \circ z \circ \dots \circ z = (0, vf^n) = (0, 0)$, what derives a contradiction. ■

Remark 3.1 *There exists an Abelian subalgebra A of the matrix metabelian Lie algebra $M_{I,\Lambda}$ so that Lemma 3.1 fails. For instance, let A be the algebra generated by $(f, u), f \neq 0$. In that case $\text{Fit}(A) = A$.*

Corollary 3.1 *The Fitting's radical of an arbitrary subalgebra of a matrix metabelian Lie algebra is Abelian.*

Notation Let $M_{I,\Lambda}$ be a matrix metabelian Lie algebra. By $M_{I,\Lambda}^0$ denote the following subset of $M_{I,\Lambda}$:

$$M_{I,\Lambda}^0 = \{(f, u) \mid f = \beta_{\alpha_1}x_{\alpha_1} + \beta_{\alpha_2}x_{\alpha_2} + \cdots + \beta_{\alpha_m}x_{\alpha_m}, \beta_{\alpha_j} \in k\}.$$

Clearly, $M_{I,\Lambda}^0$ is a subalgebra of $M_{I,\Lambda}$. In the following we shall be particularly interested in a more narrow class of matrix metabelian Lie algebras, namely, the subalgebras of $M_{I,\Lambda}^0$.

Lemma 3.2 *Let A be a subalgebra of a matrix metabelian Lie algebra $M_{I,\Lambda}^0$. Then the Fitting's radical $\text{Fit}(A)$ is a torsion-free module over the ring of polynomials.*

Proof. On behalf of Corollary 3.1, $\text{Fit}(A)$ is Abelian. Consequently, it allows a structure of a module over the ring of polynomials. Let $\{e_d, d \in \Delta\}$ be a basis of the vector space $A/\text{Fit}(A)$, let $\{(f_d, u_d), d \in \Delta\}$ be the set of preimages in A of elements of this basis. All the polynomials f_d 's are k -linear combinations of letters $\{x_\alpha, \alpha \in \Lambda\}$. Furthermore, on account of Lemma 3.1, the set of polynomials $\{f_d, d \in \Delta\}$ is linearly independent.

We need to show that $\text{Fit}(A)$ is a torsion-free module over the ring of polynomials $R = k[x_d, d \in \Delta]$. Take an arbitrary element $b \in \text{Fit}(A)$, a non-zero polynomial $g(x_d) \in R$ and consider the product $b \cdot g$. By Lemma 3.1 every element of $\text{Fit}(A)$ has the form $b = (0, v)$ and, consequently, $b \cdot g = (0, vg(f_d))$. The polynomial $g(f_d)$ is also non-zero and therefore $b \cdot g \neq 0$. ■

We next show how a free metabelian Lie algebra F over the field k is embedded into a matrix algebra. Let $\{a_\alpha, \alpha \in \Lambda\}$ be the free base of F . Then the algebra F is a subalgebra of the matrix metabelian Lie algebra $M_{\Lambda,\Lambda}^0$. The matrix algebra $M_{\Lambda,\Lambda}^0$ is constructed by the ring of polynomials $R = k[x_\alpha, \alpha \in \Lambda]$ and by the free module T with the set of free generators $\{u_\alpha, \alpha \in \Lambda\}$. Consider a subalgebra L in $M_{\Lambda,\Lambda}^0$ generated by

$$\left\{ z_\alpha = \begin{pmatrix} x_\alpha & u_\alpha \\ 0 & 0 \end{pmatrix}, \alpha \in \Lambda \right\}.$$

The cardinality of the set $\{z_\alpha, \alpha \in \Lambda\}$ of generators of the algebra L coincides with the cardinality of the free base $\{a_\alpha, \alpha \in \Lambda\}$ of F . Consequently, there exists the canonical homomorphism $\gamma : F \rightarrow M_{\Lambda,\Lambda}^0$ so that $\gamma(a_\alpha) = z_\alpha, \alpha \in \Lambda$. The following theorem shows that the algebra L is a presentation of F in a matrix algebra.

Theorem 3.1 *In the above notation, the homomorphism γ is an embedding of the free metabelian Lie algebra F into the matrix metabelian Lie algebra $M_{\Lambda, \Lambda}^0$.*

Proof. Let I be the kernel of γ . We prove that I is the trivial ideal.

By Theorem 2.2 the algebra L (as a k -vector space) is generated by normalised words from the letters $\{z_\alpha, \alpha \in \Lambda\}$. Recall that normalised words are left-normed words $z_{i_1} z_{i_2} \cdots z_{i_m}$ such that $i_1 > i_2 \leq i_3 \leq \dots \leq i_m$ (we assume that the set of indices Λ is totally ordered). To prove that I is the trivial ideal it suffices to show that the normalised words in L are linearly independent.

In the matrix representation the normalised word $z_{i_1} z_{i_2} \cdots z_{i_m}$ is presented by the following matrix

$$\begin{pmatrix} 0 & (u_{i_1} x_{i_2} - u_{i_2} x_{i_1}) \cdot x_{i_m} \cdots x_{i_3} \\ 0 & 0 \end{pmatrix}.$$

Since $(u_{i_1} x_{i_2} - u_{i_2} x_{i_1}) \cdot x_{i_m} \cdots x_{i_3}$ is a non-zero element of the free module T , this matrix is non-zero. Assume further that

$$\gamma\left(\sum_{\bar{i}} \alpha_{\bar{i}} a_{i_1} a_{i_2} \cdots a_{i_m}\right) = 0, \quad \alpha_{\bar{i}} \in k, \quad (5)$$

and show that for all multi-indices \bar{i} the coefficient $\alpha_{\bar{i}} = 0$. Obviously, we may assume that for all \bar{i} in Equation (5) $m \geq 2$. Equation (5) implies the following equality in the module T :

$$\sum_{\bar{i}} \alpha_{\bar{i}} (u_{i_1} x_{i_2} - u_{i_2} x_{i_1}) \cdot x_{i_k} \cdots x_{i_3} = 0. \quad (6)$$

Take the maximal index i_1 among all first indices in all multi-indices \bar{i} involved in Equation (5). By the definition of a normalised word, we obtain that the coefficient of u_{i_1} on the left-hand side of Equation (6) is

$$\sum_{\bar{i}=i_1, \dots, i_m} \alpha_{\bar{i}} x_{i_m} \cdots x_{i_3} \cdot x_{i_2}.$$

Since T is a free module and since u_{i_1} is an element from the basis of T , this coefficient equals zero in the ring of polynomials R . Again, by the definition of a normalised word, we obtain that if all the $\alpha_{\bar{i}}$'s are zero, provided that the

multi-index \bar{i} begins with i_1 . We proceed by induction on the first coordinate of the multi-index \bar{i} and the statement follows. \blacksquare

As a conclusion we formulate and prove several simple and important properties of elements of matrix metabelian Lie algebras.

Lemma 3.3 *Let $M_{I,\Lambda}$ be an arbitrary matrix metabelian Lie algebra and let $x = (f_1, u_1)$, $y = (f_2, u_2)$, $z = (f_3, u_3)$ be elements from $M_{I,\Lambda}$. Then*

1. *If $xyx = 0$ and $xyy = 0$ then $xy = 0$.*
2. *If $xy = 0$, $xz = 0$ and $x \neq 0$, then $yz = 0$.*

Proof. Let $x, y \in M_{I,\Lambda}$, $xyx = 0$ and $xyy = 0$. In terms of matrices two latter conditions rewrite as follows:

$$(0, (u_1 f_2 - u_2 f_1) f_1) = (0, 0) \text{ and } (0, (u_1 f_2 - u_2 f_1) f_2) = (0, 0).$$

Now the required equality is essentially immediate: $xy = (0, u_1 f_2 - u_2 f_1) = (0, 0)$.

We now prove the second statement. The conditions imposed imply:

$$u_1 f_2 = u_2 f_1, \tag{7}$$

$$u_1 f_3 = u_3 f_1. \tag{8}$$

Furthermore, $f_1 \neq 0$ and $u_1 \neq 0$. We need to show that $yz = 0$, i.e.

$$u_2 f_3 = u_3 f_2. \tag{9}$$

In the event that $f_1 \neq 0$ Equation (9) is equivalent to the following equality $u_2 f_3 f_1 = u_3 f_2 f_1$. Multiplying Equations (7) and (8) by f_3 and f_2 correspondingly we obtain $u_2 f_3 f_1 = u_3 f_2 f_1$. Now, if $f_1 = 0$ then $f_2 = f_3 = 0$ and Equation (9) also holds. \blacksquare

Corollary 3.2 *According to Theorem 3.1, Lemma 3.3 holds in the free metabelian Lie algebra.*

3.2 U -Algebras

Definition 3.1 We term a metabelian Lie algebra A over a field k a metabelian Lie U -algebra if

- $\mathbf{Fit}(A)$ is Abelian;
- $\mathbf{Fit}(A)$, treated as a module over the ring of polynomials (defined as in Section 2.2), is torsion-free.

For the sake of brevity, we sometimes call metabelian Lie U -algebras by U -algebras.

Remark 3.2 The Fitting's radical $\mathbf{Fit}(A)$ admits the structure of a module over the ring $R = k[x_\alpha, \alpha \in \Lambda]$. The system of linearly independent modulo $\mathbf{Fit}(A)$ elements $\{a_\alpha, \alpha \in \Lambda\}$ of A may vary depending on the definition of a module structure on $\mathbf{Fit}(A)$. The transformation between two such systems is a k -linear change of variables of the ring R . This implies that if for $\{a_\alpha, \alpha \in \Lambda\}$ the module $\mathbf{Fit}(A)$ is torsion-free then it is torsion-free for another such tuple of elements. Therefore the definition of a U -algebra is correct.

If A is an Abelian Lie algebra then $A = \mathbf{Fit}(A)$, thus the Fitting's radical $\mathbf{Fit}(A)$ is a module over the field k . Consequently, every Abelian Lie algebra is a U -algebra.

The connection between metabelian U -algebras and matrix metabelian Lie algebras is established by the following

Theorem 3.2 Every finitely generated metabelian Lie U -algebra is isomorphic to a subalgebra of a matrix metabelian Lie algebra $M_{I,\Lambda}^0$. Conversely, every subalgebra of the algebra $M_{I,\Lambda}^0$ is a U -algebra.

Proof. Directly from Lemmas 3.1 and 3.2 we obtain that every subalgebra A of the matrix metabelian Lie algebra $M_{I,\Lambda}^0$ is a U -algebra.

Let A be a finitely generated U -algebra. By the definition $\mathbf{Fit}(A)$ is Abelian and, therefore, admits the structure of a module over the ring of polynomials. Since A is a finitely generated algebra, by Lemma 2.6, its canonical system of generators $a_1, \dots, a_r, b_1, \dots, b_l$ is finite. Again, by the definition, $\mathbf{Fit}(A)$ is a torsion-free module over the ring $R = k[x_1, \dots, x_r]$.

Any torsion-free finitely generated module over the ring of polynomials embeds into a free module, thus $\mathbf{Fit}(A)$ embeds into the free module T over the ring R of the rank s , $s \leq l$ (see [5, 7]). Let $\varphi : \mathbf{Fit}(A) \rightarrow T$ be the correspondent embedding. Using this embedding we construct an embedding of the algebra A , $\gamma : A \rightarrow M_{s,r}^0$.

By the definition of γ , for the standard generators we set:

$$\begin{aligned}\gamma(b_i) &= (0, \varphi(b_i) \cdot (\sum_{k=1}^r x_k)), \quad i = 1, \dots, l, \\ \gamma(a_j) &= (x_j, \sum_{k=1}^r \varphi(a_j \circ a_k)), \quad j = 1, \dots, r.\end{aligned}$$

We extend the definition of γ on A following the definition of a homomorphism. We are left to check that γ is a correct homomorphism.

To prove that γ is a correct homomorphism we prove that it preserves the three types of defining relations of A given in Theorem 2.1.

The first type of defining relations of A are exactly the defining relations of $\mathbf{Fit}(A)$ treated as a module over the ring R . Every module relation of $\mathbf{Fit}(A)$ has the form: $b_1 \cdot f_1 + \dots + b_l \cdot f_l = 0$, $f_i \in R$. For every pair of indices (i, j) , $i = 1, \dots, l$ and $j = 1, \dots, r$ we obtain:

$$\gamma(b_i \circ a_j) = \gamma(b_i) \circ \gamma(a_j) = (0, \varphi(b_i) \cdot (\sum_{k=1}^r x_k) \cdot x_j).$$

Now for an arbitrary polynomial $f \in R$, applying induction on the degree of a polynomial, we obtain:

$$\gamma(b_i \cdot f) = (0, \varphi(b_i) \cdot (\sum_{k=1}^r x_k) \cdot f) = (0, \varphi(b_i \cdot f) \cdot (\sum_{k=1}^r x_k)).$$

This gives the required equality:

$$\begin{aligned}\gamma(b_1 \cdot f_1 + \dots + b_l \cdot f_l) &= \gamma(b_1 \cdot f_1) + \dots + \gamma(b_l \cdot f_l) = \\ &= (0, \varphi(b_1 \cdot f_1 + \dots + b_l \cdot f_l) \cdot (\sum_{k=1}^r x_k)) = (0, 0).\end{aligned}$$

By Lemma 3.1, $\gamma(b_i) \in \mathbf{Fit}(M_{s,r}^0)$, $i = 1, \dots, l$.

Consequently, the relations of the second type, which express that $\mathbf{Fit}(A)$ is Abelian, are preserved under the action of γ .

The third type of relations of the algebra A shows that $a_i \circ a_j \in \mathbf{Fit}(A)$ for every pair of indices (i, j) . These relations have the form: $a_i \circ a_j = b_1 \cdot g_{ij}^1 + \cdots + b_l \cdot g_{ij}^l$, $g_{ij}^p \in R$. We have already shown above that

$$\gamma(b_1 \cdot g_{ij}^1 + \cdots + b_l \cdot g_{ij}^l) = (0, \varphi(b_1 \cdot g_{ij}^1 + \cdots + b_l \cdot g_{ij}^l) \cdot (\sum_{k=1}^r x_k)).$$

We therefore are left to show that $\gamma(a_i \circ a_j) = (0, \varphi(a_i \circ a_j) \cdot (\sum_{k=1}^r x_k))$. By the means of direct computation

$$\begin{aligned} \gamma(a_i \circ a_j) &= \gamma(a_i) \circ \gamma(a_j) = (x_i, \sum_{k=1}^r \varphi(a_i \circ a_k)) \circ (x_j, \sum_{k=1}^r \varphi(a_j \circ a_k)) = \\ &= (0, \sum_{k=1}^r (\varphi(a_i \circ a_k) \cdot x_j - \varphi(a_j \circ a_k) \cdot x_i)) = (0, \varphi(a_i \circ a_j) \cdot (\sum_{k=1}^r x_k)). \end{aligned}$$

The latter equality follows from the Jacoby identity.

To prove the theorem it suffices to show that the kernel of the homomorphism γ is trivial. Let $a \in A$. We show that $a \neq 0$ implies that $\gamma(a) \neq (0, 0)$. The element a has a unique presentation in the form:

$$a = \alpha_1 a_1 + \cdots + \alpha_n a_n + b,$$

where $\alpha_i \in k$, $b \in \mathbf{Fit}(A)$. If $\alpha_1 a_1 + \cdots + \alpha_n a_n \neq 0$ then $\gamma(a) \neq (0, 0)$. Let $a = b \in \mathbf{Fit}(A)$ and $b \neq 0$. The element b can be written as $b = b_1 f^1 + \cdots + b_l f^l$, where $f^k \in R$. Now, since φ is an embedding and T is a torsion free module over R , we conclude that

$$\gamma(b) = (0, \varphi(b_1 \cdot f^1 + \cdots + b_l \cdot f^l) \cdot (\sum_{k=1}^r x_k)) \neq (0, 0).$$

We, therefore, have shown that γ is an injective homomorphism, whose restriction on the U -algebra A is an embedding of A into the matrix metabelian Lie algebra $M_{s,r}^0$ ■

Remark 3.3 *In the extreme case when A is an Abelian algebra, A embeds into $M_{l,0}^0$, where l is the dimension of the k -vector space A .*

Corollary 3.3 *Notice that free metabelian Lie algebras are U -algebras.*

Theorem 3.2 characterises finitely generated U -algebras. Further results give a description of arbitrary metabelian U -algebras.

Lemma 3.4 *Let A be a metabelian Lie U -algebra and let B be a subalgebra of A . Then B is a U -algebra.*

Proof. If B is an Abelian subalgebra of A , then, according to Remark 3.3, B is a U -algebra. We, therefore, assume that B is non-Abelian.

We show that $\text{Fit}(B) = B \cap \text{Fit}(A)$. The inclusion $\text{Fit}(B) \supseteq B \cap \text{Fit}(A)$ is obvious. The converse inclusion is at issue. Take an arbitrary element $b \in B \setminus \text{Fit}(A)$. Since B is non-Abelian, there exist two elements $c, d \in B$ so that $cd \neq 0$. Since $b \notin \text{Fit}(A)$ and since $0 \neq cd \in \text{Fit}(A)$, all the monomials of the form $cdbb \cdots b$ are non-zero. This implies that the element b is contained in neither of nilpotent ideals of the subalgebra B and consequently $b \notin \text{Fit}(B)$. This proves the reverse inclusion.

It now follows that $\text{Fit}(B)$ is Abelian. Finally, any linearly independent modulo $\text{Fit}(B)$ system of elements from B is linearly independent modulo $\text{Fit}(A)$, thus $\text{Fit}(B)$ is torsion-free. ■

Theorem 3.3 *Let A be a metabelian Lie algebra. Then A is a U -algebra if and only if every finitely generated subalgebra of A is a U -algebra.*

Proof. Let A be a U -algebra. By Lemma 3.4 every subalgebra of A is a U -algebra. We are to prove the converse. Suppose that every finitely generated subalgebra of A is a U -algebra.

We first show that $\text{Fit}(A)$ is Abelian. Let $c, d \in \text{Fit}(A)$ and let $D = \langle c, d \rangle$ be the subalgebra generated by c, d . Since $\text{Fit}(D) \supseteq D \cap \text{Fit}(A)$, both c and d lie in $\text{Fit}(D)$. By the assumption D is a U -algebra and, consequently, $cd = 0$.

We next show that $\text{Fit}(A)$ is a torsion free module over the ring of polynomials. Let $0 \neq b \in \text{Fit}(A)$, and let $f(a_1, \dots, a_n)$ be a non-zero polynomial. Set $B = \langle a_1, \dots, a_n, b \rangle$. The algebra B is a finitely generated subalgebra of A and thus is a U -algebra. Clearly, $b \in \text{Fit}(B)$. This implies that to prove the required inequality $b \cdot f \neq 0$ we need to show that the elements a_1, \dots, a_n are linearly independent modulo $\text{Fit}(B)$. Assume the converse. Let $a = \alpha_1 a_1 + \cdots + \alpha_n a_n$, $\alpha_i \in k$ be a non-trivial linear combination and

$a \in \mathbf{Fit}(B)$. We prove that in this case $a \in \mathbf{Fit}(A)$, what derives a contradiction with linear independence of the elements a_1, \dots, a_n modulo $\mathbf{Fit}(A)$. Since $\mathbf{Fit}(B)$ is Abelian and $b \in \mathbf{Fit}(B)$, we conclude that $ab = 0$. Take an arbitrary element $c \in \mathbf{Fit}(A)$. Put $C = \langle a, b, c \rangle$. The algebra C is a finitely generated subalgebra of A , thus by the assumption C is a U -algebra. Now, since $b, c \in \mathbf{Fit}(C)$, we have that $b \neq 0$, $ab = 0$ and $bc = 0$. On account of Theorem 3.2 the algebra C embeds into a matrix metabelian Lie algebra $M_{I,\Lambda}^0$. By Lemma 3.3 we conclude that $ac = 0$. Finally, on account of Lemma 2.4, $a \in \mathbf{Fit}(A)$. ■

Corollary 3.4 *Theorems 3.2 and 3.3 provide the following characterisation of metabelian Lie U -algebras:*

A metabelian Lie algebra is a U -algebra if and only if every its finitely generated subalgebra embeds into a special matrix metabelian algebra of the form $M_{I,\Lambda}^0$.

Theorem 3.4 *Let A be a U -algebra, $x, y, z \in A$. Then*

1. *if $xyx = 0$ and $xyy = 0$ then $xy = 0$.*
2. *if $xy = 0$, $xz = 0$ and $x \neq 0$ then $yz = 0$.*

Proof. Let $C = \langle x, y, z \rangle$ be a finitely generated subalgebra of the algebra A . Then, on behalf of Corollary 3.4, the algebra C embeds into an algebra of the type $M_{I,\Lambda}^0$. Consequently, using Lemma 3.3, the statement follows. ■

4 Universal Classes and Extensions of the Fitting's Radical

Let A be a U -algebra over a field k . In this section we describe two types of extensions of the Fitting's Radical of a U -algebra A over k . The first is the localisation of the Fitting's radical of A treated as an R -module by a maximal ideal $\Delta \triangleleft R$. This algebra is called Δ -local. In Section 4.2 we investigate the construction of Δ -local algebras and establish its further properties. The second type of extensions of the Fitting's radical of A is the direct module extension of the Fitting's Radical.

These constructions play an important role in our study of the universal closure of the algebra A , which is crucial in constructing algebraic geometry over Lie algebras (in fact, over arbitrary algebraic systems). Section 4.1 holds preliminary definitions and preliminary results on universal closures, other universal classes and languages.

4.1 Universal Classes

The object of this section, which arises from papers [4] and [8] is to introduce the counterparts to group-theoretic notions from [4] and [8] in the category of Lie k -algebras. We also formulate a number of results for Lie algebras, which are complete counterparts to the results of papers [4] and [8]. The proofs are similar and therefore omitted.

The standard first order language L of the theory of Lie algebras over a fixed field k consists of a symbol for multiplication ‘ \circ ’, a symbol for addition ‘ $+$ ’, a symbol for subtraction ‘ $-$ ’, a set of symbols $\{k_\alpha \mid \alpha \in k\}$ for multiplication of the elements of Lie algebras on the coefficients from the field k and a constant symbol ‘ 0 ’ which denotes zero.

For the purposes of algebraic geometry over a fixed Lie algebra A one is to study the category of A -Lie algebras, so it is more convenient to use a bigger language L_A . The language of the category of A -Lie algebras consists of the language L together with the set of constant symbols enumerated by the elements of A

$$L_A = L \cup \{c_a \mid a \in A\}.$$

A Lie algebra B over a field k is called an A -Lie algebra if and only if it contains a designated copy of A , which we shall for most part identify with A . It is clear that an A -Lie algebra B can be treated as a model of the language L_A . A homomorphism φ from an A -Lie algebra B_1 to an A -Lie algebra B_2 is an A -homomorphism of Lie algebras if it is the identity on A . The family of all A -Lie algebras together with the collection of all A -homomorphisms form a category in the obvious way.

Set $\text{Hom}_A(B_1, B_2)$ to be the set of all A -homomorphisms from the A -Lie algebra B_1 to the A -Lie algebra B_2 .

To an A -Lie algebra B we link several model-theoretical classes of A -Lie algebras.

- The variety $A\text{-var}(B)$ generated by B is the class of all A -Lie algebras that satisfy all the identities of the language L_A , satisfied by B .

- The quasivariety $A - \mathbf{qvar}(B)$ generated by B is the class of all A -Lie algebras that satisfy all the quasi identities of the language L_A , satisfied by B .
- The universal closure $A - \mathbf{ucl}(B)$ generated by B is the class of all A -Lie algebras that satisfy all the universal sentences of the language L_A , satisfied by B .

Here are the definitions of the identity, the quasi identity and the universal sentence in the language L_A .

- An A -universal sentence of the language L_A is a formula of the type

$$\forall x_1 \dots \forall x_n \left(\bigvee_{j=1}^s \bigwedge_{i=1}^t (u_{ij}(\bar{x}, \bar{a}_{ij}) = 0 \wedge w_{ij}(\bar{x}, \bar{c}_{ij}) \neq 0) \right),$$

where $\bar{x} = (x_1, \dots, x_n)$ is an n -tuple of variables, \bar{a}_{ij} and \bar{c}_{ij} are the sets of constants from the algebra A and u_{ij}, w_{ij} are the terms in the language L_A from the variables x_1, \dots, x_n . In the event that an A -universal sentence involves no constants from the algebra A this notion turns into standard notion of universal sentence in the language L .

- An A -identity of the language L_A is the formula of the type

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{i=1}^m r_i(\bar{x}, \bar{a}_{ij}) = 0 \right),$$

where $r_i(\bar{x})$ are the terms in the language L_A from the variables x_1, \dots, x_n . In the event that an A -identity involves no constants from the algebra A this notion turns into standard notion of identity of the language L .

- An A -quasi identity of the language L_A is a formula of the type

$$\forall x_1 \dots \forall x_n \left(\bigwedge_{i=1}^m r_i(\bar{x}, \bar{a}_{ij}) = 0 \rightarrow s(\bar{x}, \bar{b}) = 0 \right),$$

where $r_i(\bar{x})$ and $s(\bar{x})$ are the terms. Coefficients free analogue is the notion of quasi identity.

Remark 4.1 For our purposes we also need to consider the classes $\text{var}(B)$, $\text{qvar}(B)$ and $\text{ucl}(B)$, which, by the definition, are the variety, the quasi-variety and the universal closure generated by B in the first order language L . The classes $\text{var}(B)$, $\text{qvar}(B)$ and $\text{ucl}(B)$ are also particular cases of the classes $A - \text{var}(B)$, $A - \text{qvar}(B)$ and $A - \text{ucl}(B)$, with $A = \{0\}$.

Remark 4.2 For universal classes of A -Lie algebras the following sequence of inclusions hold

$$A - \text{ucl}(B) \subseteq A - \text{qvar}(B) \subseteq A - \text{var}(B).$$

The first inclusion is obvious, while the second one is implied by the fact that every identity is equivalent to a conjunction of a finite number of quasi-identities. For instance, the identity $\forall x_1 \dots \forall x_n (\bigwedge_{i=1}^m r_i(\bar{x}) = 0)$ is equivalent to the following set of m quasi identities $\forall x_1 \dots \forall x_n \forall y (y = y \rightarrow r_i(\bar{x}) = 0)$.

Let B_1 and B_2 be two A -Lie algebras. An A -Lie algebra B_2 is termed A -discriminated by the A -Lie algebra B_1 if for every finite subset $\{b_1, \dots, b_m\}$ of nonzero elements from the algebra B_1 there exists an A -homomorphism $\varphi : B_1 \rightarrow B_2$ such that $\varphi(b_i) \neq 0$ for every $i = 1, \dots, m$.

Theorem 4.1 Let B_1 and B_2 be two A -Lie algebras such that B_1 is A -discriminated by the A -Lie algebra B_2 . Then $B_1 \in A - \text{ucl}(B_2)$.

Proof. Recall that to prove that $B_1 \in A - \text{ucl}(B_2)$ it suffices to show that every finite submodel of the algebra B_1 A -embeds into the algebra B_2 . This is obvious, since B_1 is A -discriminated by the algebra B_2 . ■

4.2 Δ -Local Lie Algebras

Let A be a U -algebra over a field k . Let $\{z_\alpha, \alpha \in \Lambda\}$ be a maximal set of linearly independent modulo $\text{Fit}(A)$ elements from A . Denote by V the linear span of $\{z_\alpha, \alpha \in \Lambda\}$ over the field k .

Let $\Delta = \langle x_\alpha, \alpha \in \Lambda \rangle$ be the maximal ideal of the ring $R = k[x_\alpha, \alpha \in \Lambda]$ generated by the variables $\{x_\alpha, \alpha \in \Lambda\}$.

Denote by R_Δ the localisation of the ring R by Δ and by $\text{Fit}_\Delta(A)$ the localisation of the module $\text{Fit}(A)$ by the ideal Δ , i. e. the closure of $\text{Fit}(A)$

under the action of the elements of R_Δ (for definitions see [5] and [7]). Consider the direct sum $V \oplus \mathbf{Fit}_\Delta(A)$ of vector spaces over k . We next define a structure of an algebra on $V \oplus \mathbf{Fit}_\Delta(A)$. By the definition, the multiplication of the elements from V is inherited from A , the multiplication in $\mathbf{Fit}_\Delta(A)$ is trivial. Set

$$u \circ z_\alpha = u \cdot x_\alpha, \quad u \in \mathbf{Fit}_\Delta(A), \quad z_\alpha \in V, \quad u \cdot x_\alpha \in \mathbf{Fit}_\Delta(A).$$

and extend the definition of the multiplication by the elements from $\mathbf{Fit}_\Delta(A)$ on the elements from V to be linear.

One verifies that

- this multiplication turns the vector space $V \oplus \mathbf{Fit}_\Delta(A)$ into a metabelian Lie algebra which we denote by A_Δ ,
- $A_\Delta/\mathbf{Fit}(A_\Delta) \cong V$,
- $\mathbf{Fit}(A_\Delta)$ admits a structure of a module over R ,
- $\mathbf{Fit}(A_\Delta) = \mathbf{Fit}_\Delta(A)$,
- A_Δ is a U -algebra,
- the algebra A is a subalgebra of A_Δ ,
- even in the event that A is finitely generated the algebra A_Δ is not finitely generated.

In the above notation, the algebra A_Δ is termed the Δ -localisation of the algebra A .

We now point out some of the connections between the Δ -localisation and the universal closure of A .

Lemma 4.1 *Let A be a U -algebra and let A_Δ be its Δ -localisation. Assume that B is a finitely generated subalgebra of A_Δ . Then the algebra B embeds into A .*

Proof. Let $K = \{b_1, \dots, b_n\}$ be a set of generators of a subalgebra of B . We next construct an injective homomorphism $\varphi : A_\Delta \rightarrow A_\Delta$ so that $\varphi(b_i) \in A$ ($A \subseteq A_\Delta$) for each $b_i \in K$. It is straightforward, that the restriction of φ to B is an insertion of B into A , $\varphi : B \rightarrow A$. By the definition, the action of φ is the multiplication by a polynomial of the form $(1 + f)$. By Proposition 2.1, φ is a correct injective homomorphism. The correspondent polynomial is constructed as follows. Write the elements of K as sums:

$$b_i = c_i + d_i, \quad c_i \in V, \quad d_i \in \text{Fit}(A_\Delta), \quad i = 1, \dots, n.$$

The d_i 's can be presented as a sum of the elements of the form:

$$d_{ij} \frac{1}{g_{ij}}, \quad d_{ij} \in \text{Fit}(A), \quad g_{ij} \in R \setminus \Delta, \quad i = 1, \dots, n.$$

Denote by g the product of all the g_{ij} 's. Since the polynomials g_{ij} 's have non-zero free terms, the polynomial g has a non-zero free term $\alpha \neq 0$. Further, choose a polynomial $f \in \Delta$ so that $1 + f = \frac{1}{\alpha}g$ and define the homomorphism φ as the multiplication by $(1 + f)$.

By the choice of the polynomial $(1 + f)$, we have $\varphi(d_i) \in \text{Fit}(A)$ and thus $\varphi(b_i) \in A$, $i = 1, \dots, n$. Finally, since $\text{Fit}(A_\Delta)$ is a torsion-free module, the homomorphism φ is injective. \blacksquare

Proposition 4.1 *Let A be a U -algebra and let A_Δ be its Δ -localisation. Then the algebras A and A_Δ are universally equivalent: $\text{ucl}(A) = \text{ucl}(A_\Delta)$.*

Proof. To prove that two algebras are universally equivalent it suffices to prove that every finite submodel of one of the algebras embeds into the other algebra. Clearly, every finite submodel of the algebra A embeds into the algebra A_Δ . We, therefore, are left to prove the converse. Take a finite submodel of the algebra A_Δ : $K = \{b_1, \dots, b_n\}$. The elements of the set K generate a finitely generated subalgebra $B \leq A_\Delta$, which, on account of Lemma 4.1, embeds into the algebra A . Consequently, there exists an embedding of a finite submodel K of A into the algebra A_Δ . \blacksquare

We are particularly interested in the Δ -localisation of the free metabelian Lie algebra of a finite rank r . Let c_1, \dots, c_r be a linearly independent modulo $\text{Fit}(F)$ elements from F and let $C = \langle c_1, \dots, c_r \rangle$ be a subalgebra of F . By Corollary 2.4, C is the free metabelian Lie algebra of the rank r . Although, in general $C \subsetneq F$. However, for Δ -local algebras holds

Proposition 4.2 *For this notation, $C_\Delta = F_\Delta$ and $\text{Fit}(C_\Delta) = \text{Fit}(F_\Delta)$.*

Proof. The proof is required for the second equality only. Without loss of generality, we may assume that $c_i = a_i + d_i$, $i = 1, \dots, r$, where a_1, \dots, a_r is the free base of the algebra F and $d_1, \dots, d_r \in F^2$. It suffices to show that for an arbitrary pair of indices (i, j) holds $a_i a_j \in C_\Delta^2$. Let $R = k[x_1, \dots, x_r]$. Write all the products $c_i c_j$, $i > j = 1, \dots, r$ in the form

$$c_i c_j = a_i a_j \cdot f_{ij}(x_1, \dots, x_r) + \sum_{(p,q) \neq (i,j)} a_p a_q \cdot f_{pq}(x_1, \dots, x_r),$$

where $f_{ij} \in R \setminus \Delta$, $f_{pq} \in \Delta$. This gives a system of C_r^2 (the combination of r taken 2 at a time) equations over $\mathbf{Fit}(C)$ in variables $a_i a_j$, $i > j = 1, \dots, r$. The determinant of this system is a polynomial $h(x_1, \dots, x_r) \in R \setminus \Delta$. Consequently, the system is compatible over the module $\mathbf{Fit}_\Delta(C)$. ■

4.3 The Direct Module Extension of the Fitting's Radical of a U -Algebra.

In this section we introduce another type of extension of a U -algebra. As above, let A be a U -algebra over a field k . Let $\{z_\alpha, \alpha \in \Lambda\}$ be a maximal set of linearly independent modulo $\mathbf{Fit}(A)$ elements from A . Let V denote the linear span over the field k of $\{z_\alpha, \alpha \in \Lambda\}$.

Let M be a torsion free module over the ring of polynomials R . By the means of the module M we extend the algebra A to the algebra $A \oplus M$. By the definition, the algebra $A \oplus M$ is the direct sum of k -vector spaces $V \oplus \mathbf{Fit}(A) \oplus M$. To define the structure of an algebra on $V \oplus \mathbf{Fit}(A) \oplus M$ we need to introduce the multiplication on its elements. For the elements from V the multiplication is inherited from A . The multiplication of the elements from either M or $\mathbf{Fit}(A)$ on both elements from $\mathbf{Fit}(A)$ and elements from M results zero. If $b \in M$ and $z_\alpha \in V$, we set $b \circ z_\alpha = b \cdot x_\alpha$ and extend this definition of multiplication of b by elements from V to be linear. This operation turns $A \oplus M$ into a metabelian Lie algebra over k .

By the definition the canonical set of generators of $A \oplus M$ is the canonical set of generators of A together with an arbitrary set of generators of M . One verifies that

- $\mathbf{Fit}(A \oplus M) = \mathbf{Fit}(A) \oplus M$,
- the algebra $A \oplus M$ is a U -algebra.

We now point out some of the connections between the direct module extension and the universal closure of A .

Lemma 4.2 *Let M be a torsion free module over the ring of polynomials R . Then for every finite tuple u_1, \dots, u_n of elements from the module M and for every tuple of non-zero polynomials f_1, \dots, f_n from R there exists $u \in M$ so that $u \cdot f_i \neq u_i$ for any $i = 1, \dots, n$.*

Proof. Take a non-zero element $u_0 \in M$ and consider the following infinite set

$$K = \{u_0, u_0 \cdot x_\alpha, u_0 \cdot x_\alpha^2, \dots, u_0 \cdot x_\alpha^m, \dots\},$$

where x_α is an arbitrary variable from R . We next show that for any equation $u \cdot f_i = u_i$, $i = 1, \dots, n$ there exists no more than one element from K that satisfies this equation. Let $(u_0 \cdot x_\alpha^m) \cdot f_i = u_i$ and $l \neq m$. Then $(u_0 \cdot x_\alpha^l) \cdot f_i = (u_0 \cdot x_\alpha^m) \cdot f_i \cdot x_\alpha^{l-m} = u_i \cdot x_\alpha^{l-m} \neq u_i$. The latter is implied by the fact that M is a torsion-free R module. The collection of restrictions that are to be satisfied by u is finite, while the set K is infinite. In what follows that there exists a required element in K . ■

Lemma 4.3 *Let A be a U -algebra and let T_1 be a one generated torsion-free module over the ring of polynomials R . Then the algebra $A \oplus T_1$ is A -discriminated by the algebra A .*

Proof. Fix a finite number of non-zero elements from $A \oplus T_1$:

$$x_1 + u_1, \dots, x_n + u_n, x_i \in A, u_i \in T_1; u_i = t \cdot f_i,$$

where t is the generator of T_1 , $f_i \in R$. We construct an A -homomorphism $\varphi : A \oplus T_1 \rightarrow A$ so that $\varphi(x_i + u_i) \neq 0$ for any $i = 1, \dots, n$. For any $a \in A$ and some $u \in \text{Fit}(A)$, set $\varphi(a) = a$ and $\varphi(t) = u$. This map gives rise to an A -homomorphism from $A \oplus T_1$ to A in the obvious way. To show that $A \oplus T_1$ is A -discriminated by A we need to choose $u \in \text{Fit}(A)$ so that $x_i + u \cdot f_i \neq 0$ for any $i = 1, \dots, n$. For indices i such that $x_i \notin \text{Fit}(A)$ the required inequality holds whatever u is. In the event that $x_i \in \text{Fit}(A)$ and $f_i = 0$ we have $\varphi(x_i) = x_i$, i.e. the required inequality also holds. We now use Lemma 4.2 to choose the element $u \in \text{Fit}(A)$ agreeing with the conditions for the remaining indices. ■

Lemma 4.4 *For every positive integer s the free module T_s of the rank s over the ring R is discriminated by a one generated torsion-free R -module T_1 .*

Proof. Let $\{t_1, \dots, t_s\}$ be the free generators of the module T_s . The required map has the form: the element t_i goes into $t \cdot f_i$, where t is an arbitrary non-zero element from T_1 and the f_i 's are polynomials from R . The polynomials f_i 's are chosen to take the finite subset from T_s to non-zero elements. Consequently, there are only finitely many restrictions imposed on the f_i 's. Therefore, such polynomials exist. ■

Proposition 4.3 *If A is a U -algebra and M is a finitely generated module over R then the algebra $A \oplus M$ is A -discriminated by A .*

Proof. The module M embeds into the free module T_s of the rank s over the ring R (see [7], [5]). This implies that the algebra $A \oplus M$ A -embeds into the algebra $A \oplus T_s$. By Lemma 4.4 the module T_s is discriminated by the module T_1 and thus the algebra $A \oplus T_s$ is A -discriminated by the algebra $A \oplus T_1$. According to Lemma 4.3, the algebra $A \oplus T_1$ is A -discriminated by the algebra A . In what follows that the algebra $A \oplus M$ is A -discriminated by the algebra A . ■

Proposition 4.4 *Let A be a U -algebra and let M be a finitely generated module over R . Then $A - \text{ucl}(A) = A - \text{ucl}(A \oplus M)$.*

Proof. Show that every finite submodel $K = \{b_1, \dots, b_n\}$ of the algebra $A \oplus M$ A -embeds into the algebra A . Every element $b_i \in K$ decomposes into the following sum: $b_i = d_i + u_i$, where $d_i \in A$, $u_i \in M$. The tuple u_1, \dots, u_n of elements from the module M generates a submodule M_0 . Therefore, the submodel K A -embeds into the algebra $A \oplus M_0$, which, on account of Proposition 4.3, is A -discriminated by A . ■

Corollary 4.1 *Let A be a U -algebra and let M be a finitely generated module over R . Then $\text{ucl}(A) = \text{ucl}(A \oplus M)$.*

Proof. The proof is analogous to the proof of Proposition 4.4 and left to the reader. ■

The Δ -localisation and the direct module extension commute.

Lemma 4.5 *Let A be a U -algebra and let M be a finitely generated module over R . Then $(A \oplus M)_\Delta = A_\Delta \oplus M_\Delta$.*

Proof. By the means of direct verification:

$$\begin{aligned} (A \oplus M)_\Delta &= (V \oplus \mathbf{Fit}(A) \oplus M)_\Delta = V \oplus (\mathbf{Fit}(A) \oplus M)_\Delta = \\ &= V \oplus \mathbf{Fit}(A)_\Delta \oplus M_\Delta = A_\Delta \oplus M_\Delta. \end{aligned}$$
■

For the use of algebraic geometry over the free metabelian Lie algebra we next formulate some auxiliary statements about the direct module extension. By $\mathbf{Hom}_R(M, \mathbf{Fit}(A))$ denote the set of all R -homomorphisms from M to $\mathbf{Fit}(A)$.

Lemma 4.6 *In the above notation there exists a one-to-one correspondence between the set of R -homomorphisms from M to $\mathbf{Fit}(A)$ and the set of A -homomorphisms from $A \oplus M$ to A ,*

$$\mathbf{Hom}_R(M, \mathbf{Fit}(A)) \leftrightarrow \mathbf{Hom}_A(A \oplus M, A).$$

Proof. Let $\{m_\beta \mid \beta \in \mathcal{B}\}$ be a fixed system of generators of the module M . Let $\varphi \in \mathbf{Hom}_A(A \oplus M, A)$. In this notation, the A -homomorphism φ is completely defined by its values on the generators m_β of M . Let $\varphi(m_\beta) = c_\beta$. Since $bm_\beta = 0$ for any $b \in \mathbf{Fit}(A)$, we obtain $\varphi(bm_\beta) = bc_\beta = 0$. Consequently, by Lemma 2.4, $c_\beta \in \mathbf{Fit}(A)$. Moreover, every module relation between the letters m_β 's rewrites as a module relation between the letters c_β 's. In what follows that $\phi = \varphi|_M$ is a module R -homomorphism from M to $\mathbf{Fit}(A)$. The converse is also true. If $\phi \in \mathbf{Hom}_R(M, \mathbf{Fit}(A))$ then there is a unique element $\varphi \in \mathbf{Hom}_A(A \oplus M, A)$, corresponding to ϕ . This correspondence is given by the following rule:

$$\varphi(m_\beta) = \phi(m_\beta), \quad \varphi(a) = a, \quad a \in A.$$
■

Lemma 4.7 *Let M be a finitely generated module over R then for every $m \in M$, $m \neq 0$ there exists a homomorphism $\phi \in \mathbf{Hom}_R(M, \mathbf{Fit}(A))$ such that $\phi(m) \neq 0$, i.e. M is approximated by $\mathbf{Fit}(A)$.*

Proof. The module M embeds into the R -module T_s of the rank s . By Lemma 4.4, T_s is discriminated by the module T_1 . At last, clearly, T_1 embeds into $\text{Fit}(A)$. ■

Lemma 4.8 *Let M_1 and M_2 be two finitely generated modules over R . Then there exists a one-to-one correspondence between the set of R -homomorphisms from M_1 to M_2 and the set of A -homomorphisms from $A \oplus M_1$ to $A \oplus M_2$,*

$$\text{Hom}_R(M_1, M_2) \leftrightarrow \text{Hom}_A(A \oplus M_1, A \oplus M_2).$$

Moreover corresponding homomorphisms $\phi \in \text{Hom}_R(M_1, M_2)$ and $\varphi \in \text{Hom}_A(A \oplus M_1, A \oplus M_2)$ are injective or surjective simultaneously.

Proof. The proof is analogous to the one of Lemma 4.6. ■

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