A Curiosum Concerning Discrete Time Convolution

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Abstract—It is shown that the discrete time convolution of two absolutely summable nowhere zero sequences may be identically equal to zero.

Development

Consider two absolutely summable sequences \( a = \{a_n; n \in \mathbb{Z}\} \) and \( b = \{b_n; n \in \mathbb{Z}\} \) of real numbers. Further, assume that the sequences \( a \) and \( b \) are nowhere zero. Does it follow that the discrete time convolution \( a * b \) is nowhere zero? Does it follow that \( a * b \) is nonzero on some nonempty subset of \( \mathbb{Z} \)? From a linear systems viewpoint, does an absolutely summable nowhere zero input to a discrete time linear time-invariant system described via discrete time convolution with a fixed absolutely summable nowhere zero sequence result in an output which is nonzero somewhere? The following development, inspired by [1, pp. 354–356], addresses these questions.

To begin, we will use the following notation. For an absolutely summable sequence of real numbers \( a = \{a_n; n \in \mathbb{Z}\} \), let \( T_a \) map absolutely summable sequences of real numbers into absolutely summable sequences of real numbers via

\[
[T_a(a)]_n = \sum_{k=-\infty}^{\infty} a_k a_{n-k}
\]

where \( a = \{a_n; n \in \mathbb{Z}\} \) is any absolutely summable sequence of real numbers. For any two absolutely summable sequences of real numbers...
numbers $\alpha = \{ \alpha_n : n \in \mathbb{Z} \}$ and $\beta = \{ \beta_n : n \in \mathbb{Z} \}$, it follows that

$$[(T_a \circ T_b)(a)]_k = T_a \left( \sum_{n=-\infty}^{\infty} \beta_n a_{k-n} \right)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \alpha_m \beta_n a_{k-n-m}$$

$$= \sum_{n=-\infty}^{\infty} \lambda_n a_{k-n}$$

where we define $\lambda_n = \sum_{k=-\infty}^{\infty} \alpha_k \beta_n$. Finally, note that for any two absolutely summable sequences of real numbers $a$ and $b$, it follows via Fubini’s theorem that $T_a(a) \ast T_b(b) = (T_a \circ T_b)(a \ast b)$.

**Theorem 1:** Letting the above paragraph set notation, there exist two nonidentically zero absolutely summable sequences of real numbers $a$ and $\beta$ such that for any absolutely summable sequences of real numbers $a$ and $b$, $T_a(a) \ast T_b(b) = 0$.

**Proof:** Recall that the function $|\cos(x)|$ is expressible as a Fourier series given by $\sum_{m=-\infty}^{\infty} c_m \exp(\imath m x)$ where $\imath$ denotes the imaginary unit and where it follows easily that $c_m = 0$ if $n$ is odd and $c_m = (2/\pi) \left[ (-1)^{m/2} / (1 - n^2) \right]$ if $n$ is even. Further, if we define

$$f_1(x) = \frac{1}{2} \left( |\cos(x)| + \cos(x) \right)$$

and

$$f_2(x) = \frac{1}{2} \left( |\cos(x)| - \cos(x) \right)$$

then $f_1(x) f_2(x) = 0$, so

$$f_1(x) = \sum_{n=-\infty}^{\infty} \alpha_n \exp(\imath n x)$$

and

$$f_2(x) = \sum_{n=-\infty}^{\infty} \beta_n \exp(\imath n x)$$

where

$$\alpha_n = \begin{cases} \frac{1}{\pi} \left[ (-1)^{n/2} / (1 - n^2) \right] & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta_n = \begin{cases} \frac{-1}{4} & \text{if } n = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

But

$$f_1(x) f_2(x) = 0 = \sum_{n=-\infty}^{\infty} \alpha_n \exp(\imath n x) \sum_{m=-\infty}^{\infty} \beta_m \exp(\imath m x)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \alpha_n \beta_m \exp(\imath (n + m) x)$$

$$= \sum_{n=-\infty}^{\infty} \lambda_n \exp(\imath n x)$$

where $\lambda_n = \sum_{n=-\infty}^{\infty} \alpha_n \beta_n$ as before. Further, since this sum is equal to zero, it follows that $\lambda_n = 0$ for every integer $n$. Thus, it follows that $T_a(a) \ast T_b(b) = (T_a \circ T_b)(a \ast b) = 0$ for any absolutely summable sequences of real numbers $a$ and $b$.

**Theorem 2:** There exist two nowhere zero absolutely summable sequences of real numbers such that their discrete time convolution is identically equal to zero.

**Proof:** Let $f(k)$ equal one when $k$ is either zero or one, and equal zero otherwise. Then in the proof of Theorem 1, choose $a_k = b_k = f(k)$. Then it follows that $T_a(a)(k) = \sum_{n=-\infty}^{\infty} \alpha_n f(k - n)$ and $T_b(b)(k) = \sum_{n=-\infty}^{\infty} \beta_n f(k - n)$ define nowhere zero absolutely summable sequences such that $T_a(a) \ast T_b(b) = 0$.

**Q.E.D.**

**Conclusion**

In conclusion, we hope that the result of this paper will be helpful to those in the engineering community attempting to identify the input to a discrete time linear time-invariant system via some operations on the output, such as in deconvolution problems. Further, this result should be of interest to those concerned with the analysis of cascaded systems in that two discrete time linear time-invariant systems each characterized by discrete time convolution with a fixed nowhere zero absolutely summable sequence can, when cascaded, result in a no-pass discrete time linear time-invariant system.

**References**