Maximal Ratio Combining Analysis of the $\eta-\mu$ Generalized Fading Channels with Integer $\mu$

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Abstract—In this paper, we introduce a novel performance analysis of the $\eta-\mu$ generalized radio fading channels with integer value of the $\mu$ fading parameter, i.e., with even number of multipath clusters. This fading model includes the other fading models as special cases such as the Nakagami-$m$, the Hoyt, and the Rayleigh. We obtain novel unified and generic simple closed-form expressions for the Average Bit Error Rates and Ergodic Channel Capacity in the Additive White Generalized Gaussian Noise (AWGGN), which includes the AWGN, the Laplacian, the Gamma, and the impulsive noise as special cases, deploying Maximal Ratio Combining Diversity reception. Numerical simulation results corroborate the generality and accuracy of the derived unified expressions under the different test scenarios.

Keywords—$\eta-\mu$ fading; Additive White Generalized Gaussian Noise; Maximal Ratio Combining; Bit Error Rates; Ergodic Capacity; Unified Expressions.

I. INTRODUCTION

Wireless propagation is characterized by various effects that degrade the performance of wireless systems, e.g., shadowing and multipath fading [1] [2] [3]. Statistical distributions are typically used to characterize on these effects. Many statistical models exist that describe the characteristics of the radio signal. Rayleigh, Rice, Nakagami-$m$, Weibull, and Hoyt are the well-known models that describe the short term variations. However, in some situations, none of these distributions seem to effectively fit measured data [4]. In fact, some published worked questioned the use of the Nakagami-$m$ distribution as its tail does not seem to fit adequately experimental data [5]. In this context, the need for new statistical models became more evident. The efforts are directed to extend the existing fading models along different directions to obtain more flexibility and better fitting results. M. D. Yacoub proposed in [6] a new fading model that gained much momentum in the study of wireless systems, namely the $\eta-\mu$ generalized fading distribution to model non-line-of-sight scenarios. This model is distinct due to the remarkable flexibility offered by its parameters, which ultimately provide rather good fitting to experimental measurements of realistic and practical fading scenarios, particularly at the tail portion. The practicality of these distributions is also noticed by the fact that they include, as special cases, most of the widely used fading distributions.

The $\eta-\mu$ distribution follows the pattern of the multi-cluster analysis by including 2 formats. In format 1, the in-phase and quadrature components are independent from each other and have different powers, while the other format coincides with the $\lambda-\mu$ model, in which the in-phase and quadrature components are correlated with identical powers. Moreover, it was shown that $\eta-\mu$ model can accurately approximate the sum of independent non-identical (i.n.d.) Hoyt envelopes having arbitrary mean powers and arbitrary fading degrees [7] and the sum of i.i.d. $\eta-\mu$ envelopes [8]. The sum of i.i.d. $\eta-\mu$ power variables is also an $\eta-\mu$ power variable [6], and the sum of i.n.d. $\eta-\mu$ power variables is the sum of gamma variables with properly chosen parameters [9]. Some different forms of the probability density function of this fading model were also obtained in [10].

In the context of performance evaluation, this model was used only recently. The Level-Crossing Rate (LCR) and Average Fading Duration (AFD) were obtained in [11]. Average Channel Capacity (ACC) expressions for this fading channel using single branch receiver were obtained in [12] and in [10] for integer $\mu$. The moment generating function (MGF) for this generalized distribution was obtained in [13] and [14], in which the Average Bit Error Rates (ABER) for BPSK were numerically evaluated. The performance analysis of the ABER in multi-branch diversity reception that is usually utilized to combat fading [15] was also considered for the $\eta-\mu$ fading in [16] for specific modulation schemes. Most importantly, all the carried ABER analysis in the literature considers the AWGN as the noise model, and being completely separate and independent from the ACC analysis, and result in complicated expressions in the form of an infinite series or using complicated functions (e.g., Meijer-G).

To circumvent these issues, in this paper we present novel unified expressions for the ABER and ACC analysis over $\eta-\mu$ generalized fading channel subject to the Additive White Generalized Gaussian Noise (AWGGN), that includes the Gaussian, the Laplacian, the impulsive, and the Gamma noise models as special cases, deploying Maximal Ratio Combiner (MRC) receiver for coherent digital modulation schemes. To simplify the carried analysis, we assume an even number of multipath clusters and utilize the exponential approximations of the generalized $Q$-function and the $\log_2(\cdot)$ function. The newly derived unified expressions are very simple, accurate, and generic as they incorporate, as special cases, several others available in the literature.

The remaining part of this paper is structured as follows. The statistical derivations of the MRC output SNR and the used approximations are presented in section II. The ABER and ACC unified expressions for these generalized fading models in AWGGN environment are derived in section III. Sample simulation results for different test scenarios are shown in section IV. The paper findings and contributions are then summarized in section V.
II. STATISTICAL DERIVATIONS

A. The $\eta - \mu$ Fading Distribution

The power Probability Density Function (PDF) of the two fading models, $\eta - \mu$ and $\lambda - \mu$, is given in [6] [10] [14] by:

$$f_R(y) = \frac{2^{2\hat{\mu}+0.5\hat{\mu}}}{\Gamma(\hat{\mu})}\hat{\mu}^{-0.5}y^{\hat{\mu}-0.5}e^{-\frac{2 y}{\hat{\mu}}}, \quad 0 < y < \infty,$$

(1)

where $I_\mu(x)$ is the modified Bessel function of the first type [17], and $\hat{\mu}$ and $H$ are shown in Table I for the two fading formats. In these two generalized fading formats, the physical parameters $\hat{\mu}$, in format I, is the power ratio of the in-phase and quadrature scattered waves in each multipath channel, while in format II, is the correlation coefficient between the in-phase and quadrature scattered waves in each multipath channel, and $2\mu$ is the number of multipath channels.

**TABLE I: h and H VALUES FOR THE $\eta - \mu$ TWO FORMATS**

<table>
<thead>
<tr>
<th>Format</th>
<th>$h$</th>
<th>$H$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$(\eta^{-1} + \eta + 2)/\eta$</td>
<td>$(\eta^{-1} - 1)/\eta$</td>
<td>$0 &lt; \eta &lt; \infty$</td>
</tr>
<tr>
<td>II</td>
<td>$(2 - \eta)^{-1}$</td>
<td>$\eta(2 - \eta)^{-1}$</td>
<td>$1 &lt; \eta &lt; 1$</td>
</tr>
</tbody>
</table>

This generalized distribution include other fading models as special cases such as the Nakagami-$m$, the Hoyt, and the Rayleigh. The detailed relation is given in [6].

B. Maximum Ratio Combining Analysis

i. The MRC analysis with i.i.d. $\eta - \mu$ fading

Assuming that signals are transmitted over $L$ independently and identically distributed (i.i.d.) $\eta - \mu$ fading channels with MRC diversity, the corresponding PDF of the receiver output SNR is then given by [18]:

$$f_{MRC}(y) = \frac{2^{2\mu + 1/2}}{\Gamma(\mu + 1/2)}y^{\mu - 1}e^{-y/2}, \quad \mu > 1,$$

(2)

where $\hat{\mu}$ = $L\mu$, and $\xi = L\hat{\gamma}$, where $L$ denotes the number of diversity branches used.

**Proof:** following the previous assumptions, the instantaneous SNR of the MRC combiner output is given by [15]:

$$y = \sum_{i=1}^{L} y_i.$$

(4)

If the Moment Generating Function (MGF) for any of the MRC receiver branches is given, as in [14], by:

$$M_y(s) = \left[\frac{2\mu^2 h}{2(\hat{\mu}+\mu+1)}\right]^\mu,$$

(5)

and assuming that the average SNR for all $L$ branches to be the same, i.e. $\hat{y_i} = \hat{y_1} = \hat{y_2} = \cdots = \hat{y}_N$, the MGF of the MRC diversity output is then given as

$$M_{MRC}(s) = \prod_{i=1}^{L} M_y(s) = \left[\frac{2\mu^2 h}{2(\hat{\mu}+\mu+1)}\right]^\mu,$$

(6)

and defining $\bar{\mu} = L\mu$, so that $\mu = \bar{\mu}/L$, and $\xi = L\hat{\gamma}$, then (6) can be rewritten as

$$M_{MRC}(s) = \left[\frac{2\mu^2 h}{2(\hat{\mu}+\mu+1)}\right]^\mu.$$

(7)

By comparing (5) and (7), it can be deduced that the PDF of the output SNR with $L$ diversity branches MRC receiver is given by:

$$f_{MRC}(y) = \frac{2^{2\bar{\mu} + 1/2}}{\Gamma(\bar{\mu} + 1/2)}y^{\bar{\mu} - 1}e^{-y/2}, \quad \bar{\mu} > 1.$$n

(8)

This new generic PDF for this generalized fading distribution can be mapped to any of the presented equivalent PDF forms in [10] and their special cases.

For integer $\mu$, the corresponding PDF is given in [10] by:

$$f_{MRC}(y) = \sum_{k=0}^{n-1} \left[\frac{\mu}{k} \right]^{k} \left(1 - \frac{1}{\mu}\right)^{k} \psi^{y} e^{-\psi y} L(\xi, \psi),$$

(9)

where $L(\cdot, \cdot)$ denotes the incomplete gamma function and

$$\psi = \left[\frac{\mu}{k - \mu - 1}\right]^{k} \left(\frac{\mu - 1}{\mu}\right)^{k}, \quad m = \bar{\mu} - k, \quad \beta = \frac{\mu}{k},$$

$\xi = k + \bar{\mu}, \Omega_2 = \frac{\xi}{2(\hat{\mu}+\xi)}, \Omega_3 = \frac{\xi}{2(\hat{\mu}+\xi)}, \text{ and } p = \frac{1}{\mu}.$

The PDF in (9) will be utilized to derive simple expressions for the ABER and ACC for multichannel MRC receiver in the next section.

III. THE UNIFIED PERFORMANCE EVALUATION

A. ABER Analysis of MRC in $\eta - \mu$ and AWGGN

The average bit error rate due to a fading channel can be evaluated by averaging the bit error rate of the noise channel using the fading PDF [19] [20]. The averaging process can be then expressed as:

$$P_e = \mathcal{A} \int_0^{\infty} f_R(y) Q_u(\sqrt{\gamma}) dy,$$

(10)

where $f_R(y)$ is the fading PDF, $Q_u(\sqrt{\gamma})$ is the AWGGN, given in [1] by:

$$Q_u(x) = \frac{\alpha^2/a}{4(1/2)^2} \int_0^\alpha e^{-\alpha^2/4 \mu^2} \frac{du}{u},$$

(11)

with $\alpha = \sqrt{(1/2)^2} \sqrt{2} = 1/a$, and $\mathcal{A}$ and $\mathcal{B}$ are modulation dependent. Table II and III illustrate how the noise special models can be achieved from (11) and the $\mathcal{A}$ and $\mathcal{B}$ values for the different modulation schemes, respectively.

**TABLE II: RELATION BETWEEN $Q_u(x)$ AND SPECIAL NOISE MODELS.**

<table>
<thead>
<tr>
<th>Noise Dist.</th>
<th>Impulsive</th>
<th>Gamma</th>
<th>Laplacian</th>
<th>Gaussian</th>
<th>Uniform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.0</td>
<td>0.5</td>
<td>1.0</td>
<td>2.0</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

**TABLE III: $\mathcal{A}$ AND $\mathcal{B}$ VALUES FOR DIFFERENT MODULATIONS**

<table>
<thead>
<tr>
<th>Modulation Scheme</th>
<th>Average SER</th>
<th>$\mathcal{A}$</th>
<th>$\mathcal{B}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>$Q_u(\sqrt{2})$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>QPSK,4-QAM</td>
<td>$Q_u(\sqrt{2})$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>M-PAM</td>
<td>$2Q_u\left(\frac{4}{M-1}\right)$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>M-PSK</td>
<td>$2Q_u\left(\frac{4}{M-1}\right)$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Rectangular M-QAM</td>
<td>$\frac{\sqrt{2\pi}}{M-1}\left(\frac{4}{M-1}\right)$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Non-Rectangular M-QAM</td>
<td>$\frac{3}{M-1}$</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>
Using (9), then (10) can be written as

$$P_e = \sum_{k=0}^{\mu-1} A_k \psi_{\alpha} \int_0^\infty y^{m-1} e^{-\beta y} \mathcal{L}(\xi, p y) Q_a(\sqrt{\beta y}) dy, \quad (11)$$

and utilizing the exponential $Q_a(\sqrt{\xi})$ approximation from [1], given as:

$$Q_a(\sqrt{\xi}) \approx \sum_{i=1}^{4} a_i e^{-\lambda_i x}, \quad (12)$$

with $\alpha_i$ and $\lambda_i$ being given in Table V, then (11) can then be written in the form:

$$P_e = \sum_{k=0}^{\mu-1} \sum_{i=1}^{4} a_i \psi_{\alpha} \int_0^\infty y^{m-1} e^{-\beta y} \mathcal{L}(\xi, p y) Q_a(\sqrt{\beta y}) dy, \quad (13)$$

with $\beta = [\beta + \lambda_i B]$, with $\alpha_i$ and $\lambda_i$ being given in [1], which can be evaluated in a simple closed-form as:

$$P_e = \sum_{k=0}^{\mu-1} \sum_{i=1}^{4} a_i \psi_{\alpha} \frac{p F_1(1, m + \xi, [\xi + 1], -p \beta)}{\beta^{m+\xi}}, \quad (14)$$

where $\psi = A \alpha \psi\left[p \int_{0}^{1} \frac{1}{\xi + 1} \frac{d\xi}{\xi + 1}\right]$ and $\int_{0}^{\infty} F_1(1, m + \xi, [\xi + 1], -p \beta)$ is the confluent hypergeometric function [17]. This new derived expression in (14) is simple, generic and generalized for the ABER analysis in the $\eta - \mu$ fading channels deploying MRC reception in AWGN with integer value of $\mu$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta_{\alpha}$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>$\alpha_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>44.920</td>
<td>126.460</td>
<td>389.400</td>
<td>96.540</td>
<td>0.130</td>
<td>2.311</td>
<td>12.52</td>
</tr>
<tr>
<td>1.1</td>
<td>0.0685</td>
<td>0.202</td>
<td>0.318</td>
<td>0.255</td>
<td>0.211</td>
<td>2.185</td>
<td>0.857</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0991</td>
<td>0.149</td>
<td>0.136</td>
<td>0.125</td>
<td>0.241</td>
<td>0.732</td>
<td>10.57</td>
</tr>
<tr>
<td>2</td>
<td>0.126</td>
<td>1.047</td>
<td>1.125</td>
<td>0.442</td>
<td>0.939</td>
<td>0.833</td>
<td>0.994</td>
</tr>
</tbody>
</table>

In section IV, the simulation results presented therein is based on (14).

B. ACC Analysis in $\eta - \mu$ fading Channel

The Average Channel Capacity (ACC) in fading channels can be obtained by averaging the capacity of AWGN channel over the fading PDF [15], i.e.

$$C = \int_0^\infty \log_2(1 + \gamma) f_\gamma(\gamma) d\gamma. \quad (15)$$

One can directly notice that (10) and (15) are very similar. If the $\log_2(\cdot)$ exponential approximation given in [1] is utilized, then the $C$ will be of exactly the same form, as that in (14). Hence, $C$ is also given by:

$$C = \sum_{i=1}^{4} \sum_{k=0}^{\mu-1} \omega_{\alpha} \psi_{\alpha} \int_0^\infty y^{m-1} e^{-\beta y} \mathcal{L}(\xi, p y) Q_a(\sqrt{\beta y}) dy, \quad (16)$$

with the values of the fitting parameters being replaced accordingly. The use of the exponential approximations in (14) and (16) allow the ABER and ACC expressions to be unified and given in simple closed-forms, avoiding any analytical complications.

IV. SIMULATION RESULTS

This section illustrates the simulation results for sample test scenarios of the derived unified expressions, and compares these results with numerical simulations. We will consider the test cases to be as generalized as possible (in terms of the modulation schemes and their order, receiver diversity antenna, noise and fading environments, etc...).

We first present sample ABER curves, and not Average Symbol Error Rates (ASER), and then few sample ACC scenarios.

The ABER for different modulation schemes of different constellation orders in $\eta - \mu$ fading subject to different noise environments (defined by $\alpha$ value as in Table II), using 3-branches ($L=3$) MRC reception, are simulated. The results are shown in Fig. 1, where the solid lines represent the numerical simulation curves and the dots correspond to (14). One can see that the dots overlay the numerical curves confirming the accuracy of (14). As a second test, we assume the same fading condition of the first test without diversity ($L=1$), following the same modulation schemes and noise environments of the first test. The results, illustrated in Fig. 2, shows that the exact matching between the numerical curves and the generated results from (14).

Another test scenarios consider the special cases of the Nakagami-$m$ fading and Hoyt fading with 3-branches MRC diversity and without diversity, assuming AWGN, with the results being shown in Fig. 3 and Fig. 4, respectively (refer to [6] for the parameter mapping, for which Nakagami-$m$ is obtained as $\eta \rightarrow 0, \mu = m$, and for Hoyt $\eta = [1 - q]/[1 + q]$ and $\mu = 1$). One can clearly see that the approximate plots agree with the numerical simulated ones.

As for the ACC, two test cases are considered and illustrated here. The first test utilizes (16) to generate the ACC plots for the $\eta - \mu$ (format I) fading, whereas in the second test, (16) is used to generate the ACC curves for the $\eta - \mu$ fading with the same parameters values but using format II. The results are shown in Fig. 5 and 6, respectively. One can see the accurate matching as the dots generated using (16) overlay the exact ACC plots. It’s concluded from ABER and ACC resulting plots that the expressions in (14) and (16) are very accurate, unified and generic for ABER and ACC in these fading models and their special cases.

V. CONCLUSION

In this paper, we presented novel performance analysis of the $\eta - \mu$ generalized radio fading channels with integer value of the $\mu$ fading parameter, i.e. with even number of multipath clusters, which includes other fading models as special cases such as the Nakagami-$m$, the Hoyt, and the Rayleigh. New unified and generic simple closed-form expressions for the Ergodic Channel Capacity and the Average Bit Error Rates using MRC reception in Additive White Generalized Gaussian Noise (AWGGN), which includes the Gaussian, the Laplacian, the Gamma, and the impulsive noise models as special cases, were derived. Numerical simulations proved the generality and accuracy of the derived unified expressions under the different test scenarios.
REFERENCES


