The Laplace derivative

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Abstract. A function $f : \mathbb{R} \to \mathbb{R}$ is said to have the n-th Laplace derivative on the right at x if f is continuous in a right neighborhood of x and there exist real numbers $\alpha_0, \dots, \alpha_{n-1}$ such that $s^{n+1} \int_0^\delta e^{-st}[f(x+t) - \sum_{i=0}^{n-1} \alpha_i t^i/i!]\,dt$ converges as $s \to +\infty$ for some $\delta > 0$. There is a corresponding definition on the left. The function is said to have the n-th Laplace derivative at x when these two are equal, the common value is denoted by $f_{\langle n\rangle}(x)$.

In this work we establish the basic properties of this new derivative and show that, by an example, it is more general than the generalized Peano derivative; hence the Laplace derivative generalizes the Peano and ordinary derivatives.

Keywords: Peano derivative, generalized Peano derivative, Laplace derivative, Laplace transform, Tauberian theorem

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1. Introduction

The Peano derivative and its generalizations have received considerable attention, see for example [1], and [3]–[16].

Lee's generalized Peano derivative [8] is of interest in this work. A continuous function f has the generalized Peano derivative at a point x, denoted by $f_{[1]}(x)$, if some k-th primitive of f has the $(1+k)$-th Peano derivative at x. In [14] it was shown that this implies that

$$\lim_{s\to+\infty} s^{k+2} \int_0^\delta e^{-st}[f^{(-k)}(x+t) - f(x)t^k/k!]\,dt = f_{[1]}(x)$$

for every $\delta > 0$, where $f^{(-k)}$ is the k-th primitive of f to be defined in Section 2. Integrating by parts k times shows that

$$\lim_{s\to+\infty} s^2 \int_0^\delta e^{-st}[f(x+t) - f(x)]\,dt = f_{[1]}(x). \tag{1}$$

The statement "$f_{[1]}(x)$ exists implies (1)" is an Abelian theorem in Laplace transform theory. In this work we show that (as is usually the case) the converse

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does not hold. Namely, that $s^2\int_0^\delta e^{-st}[f(x+t)-f(x)]\,dt$ may converge as $s\to+\infty$ for some $\delta>0$ even though $f_{[1]}(x)$ does not exist.

This suggests that the limit behavior of the Laplace transform could be used to define a generalized derivative. We use this idea to define the Laplace derivative of order $n\in\mathbb{Z}^+$ and show that it generalizes the generalized Peano derivative.

2. Preliminaries

We denote the real numbers and integers by $\mathbb{R}$ and $\mathbb{Z}$ respectively. Then $\mathbb{R}^+$, $\mathbb{Z}^+$ and $\mathbb{R}_+$, $\mathbb{Z}_+$ denote the positive and non-negative elements respectively. When we say exists, we mean exists finite. Unless otherwise specified, function means real valued function of a real variable, and given two functions f and g, we write $f\sim g$ as $x\to x_0^+$ if $\lim_{x\to x_0^+} f(x)/g(x)=1$.

We use $f^{(-k)}$ to denote the particular k-th primitive of a function f given by

$$f^{(-k)}(x)=\int_\xi^x f^{(-k+1)}(t)\,dt \quad \text{for} \quad k\in\mathbb{Z}^+ \quad \text{and} \quad x\in\mathbb{R},$$

where $f^{(0)}=f$ and $\xi\in\mathbb{R}$ is fixed. There is no loss in generality since, when a result mentioned in this work depends on a primitive, it will be true that the result is independent of which primitive is taken (see [8]).

It will happen that $\sin(x)$ or $\cos(x)$ appears in an expression according to whether a certain integer is even or odd. It will simplify the exposition if we adopt the convention that the notation $\mathcal{S}(x)$ will denote the appropriate trigonometric function.

3. The Laplace derivative

We say that a function f has the *n-th Peano derivative* (*PD*) *at* x, $n\in\mathbb{Z}^+$, if there exist real numbers $f_{(1)}(x),\dots,f_{(n-1)}(x)$ such that

$$\frac{f(x+h)-f(x)-f_{(1)}(x)h-\cdots-f_{(n-1)}(x)h^{n-1}/(n-1)!}{h^n/n!}$$

converges as $h\to0$. In this case the limit is denoted by $f_{(n)}(x)$; for convenience we define $f_{(0)}(x):=f(x)$ (see [13]).

We say that f has the n-th *generalized Peano derivative* (*GPD*) *at* x, $n\in\mathbb{Z}^+$ if there exists a nonnegative integer k such that a k-th primitive of f has the $(n+k)$-th Peano derivative at x. We denote the result by $f_{[n]}(x)$ and for convenience we define $f_{[0]}(x):=f(x)$ (see [8], [10]). From our definition of the primitive, it follows that this is equivalent to saying that f has the n-th GPD at x if there exists a nonnegative integer k and real numbers $f_{[1]}(x), \dots, f_{[n-1]}(x)$ such that

$$(2)\qquad \frac{\begin{array}{r} f^{(-k)}(x+h) - f(x)h^k/k! - f_{[1]}(x)h^{1+k}/(1+k)! - \cdots \\ \cdots - f_{[n-1]}(x)h^{n-1+k}/(n-1+k)! \end{array}}{h^{n+k}/(n+k)!}$$

converges as $h \to 0$.

The lim inf (lim sup) as $h \to 0$ of the quotient in (2) will be denoted by $l_k f_{(n)}(x)$ $(u_k f_{(n)}(x))$ or even l_k (u_k) when there is no possibility of confusion. Notice that l_0 and u_0 are just the n-th Peano derivatives of f and the standard proof of l'Hospital's rule gives the monotonicity property ([10]):

$$(3)\qquad l_0 \le l_1 \le l_2 \le \cdots \le u_2 \le u_1 \le u_0.$$

It follows that we can write $\lim_{k\to\infty} l_k = l$ and $\lim_{k\to\infty} u_k = u$. Then Theorem 2 in [14] implies that f has the n-th GPD at x if and only if $l = u \in \mathbb{R}$; in this case $f_{[n]}(x) = l$. In the proof of Theorem 2 we see that $l = u \in \mathbb{R}$ implies that

$$(4)\qquad s^{n+1} \int_0^\delta e^{-st}[f(x+t) - \sum_{i=0}^{n-1} f_{[i]}(x)t^i/i!]\,dt \quad \begin{array}{l}\text{converges as } s \to +\infty \\ \text{for some } \delta \in \mathbb{R}^+\end{array}$$

and that the limit is l. The converse, namely that (4) implies that $l = u \in \mathbb{R}$, was not considered in [14]. The main result of this work is that the converse is not true and is most easily stated after we make the following definitions.

Definition 3.1. *A function $f : \mathbb{R} \to \mathbb{R}$ has the n-th Laplace derivative on the right [left] at x if f is continuous in a right [left] neighborhood of x and there exist numbers $\alpha_0, \dots, \alpha_{n-1}$ such that* $s^{n+1}\int_0^\delta e^{-st}[f(x+t) - \sum_{i=0}^{n-1}\alpha_i t^i/i!]\,dt[(-s)^n s\int_{-\delta}^0 e^{st}[f(x+t) - \sum_{i=0}^{n-1}\alpha_i t^i/i!]\,dt]$ *converges as $s \to +\infty$ for some $\delta > 0$. In this case the limit is denoted by $f_{\langle n,+\rangle}(x)$ $[f_{\langle n,-\rangle}(x)]$.*

Definition 3.2. *A continuous function $f : \mathbb{R} \to \mathbb{R}$ has the n-th Laplace derivative at x if $f_{\langle n,+\rangle}(x) = f_{\langle n,-\rangle}(x)$. In this case the common value is denoted by $f_{\langle n\rangle}(x)$.*

We may now state the main result.

Theorem 3.3. *There exists a continuous function which has the Laplace derivative at a point and does not have the generalized Peano derivative there. Furthermore, the Laplace derivative of this function is not the generalized Peano derivative of any continuous function.*

The function guaranteed by this theorem is such that (4) holds (with limit equal to 0) while $l = -\infty$ and $u = +\infty$. This is the only possibility; namely, if (4) holds and any l_k (or u_k) is finite, then the Tauberian theorem [2] implies that $l_{k+1} = u_{k+1}$ and that the common value equals the limit in (4).

Finally, we will show that, if the Laplace derivative exists at a point $x \in \mathbb{R}$, then it is well defined and that the associated numbers $\alpha_0, \dots, \alpha_{n-1}$ are uniquely determined with $\alpha_0 = f_{\langle 0 \rangle}(x) := f(x)$ and $\alpha_i = f_{\langle i \rangle}(x)$ for $i = 1, \dots, n-1$.

4. Proof of the main result

For each integer $m \geq 2$ define the function $\phi_m : \mathbb{R}_+ \to \mathbb{R}$ by $\phi_m(0) = 0$ and $\phi_m(x) = x^{[1/m+m+1]} \sin(x^{-1/m})$ otherwise. Since each ϕ_m is m-times continuously differentiable on $\mathbb{R}_+$, it makes sense to define constants $c_{m,j}$ for integer j, $0 \leq j \leq m$ by

$$\phi_m^{(j)}(x) \sim c_{m,j}\, x^{[1/m+(m-j)(1+1/m)]} \mathcal{S}(x^{-1/m}) \quad \text{as} \quad x \to 0^+.$$

Now for each integer $m \geq 2$ define $f_m : \mathbb{R} \to \mathbb{R}$ by

$$f_m(x) = \begin{cases} \phi_m^{(m)}(|x|)/\beta_m & \text{if } \ 0 \leq |x| \leq \alpha_m, \\ 0 & \text{if } \ \alpha_m < |x|, \end{cases}$$

where α_m is the largest zero of $\phi_m^{(m)}$ less than $1/m$ and β_m is a positive constant to be specified shortly. The functions defined in this way are continuous on $\mathbb{R}$ since $\phi_m^{(m)}(\alpha_m) = 0$ and have support in $[-1/m, 1/m]$. In order to simplify the exposition, we use this definition of f_m since it is adequate to prove the first part of the theorem. However, to prove the second part, we will need to modify the definition slightly; we do this later.

Notice that $f_m^{(-k)}(x) = (\phi_m^{(m)})^{(-k)}(x)/\beta_m = \phi_m^{(m-k)}(x)/\beta_m$, with $\xi = 0$, for $0 \leq k \leq m$ and $0 \leq x \leq \alpha_m$. Thus,

$$f_m^{(-k)}(x) \sim \frac{c_{m,m-k}}{\beta_m} x^{[1/m+k(1+1/m)]}\, \mathcal{S}(x^{-1/m}) \quad \text{as} \quad x \to 0^+. \tag{5}$$

Since $f_m(0) = 0$, the quotient in (2) takes the form $(1+k)! f_m^{(-k)}(x)/x^{1+k}$ for the $n = 1$ case. It is easy to check that the smallest integer k such that $l_k = u_k$ is m. Thus, we have that $(f_m)_{(1+m)}^{(-m)}(0) = (f_m)_{[1]}(0) = 0$. Using (1), we find that

$$\lim_{s \to +\infty} s^2 \int_0^\infty e^{-st} f_m(t)\, dt = 0. \tag{6}$$

In addition, since $s^2 \int_0^\infty e^{-st} f_m(t)\, dt$ is a continuous function of s that converges to zero as s tends to zero from the right, it is bounded on $\mathbb{R}_+$. Hence, we may choose each β_m such that $|f_m(x)| \leq 1$ on $\mathbb{R}$ and $|s^2 \int_0^\infty e^{-st} f_m(t)\,dt| \leq 1$ for $s \in \mathbb{R}_+$. We define our example function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{m=2}^{\infty} a_m f_m(x)$$

for all x where $\{a_m\}_{m=2}^\infty$ is a sequence of positive numbers which we initially require to satisfy the condition that

$$a_2 \leq 1 \quad \text{and} \quad a_{m+1} < a_m/5 \quad \text{for each integer} \quad m \geq 2. \tag{7}$$

Then, since $|f_m| \leq 1$ for all m, the previous sum converges uniformly on $\mathbb{R}$ to a continuous function with support in $[-1, 1]$ and $|f(x)| \leq 5/4$ on $\mathbb{R}$.

Now we show that $f_{\langle 1\rangle}(0) = 0$ and, since f is symmetric, it will suffice to show that $f_{\langle 1,+\rangle}(0) = 0$. Let $s, \delta \in \mathbb{R}^+$ be arbitrary, then

$$s^2 \int_0^\delta e^{-st} f(t)\,dt = s^2 \int_0^\delta e^{-st} \sum_{m=2}^{\infty} a_m f_m(t)\,dt = \sum_{m=2}^{\infty} a_m\, s^2 \int_0^\delta e^{-st} f_m(t)\,dt$$

by the bounded convergence theorem. Furthermore, since $|s^2 \int_0^\delta e^{-st} f_m(t)\,dt| \leq 1$ for all $s \in \mathbb{R}_+$, the convergence is uniform with respect to s. This and (6) imply that

$$f_{\langle 1,+\rangle}(0) = \lim_{s\to+\infty} s^2 \int_0^\delta e^{-st} f(t)\,dt = \sum_{m=2}^{\infty} a_m \lim_{s\to+\infty} s^2 \int_0^\delta e^{-st} f_m(t)\,dt = 0.$$

Preparing to show that $f_{[1]}(0)$ does not exist, we introduce two additional sequences $\{x_m\}_{m=2}^\infty$ and $\{y_m\}_{m=2}^\infty$ satisfying

$$0 < \cdots < x_{m+1} < y_m < x_m < \cdots < y_2 < x_2 \leq 1 \tag{8}$$

and simultaneously complete the definition of the sequence $\{a_m\}_{m=2}^\infty$.

Set $x_1 = y_1 = 1 = a_2$. Then, for an integer $m \geq 2$, assume that $x_1, x_2, \ldots, x_{m-1}$ and $y_1, y_2, \ldots, y_{m-1}$ and $a_2, a_3, \ldots, a_m$ have been chosen. We will choose x_m, y_m and a_{m+1}.

First we choose $x_m \in (0, y_{m-1})$ such that for each $x \in (0, x_m]$ we have

$$|a_{k+2} f_{k+2}^{(-k)}(x) + \cdots + a_{m-1} f_{m-1}^{(-k)}(x)| < \frac{1}{4} \frac{a_m\,|c_{m,m-k}|}{\beta_m} x^{[1/m+k(1+1/m)]} \tag{9}$$

for each integer k, $0 \leq k \leq m-3$ (an empty condition when $m = 2$, in which case we choose $x_2 = 1/2 < y_1 = 1$). This is possible for $m \geq 3$ as follows. Let m' and k be integers such that $2 \leq m' < m$ and $0 \leq k \leq m'$. Then we have

$$f_{m'}^{(-k)}(x) \sim \frac{c_{m',m'-k}}{\beta_{m'}} x^{[1/m'+k(1+1/m')]}\, \mathcal{S}(x^{-1/m'}) \quad \text{as} \quad x \to 0^+.$$

Since $m' < m$ implies that $1/m + k(1 + 1/m) < 1/m' + k(1 + 1/m')$, the left hand side of (9) can be made smaller than any fixed constant multiple of $x^{[1/m+k(1+1/m)]}$ for all sufficiently small x.

Now we choose $y_m \in (0, x_m)$ such that for each integer $k = 0, \ldots, m-2$ there exist points z^+ and z^- (depending on k) in (y_m, x_m) such that

$$a_m f_m^{(-k)}(z^+) > +\frac{3}{4}\frac{a_m\,|c_{m,m-k}|}{\beta_m}(z^+)^{[1/m+k(1+1/m)]} > +3m\frac{(z^+)^{1+k}}{(1+k)!}$$

and

(10)
$$a_m f_m^{(-k)}(z^-) < -\frac{3}{4}\frac{a_m\,|c_{m,m-k}|}{\beta_m}(z^-)^{[1/m+k(1+1/m)]} < -3m\frac{(z^-)^{1+k}}{(1+k)!}\,.$$

Property (5) implies the left pair of inequalities, while the right pair of inequalities follow from $x^{[1/m+k(1+1/m)]-(1+k)} \to +\infty$ as $x \to 0^+$ since $k + 2 \le m$ implies that $1/m + k(1 + 1/m) < 1 + k$.

Now we choose a_{m+1} such that

$$0 < a_{m+1} < \frac{a_m}{5}\min\left\{1, \frac{|c_{m,0}|}{\beta_m}, \ldots, \frac{|c_{m,m}|}{\beta_m}\right\} y_m^{[1/m+m+1]}.$$

Then, for $x \in (y_m, x_m)$, we have

$$|a_{m+1}f_{m+1}^{(-k)}(x) + a_{m+2}f_{m+2}^{(-k)}(x) + \cdots| \le \sum_{m'=m+1}^{\infty} a_{m'}|f_{m'}^{(-k)}(x)| \le \sum_{m'=m+1}^{\infty} a_{m'}$$

since $|f_m| \le 1$ implies that $|f_m^{(-k)}| \le 1$ on $[0,1]$ for all $k \in \mathbb{Z}_+$. Hence

(11)
$$\begin{aligned}|a_{m+1}f_{m+1}^{(-k)}(x) + \cdots| &\le a_{m+1} + (\frac{1}{5})a_{m+1} + (\frac{1}{5})^2 a_{m+1} + \cdots = \frac{5}{4}a_{m+1}\\ &< \frac{5}{4}\frac{a_m}{5}\min\{1, \frac{|c_{m,0}|}{\beta_m}, \ldots, \frac{|c_{m,m}|}{\beta_m}\}y_m^{[1/m+m+1]}\\ &< \frac{1}{4}\frac{a_m}{\beta_m}|c_{m,m-k}|x^{[1/m+k(1+1/m)]}\end{aligned}$$

since $x \in (y_m, x_m)$ and $k \le m-2$. Thus, by the principle of recursive definition, we have sequences $\{a_m\}_{m=2}^{\infty}$, $\{x_m\}_{m=2}^{\infty}$ and $\{y_m\}_{m=2}^{\infty}$ which satisfy conditions (7), (8), (9), (10), and (11) for all integers $m \ge 2$.

This completes our preparations and, as before, it will suffice to show that $f_{[1,+]}(0)$ does not exist. Since $f(0) = 0$, the quotient in (2) takes the form

$(1+k)!f^{(-k)}(x)/x^{1+k}$, $k \in \mathbb{Z}_+$, and as previously mentioned we must show that $l_k = -\infty$ and $u_k = +\infty$ for each $k \in \mathbb{Z}_+$. Since the arguments are similar, we only show the latter; namely that

$$u_k = \limsup_{x\to 0^+} \frac{f^{(-k)}(x)}{x^{1+k}/(1+k)!} = +\infty \quad \text{for each} \quad k \in \mathbb{Z}_+.$$

Taking advantage of the monotonicity property, (3), it suffices to consider an arbitrary integer $k \geq 2$. Since $x \in [0,1]$, the bounded convergence theorem implies that we may integrate term by term to obtain

$$\begin{aligned} u_k &= \limsup_{x\to 0^+} \sum_{m=2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!} \\ &\geq \sum_{m=2}^{k} a_m \liminf_{x\to 0^+} \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!} + \liminf_{x\to 0^+} a_{1+k} \frac{f_{1+k}^{(-k)}(x)}{x^{1+k}/(1+k)!} \\ &\qquad + \limsup_{x\to 0^+} \sum_{m=k+2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!}\,. \end{aligned}$$

Using (5), we have $\lim_{x\to 0^+}(1+k)!f_m^{(-k)}(x)/x^{1+k} = 0$, for $m = 2,\dots,k$, and $\liminf_{x\to 0^+}(1+k)!f_{1+k}^{(-k)}(x)/x^{1+k} = -(1+k)!\,|c_{1+k,1}|/\beta_{1+k}$. Hence

$$\begin{aligned} u_k &\geq \limsup_{x\to 0^+} \sum_{m=k+2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!} - a_{1+k}\frac{(1+k)!\,|c_{1+k,1}|}{\beta_{1+k}} \\ &\geq \limsup_{k+2<l\to\infty} \max_{x\in(y_l,x_l)} \left\{ \sum_{m=k+2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!} \right\} - a_{1+k}\frac{(1+k)!\,|c_{1+k,1}|}{\beta_{1+k}}\,. \end{aligned}$$

To estimate this sum, let l be an integer such that $l > k+2$. Then

$$|a_{k+2}f_{k+2}^{(-k)}(x) + \cdots + a_{l-1}f_{l-1}^{(-k)}(x)| < \frac{1}{4}\frac{a_l|c_{l,l-k}|}{\beta_l}x^{[1/l+k(1+1/l)]}$$

for every $x \in (y_l, x_l)$ follows from (9). Furthermore,

$$|a_{l+1}f_{l+1}^{(-k)}(x) + \cdots| < \frac{1}{4}\frac{a_l|c_{l,l-k}|}{\beta_l}x^{[1/l+k(1+1/l)]}$$

for every $x \in (y_l, x_l)$ follows from (11).

According to (10) there exists a point $z^+ \in (y_l, x_l)$ such that

$$a_l f_l^{(-k)}(z^+) > +\frac{3}{4}\frac{a_l|c_{l,l-k}|}{\beta_l}(z^+)^{[1/l+k(1+1/l)]} > +3l\frac{(z^+)^{1+k}}{(1+k)!}\,.$$

Thus we have

$$\begin{aligned}\sum_{m=k+2}^{\infty} a_m f_m^{(-k)}(z^+) &\geq -\left|\sum_{m=k+2}^{l-1} a_m f_m^{(-k)}(z^+)\right| + a_l f_l^{(-k)}(z^+) \\ &\qquad - \left|\sum_{m=l+1}^{\infty} a_m f_m^{(-k)}(z^+)\right| \\ &\geq \left[-\frac{1}{4}+\frac{3}{4}-\frac{1}{4}\right]\frac{a_l|c_{l,l-k}|}{\beta_l}(z^+)^{[1/l+k(1+1/l)]} \\ &= \frac{1}{4}\frac{a_l|c_{l,l-k}|}{\beta_l}(z^+)^{[1/l+k(1+1/l)]} > l\frac{(z^+)^{1+k}}{(1+k)!}\,.\end{aligned}$$

Hence

$$\max_{x\in(y_l,x_l)}\left\{\sum_{m=k+2}^{\infty} a_m \frac{f_m^{(-k)}(x)}{x^{1+k}/(1+k)!}\right\} \geq \frac{l(z^+)^{1+k}/(1+k)!}{(z^+)^{1+k}/(1+k)!} = l.$$

Thus

$$u_k \geq \limsup_{k+2<l\to\infty}\; l - a_{1+k}\frac{(1+k)!\,|c_{1+k,1}|}{\beta_{1+k}} = +\infty.$$

In order to prove the second part of the theorem we need the functions f_m to be differentiable on $\mathbb{R}-\{0\}$. We modify the definition of f_m as follows.

First, define $\psi_{a,b}:\mathbb{R}_+\to\mathbb{R}$ for $a,b\in\mathbb{R}^+$, $a<b$, by

$$\psi_{a,b}(x) = \begin{cases} 1 & \text{if } 0\leq x<a, \\ F_{a,b}(x) & \text{if } a\leq x<b, \\ 0 & \text{if } b\leq x<\infty, \end{cases}$$

where $F_{a,b} = [2x^3-3ax^2-3bx^2+6abx+b^3-3ab^2]/(b-a)^3$. Observe that $\psi_{a,b}$ is continuously differentiable on $\mathbb{R}_+$.

Next, let α'_m be the largest zero of $\phi_m^{(m)}$ less than α_m. Finally, define

$$f_m(x) = \frac{\phi_m^{(m)}(|x|)\psi_{\alpha'_m,\alpha_m}(|x|)}{\beta_m}\,.$$

Now, each f_m is differentiable on $\mathbb{R} - \{0\}$ and has support contained in $[-1/m, 1/m]$. Thus in a neighborhood of any $x \neq 0$, f is the sum of at most finitely many non-zero differentiable functions and hence $f_{\langle 1\rangle}(x) = f^{(1)}(x)$.

With this new definition we repeat the construction of f and observe that equation (5) (for $0 \le x \le \alpha'_m$) remains valid, as does the remainder of the proof of the first part, although the constants a_m, β_m, x_m, and y_m may change.

Proceeding toward a contradiction, suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and such that $g_{[1]} = f_{\langle 1\rangle}$ on $\mathbb{R}$. Since $f_{\langle 1\rangle} = f^{(1)}$ on $\mathbb{R} - \{0\}$, g is differentiable on $\mathbb{R} - \{0\}$, and hence, without loss of generality, we may write $g = f$ on $\mathbb{R} - \{0\}$. Since f and g are continuous on $\mathbb{R}$, it must be $f \equiv g$ which is impossible since $f_{[1]}(0)$ does not exist. Hence no such g exists which shows that $f_{\langle 1\rangle}$ is not the generalized Peano derivative of any continuous function. □

5. Properties of the Laplace derivative

The Laplace derivative is well-defined as a result of the following lemma.

Lemma 5.1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be integrable and $n \in \mathbb{Z}_+$. If there exists $0 < \delta_0 < b - a$ and $\alpha \in \mathbb{R}$ such that*

$$\lim_{s\to+\infty} s^{1+n} \int_0^{\delta_0} e^{-st} f(a+t)\, dt = \alpha,$$

then the same is true for each $0 < \delta < b - a$ replacing δ_0.

Proof: The result is a consequence of the fact that $\lim_{s\to+\infty} s^{1+n} \int_c^d e^{-st} f(a+t)\, dt = 0$ for every $0 < c < d < b - a$. □

We need the following two lemmas to prove that the Laplace derivative is uniquely defined.

Lemma 5.2. *Let $\delta \in \mathbb{R}^+$ and $p, q \in \mathbb{Z}_+$. Then*

$$s^q \int_0^{\delta} e^{-st} t^p\, dt = p!\, s^{q-p-1} + \epsilon(s),$$

where $\epsilon(s) \to 0$ as $s \to +\infty$.

Proof: For $p, q \in \mathbb{Z}_+$, $s^q \int_0^{\delta} e^{-st} t^p\, dt = s^{q-p-1} \int_0^{s\delta} e^{-\tau} \tau^p\, d\tau$ for any $\delta \in \mathbb{R}^+$, where $\tau = st$. Since $\int_0^{\infty} e^{-\tau}\tau^p\, d\tau = p!$ and $\lim_{s\to+\infty} s^{q-p-1} \int_{s\delta}^{\infty} e^{-\tau}\tau^p\, d\tau = 0$, we have the desired result. □

Lemma 5.3. *Let $n \in \mathbb{Z}_+$ and $\epsilon : \mathbb{R}_+ \to \mathbb{R}$ be continuous with $\epsilon(0) = 0$. Then for all $\delta_0 \in \mathbb{R}^+$*

$$\lim_{s \to +\infty} s^{1+n} \int_0^{\delta_0} e^{-st} t^n \epsilon(t)\, dt = 0.$$

Proof: Let $\delta_0 \in \mathbb{R}^+$ be given. By the continuity of $\epsilon(t)$ there exists $M \in \mathbb{R}^+$ such that $|\epsilon(t)|$ is bounded by M on $[0, \delta_0]$. Let $\epsilon \in \mathbb{R}^+$ be arbitrary and choose $\delta \in (0, \delta_0)$ such that $|\epsilon(t)| < \epsilon$ on $[0, \delta]$. Then we have

$$\begin{aligned} |s^{1+n} \int_0^{\delta_0} e^{-st} t^n \epsilon(t)\, dt| &\le s^{1+n} \int_0^{\delta_0} e^{-st} t^n |\epsilon(t)|\, dt \\ &\le s^{1+n} \int_0^{\delta} e^{-st} t^n \epsilon\, dt + s^{1+n} \int_\delta^{\delta_0} e^{-s\delta} \delta_0^n M\, dt. \end{aligned} \tag{12}$$

By Lemma 5.2 the right hand side of (12) converges to $n!\epsilon + 0$ as $s \to +\infty$. The result follows since ϵ was arbitrary. □

Theorem 5.4 (Uniqueness Theorem)**.** *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be continuous, $x \in [a,b]$, and $n \in \mathbb{Z}^+$. If $f_{\langle n \rangle}(x)$ exists with associated numbers $\alpha_0, \dots, \alpha_{n-1}$, then the numbers are uniquely determined with $\alpha_0 = f(x) = f_{\langle 0 \rangle}(x)$ and $\alpha_i = f_{\langle i \rangle}(x)$ for $i = 1, \dots, n-1$.*

Proof: There is no loss in generality in proving the theorem only for the right hand one-sided derivative at $x \in [a, b)$. We know that there exists $\delta \in \mathbb{R}^+$ and $\epsilon_x : \mathbb{R}^+ \to \mathbb{R}$ such that $\epsilon_x(s) \to 0$ as $s \to +\infty$ and

$$f_{\langle n,+ \rangle}(x) + \epsilon_x(s) = s^{1+n} \int_0^{\delta} e^{-st} [f(x+t) - \sum_{i=0}^{n-1} \alpha_i t^i / i!]\, dt$$

for all $s \in \mathbb{R}^+$. Let m be an integer such that $1 \le m < n$ and $s \in \mathbb{R}^+$, then

$$\begin{aligned} \frac{f_{\langle n,+ \rangle}(x)}{s^{n-m}} + \frac{\epsilon_x(s)}{s^{n-m}} &= s^{1+m} \int_0^{\delta} e^{-st} [f(x+t) - \sum_{i=0}^{m-1} \alpha_i t^i / i!]\, dt \\ &\quad - s^{1+m} \int_0^{\delta} e^{-st} \alpha_m t^m / m!\, dt - s^{1+m} \int_0^{\delta} e^{-st} \sum_{i=m+1}^{n-1} \alpha_i t^i / i!\, dt. \end{aligned}$$

Letting $s \to +\infty$, and using Lemma 5.2 we obtain that $0 = f_{\langle m,+ \rangle}(x) - \alpha_m - 0$.

The $m = 0$ case, namely $\alpha_0 = f(0)$, follows from Lemma 5.3, with $n = 0$ and $\epsilon(t) = f(x+t) - f(x)$. □

Theorem 5.5 (Regularity Theorem). *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be continuous, $x \in [a,b]$, and $n \in \mathbb{Z}^+$. Then,*

(i) *if $f_{[n]}(x)$ exists, then $f_{\langle n\rangle}(x)$ exists and $f_{[n]}(x) = f_{\langle n\rangle}(x)$;*

(ii) *if $f_{(n)}(x)$ exists, then $f_{\langle n\rangle}(x)$ exists and $f_{[n]}(x) = f_{(n)}(x) = f_{\langle n\rangle}(x)$;*

(iii) *if $f^{(n)}(x)$ exists, then $f_{\langle n\rangle}(x)$ exists and $f_{[n]}(x) = f_{(n)}(x) = f^{(n)}(x) = f_{\langle n\rangle}(x)$; and*

(iv) *for each $k, n \in \mathbb{Z}_+$, $f^{(-k)}_{\langle n+k\rangle}(x)$ exists if and only $f_{\langle n\rangle}(x)$ does, in which case they are equal.*

Proof: There is no loss in generality in proving the theorem only for the right hand one-sided derivatives at $x \in [a,b)$.

Beginning with (ii), suppose that $f_{(n,+)}(x)$ exists. Since $f(x) = f_{(0,+)}(x) = f_{\langle 0,+\rangle}(x)$ there is a positive integer $m \leq n$ such that $f_{(j,+)}(x) = f_{\langle j,+\rangle}(x)$ for $j = 0 \ldots m-1$. Since $f_{(m,+)}(x)$ exists, there is a continuous $\epsilon_x : \mathbb{R}_+ \to \mathbb{R}$ such that

$$f(x+t) - \sum_{i=0}^{m-1} f_{\langle i,+\rangle}(x)t^i/i! = (t^m/m!)[f_{(m,+)}(x) + \epsilon_x(t)]$$

for all $t \in \mathbb{R}_+$, where $\epsilon_x(t) \to 0$ as $t \to 0^+$ and $\epsilon_x(0) = 0$. Then, using Lemmas 5.1, 5.2 and 5.3, there exists $0 < \delta_0 < b - x$ such that

$$\begin{aligned} f_{(m,+)}(x) &= \lim_{s\to+\infty} s^{1+m} \int_0^{\delta_0} e^{-st} t^m [f_{(m,+)}(x) + \epsilon_x(t)]/m!\, dt \\ &= \lim_{s\to+\infty} s^{1+m} \int_0^{\delta_0} e^{-st} [f(x+t) - \sum_{i=0}^{m-1} f_{\langle i,+\rangle}(x)t^i/i!]\, dt \\ &= f_{\langle m,+\rangle}(x). \end{aligned}$$

The proof of (ii) is completed by repeating this calculation $n - m + 1$ times and recalling the well known fact that the existence of $f_{(n,+)}(x)$ implies that $f_{[n,+]}(x) = f_{(n,+)}(x)$.

Statement (iii) is now clear since it is well known that if $f^{(n,+)}(x)$ exists then $f^{(n,+)}(x) = f_{(n,+)}(x)$ and, hence, (ii) completes the proof.

To see (i), assume that $f_{[n,+]}(x)$ exists. Then the definition of $f_{[n,+]}(x)$ implies that there exists $k \in \mathbb{Z}_+$ such that $f_{[n,+]}(x) = f^{(-k)}_{(n+k,+)}(x)$. Now, from (ii), we have that $f_{[n,+]}(x) = f^{(-k)}_{\langle n+k,+\rangle}(x)$ and the result will follow from (iv).

To see (iv), we proceed by induction on $n \in \mathbb{Z}_+$. Since f is continuous, (iii) implies the $n = 0$ case by observing that $f(x) = (f^{(-k)})^{(k,+)}(x) = f^{(-k)}_{\langle k,+\rangle}(x)$ for all $k \in \mathbb{Z}_+$.

Now, suppose that for some $n \in \mathbb{Z}^+$, (iv) holds for $0, \dots, n-1$. We want to show that for each $k \in \mathbb{Z}_+$, $f^{(-k)}_{\langle n+k,+\rangle}(x)$ exists if and only $f_{\langle n,+\rangle}(x)$ does, in which case they are equal.

Suppose that $f^{(-k)}_{\langle n+k,+\rangle}(x)$ exists for some $k \in \mathbb{Z}^+$ since there is nothing to prove if $k = 0$. We have

$$f^{(-k)}_{\langle n+k,+\rangle}(x) = \lim_{s \to +\infty} s^{1+n+k} \int_0^\delta e^{-st}[f^{(-k)}(x+t) - \sum_{i=k}^{k+n-1} f^{(-k)}_{\langle i,+\rangle}(x) t^i/i!]\, dt$$

for some $0 < \delta < b - x$, where $f^{(-k)}_{\langle i,+\rangle}(x) = 0$, $i = 0 \dots k-1$, by (iii) and our definition of $f^{(-k)}(x)$. If we integrate by parts and let s tend to infinity we obtain

$$\begin{aligned} f^{(-k)}_{\langle n+k,+\rangle}(x) &= \lim_{s \to +\infty} s^{n+k} \int_0^\delta e^{-st}[f^{(-k+1)}(x+t) - \sum_{i=k-1}^{k+n-2} f^{(-k+1)}_{\langle i,+\rangle}(x) t^i/i!]\, dt \\ &= f^{(-k+1)}_{\langle n+k-1,+\rangle}(x), \end{aligned}$$

where we have used our induction hypothesis to write $f^{(-k)}_{\langle i,+\rangle}(x) = f^{(-k+1)}_{\langle i-1,+\rangle}(x)$ for $i = k, \dots, k+n-1$. Hence, integrating by parts k times, we find that $f^{(-k)}_{\langle n+k,+\rangle}(x) = f_{\langle n,+\rangle}(x)$ as required.

Conversely, suppose that $f_{\langle n,+\rangle}(x)$ exists. We have

$$f_{\langle n,+\rangle}(x) = \lim_{(s \to +\infty)} s^{1+n} \int_0^\delta e^{-st}[f(x+t) - \sum_{i=0}^{n-1} f_{\langle i,+\rangle}(x) t^i/i!]\, dt$$

for some $\delta \in \mathbb{R}^+$. Integrate by parts and let s tend to infinity to obtain that

$$\begin{aligned} f_{\langle n,+\rangle}(x) &= \lim_{(s \to +\infty)} s^{1+n+1} \int_0^\delta e^{-st}[f^{(-1)}(x+t) - \sum_{i=1}^{n} f^{(-1)}_{\langle i,+\rangle}(x) t^i/i!]\, dt \\ &= f^{(-1)}_{\langle n+1,+\rangle}(x). \end{aligned}$$

Integrating by parts k times we obtain that $f_{\langle n,+\rangle}(x) = f^{(-k)}_{\langle n+k,+\rangle}(x)$ as required. Thus, by induction, (iv) is true completing the proof of the theorem. □

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